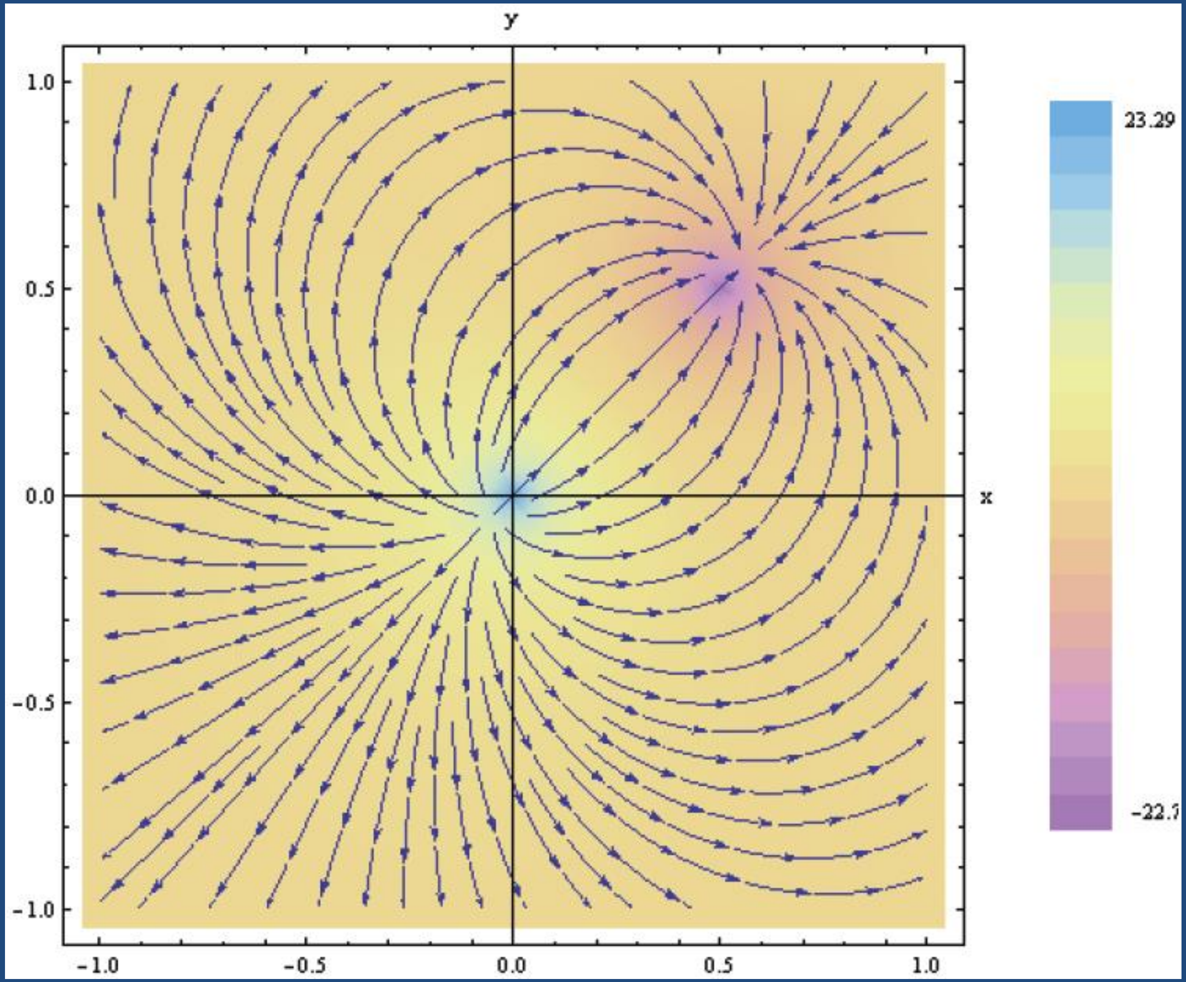


3. VECTOR ANALYSIS

7e Applied EM by Ulaby and Ravaioli

Chapter 3 Overview

Chapter Contents

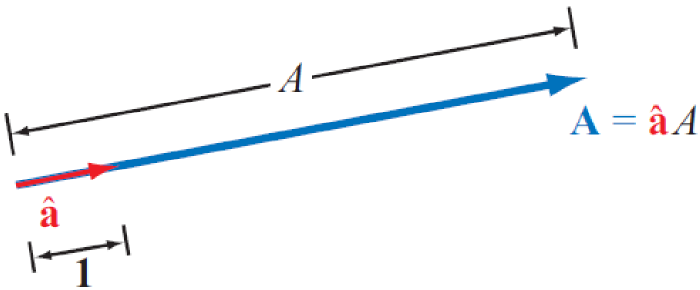
Objectives

Upon learning the material presented in this chapter, you should be able to:

1. Use vector algebra in Cartesian, cylindrical, and spherical coordinate systems.
2. Transform vectors between the three primary coordinate systems.
3. Calculate the gradient of a scalar function and the divergence and curl of a vector function in any of the three primary coordinate systems.
4. Apply the divergence theorem and Stokes's theorem.

https://www.youtube.com/watch?v=rB83DpBJQsE

Laws of Vector Algebra

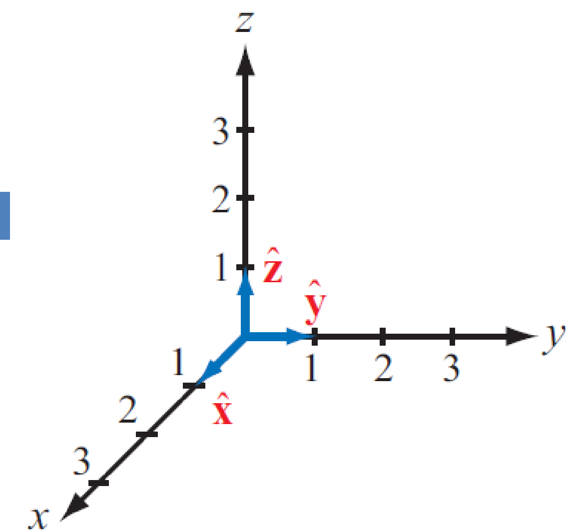


$$\mathbf{A} = \hat{\mathbf{a}}|\mathbf{A}| = \hat{\mathbf{a}}A$$

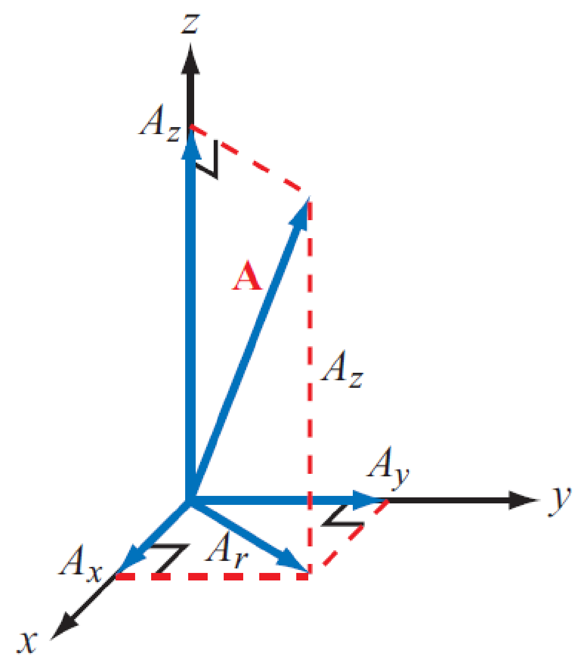
$$\mathbf{A} = \hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$$

$$A = |\mathbf{A}| = \sqrt[+]{A_x^2 + A_y^2 + A_z^2}$$

$$\hat{\mathbf{a}} = \frac{\mathbf{A}}{A} = \frac{\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z}{\sqrt[+]{A_x^2 + A_y^2 + A_z^2}}$$



(a) Base vectors



(b) Components of **A**

Properties of Vector Operations

Equality of Two Vectors

$$\mathbf{A} = \hat{\mathbf{a}}A = \hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z, \tag{3.6a}$$

$$\mathbf{B} = \hat{\mathbf{b}}B = \hat{\mathbf{x}}B_x + \hat{\mathbf{y}}B_y + \hat{\mathbf{z}}B_z, \tag{3.6b}$$

then $\mathbf{A} = \mathbf{B}$ if and only if $A = B$ and $\hat{\mathbf{a}} = \hat{\mathbf{b}}$, which requires that $A_x = B_x$, $A_y = B_y$, and $A_z = B_z$.

Equality of two vectors does not necessarily imply that they are identical; in Cartesian coordinates, two displaced parallel vectors of equal magnitude and pointing in the same direction are equal, but they are identical only if they lie on top of one another.

Commutative property

$$\mathbf{C} = \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$$

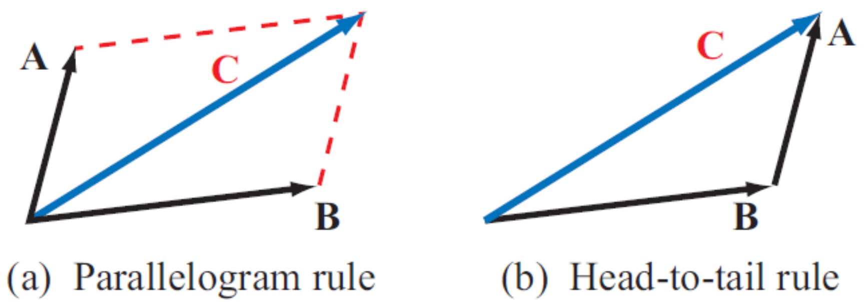
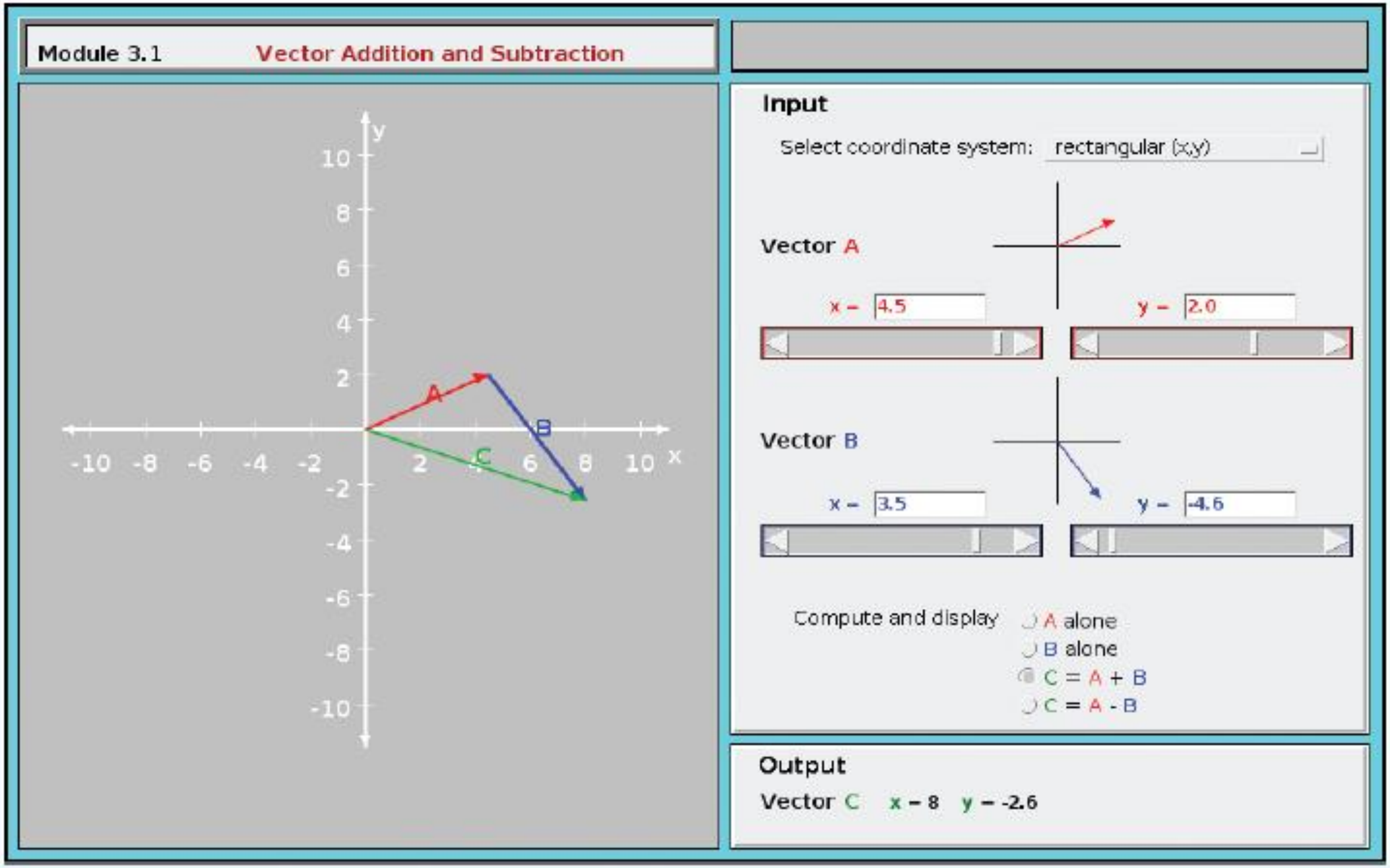


Figure 3-3: Vector addition by (a) the parallelogram rule and (b) the head-to-tail rule.

Module 3.1 Vector Addition and Subtraction Display two vectors in rectangular or cylindrical coordinates, and compute their sum and difference.



Position & Distance Vectors

Position Vector: From origin to point P

$$\mathbf{R}_1 = \overrightarrow{OP_1} = \hat{\mathbf{x}}x_1 + \hat{\mathbf{y}}y_1 + \hat{\mathbf{z}}z_1$$

$$\mathbf{R}_2 = \overrightarrow{OP_2} = \hat{\mathbf{x}}x_2 + \hat{\mathbf{y}}y_2 + \hat{\mathbf{z}}z_2$$

Distance Vector: Between two points

$$\begin{aligned}\mathbf{R}_{12} &= \overrightarrow{P_1P_2} \\ &= \mathbf{R}_2 - \mathbf{R}_1 \\ &= \hat{\mathbf{x}}(x_2 - x_1) + \hat{\mathbf{y}}(y_2 - y_1) + \hat{\mathbf{z}}(z_2 - z_1)\end{aligned}$$

the distance d between P_1 and P_2 equals the magnitude of $\mathbf{R}_{12}$:

$$\begin{aligned}d &= |\mathbf{R}_{12}| \\ &= [(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]^{1/2}. \qquad (3.12)\end{aligned}$$

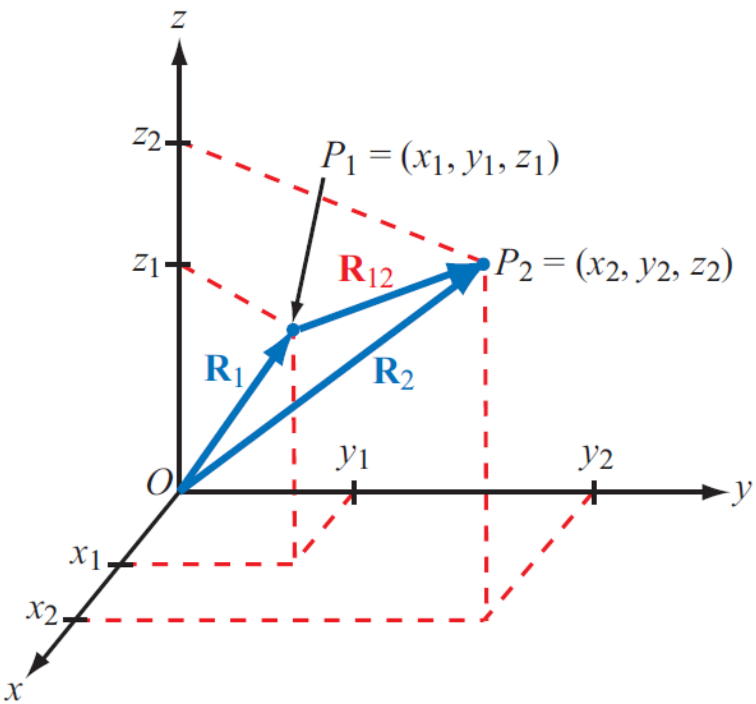


Figure 3-4: Distance vector $\mathbf{R}_{12} = \overrightarrow{P_1P_2} = \mathbf{R}_2 - \mathbf{R}_1$, where $\mathbf{R}_1$ and $\mathbf{R}_2$ are the position vectors of points P_1 and P_2, respectively.

Vector Multiplication: Scalar Product or "Dot Product"

$$\mathbf{A} \cdot \mathbf{B} = AB\cos\theta_{AB}$$

$$A = |\mathbf{A}| = \sqrt[+]{\mathbf{A} \cdot \mathbf{A}}$$

$$\theta_{AB} = \cos^{-1}\left[\frac{\mathbf{A} \cdot \mathbf{B}}{\sqrt[+]{\mathbf{A} \cdot \mathbf{A}}\ \sqrt[+]{\mathbf{B} \cdot \mathbf{B}}}\right]$$

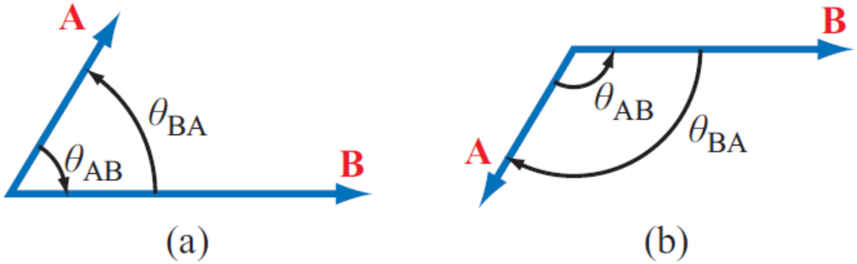


Figure 3-5: The angle θ_{AB} is the angle between **A** and **B**, measured from **A** to **B** between vector tails. The dot product is positive if $0 \le \theta_{AB} < 90°$, as in (a), and it is negative if $90° < \theta_{AB} \le 180°$, as in (b).

$$\hat{\mathbf{x}} \cdot \hat{\mathbf{x}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{z}} = 1,$$

$$\hat{\mathbf{x}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{z}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{x}} = 0.$$

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} \quad \text{(commutative property)},$$

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C} \quad \text{(distributive property)}$$

If $\mathbf{A} = (A_x, A_y, A_z)$ and $\mathbf{B} = (B_x, B_y, B_z)$, then

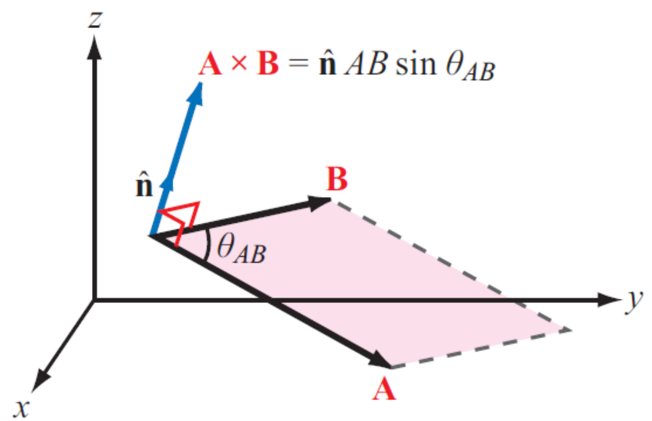
$$\mathbf{A} \cdot \mathbf{B} = (\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z) \cdot (\hat{\mathbf{x}}B_x + \hat{\mathbf{y}}B_y + \hat{\mathbf{z}}B_z).$$

Hence:

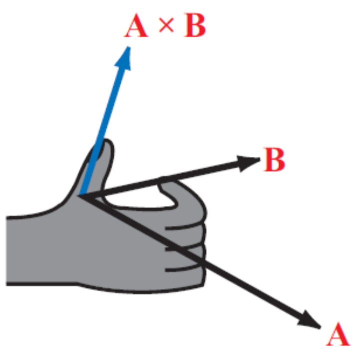
$$\mathbf{A} \cdot \mathbf{B} = A_xB_x + A_yB_y + A_zB_z.$$

Vector Multiplication: Vector Product or "Cross Product"

$$\mathbf{A} \times \mathbf{B} = \hat{\mathbf{n}}\, AB \sin\theta_{AB}$$



(a) Cross product



(b) Right-hand rule

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \qquad \text{(anticommutative)}$$

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C} \qquad \text{(distributive)}$$

$$\mathbf{A} \times \mathbf{A} = 0$$

$$\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}}, \qquad \hat{\mathbf{y}} \times \hat{\mathbf{z}} = \hat{\mathbf{x}}, \qquad \hat{\mathbf{z}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}}. \qquad (3.25)$$

Note the cyclic order ($xyzxyz\ldots$). Also,

$$\hat{\mathbf{x}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}} \times \hat{\mathbf{z}} = 0. \qquad (3.26)$$

If $\mathbf{A} = (A_x, A_y, A_z)$ and $\mathbf{B} = (B_x, B_y, B_z)$,

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}.$$

Example 3-1: Vectors and Angles

In Cartesian coordinates, vector **A** points from the origin to point $P_1 = (2, 3, 3)$, and vector **B** is directed from P_1 to point $P_2 = (1, -2, 2)$. Find

(a) vector **A**, its magnitude A, and unit vector $\hat{\mathbf{a}}$,
(b) the angle between **A** and the y-axis,
(c) vector **B**,
(d) the angle θ_{AB} between **A** and **B**, and
(e) the perpendicular distance from the origin to vector **B**.

Solution: (a) Vector **A** is given by the position vector of $P_1 = (2, 3, 3)$ as shown in Fig. 3-7. Thus,

$$\mathbf{A} = \hat{\mathbf{x}}2 + \hat{\mathbf{y}}3 + \hat{\mathbf{z}}3,$$

$$A = |\mathbf{A}| = \sqrt{2^2 + 3^2 + 3^2} = \sqrt{22}\,,$$

$$\hat{\mathbf{a}} = \frac{\mathbf{A}}{A} = (\hat{\mathbf{x}}2 + \hat{\mathbf{y}}3 + \hat{\mathbf{z}}3)/\sqrt{22}\,.$$

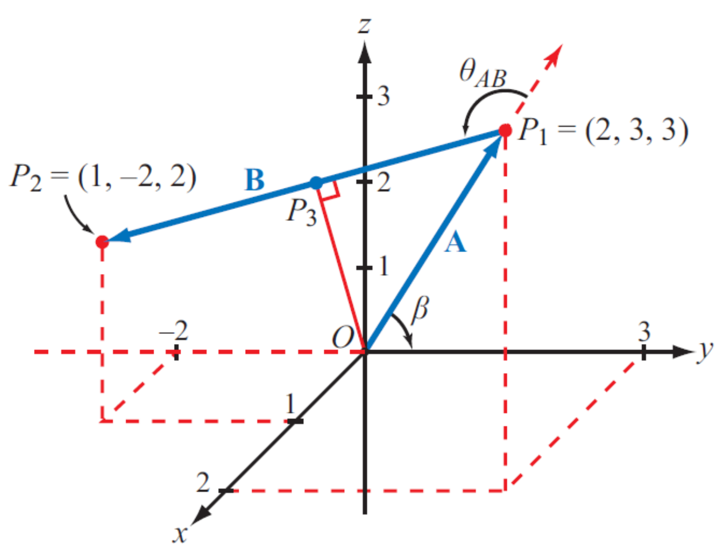


Figure 3-7: Geometry of Example 3-1.

(b) The angle β between **A** and the y-axis is obtained from

$$\mathbf{A} \cdot \hat{\mathbf{y}} = |\mathbf{A}||\hat{\mathbf{y}}| \cos\beta = A\cos\beta,$$

or

$$\beta = \cos^{-1}\left(\frac{\mathbf{A} \cdot \hat{\mathbf{y}}}{A}\right) = \cos^{-1}\left(\frac{3}{\sqrt{22}}\right) = 50.2^\circ.$$

(c)

$$\mathbf{B} = \hat{\mathbf{x}}(1-2) + \hat{\mathbf{y}}(-2-3) + \hat{\mathbf{z}}(2-3) = -\hat{\mathbf{x}} - \hat{\mathbf{y}}5 - \hat{\mathbf{z}}.$$

(d)

$$\theta_{AB} = \cos^{-1}\left[\frac{\mathbf{A} \cdot \mathbf{B}}{|\mathbf{A}||\mathbf{B}|}\right] = \cos^{-1}\left[\frac{(-2-15-3)}{\sqrt{22}\,\sqrt{27}}\right] = 145.1^\circ.$$

(e) The perpendicular distance between the origin and vector **B** is the distance $|\overrightarrow{OP_3}|$ shown in Fig. 3-7. From right triangle OP_1P_3,

$$|\overrightarrow{OP_3}| = |\mathbf{A}|\sin(180^\circ - \theta_{AB}) = \sqrt{22}\,\sin(180^\circ - 145.1^\circ) = 2.68.$$

Triple Products

Scalar Triple Product

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B}).$$

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

Vector Triple Product

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B}),$$

which is known as the "bac-cab" rule.

Example 3-2: Vector Triple Product

Given $\mathbf{A} = \hat{\mathbf{x}} - \hat{\mathbf{y}} + \hat{\mathbf{z}}2$, $\mathbf{B} = \hat{\mathbf{y}} + \hat{\mathbf{z}}$, and $\mathbf{C} = -\hat{\mathbf{x}}2 + \hat{\mathbf{z}}3$, find $(\mathbf{A}\times\mathbf{B})\times\mathbf{C}$ and compare it with $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$.

Solution:

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 1 & -1 & 2 \\ 0 & 1 & 1 \end{vmatrix} = -\hat{\mathbf{x}}3 - \hat{\mathbf{y}} + \hat{\mathbf{z}}$$

and

$$(\mathbf{A}\times\mathbf{B})\times\mathbf{C} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ -3 & -1 & 1 \\ -2 & 0 & 3 \end{vmatrix} = -\hat{\mathbf{x}}3 + \hat{\mathbf{y}}7 - \hat{\mathbf{z}}2.$$

A similar procedure gives $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \hat{\mathbf{x}}2 + \hat{\mathbf{y}}4 + \hat{\mathbf{z}}$.

Hence:

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) \neq (\mathbf{A}\times\mathbf{B})\times\mathbf{C}$$

Cartesian Coordinate System

Differential length vector

$$d\mathbf{l} = \hat{\mathbf{x}}\, dl_x + \hat{\mathbf{y}}\, dl_y + \hat{\mathbf{z}}\, dl_z = \hat{\mathbf{x}}\, dx + \hat{\mathbf{y}}\, dy + \hat{\mathbf{z}}\, dz, \quad (3.34)$$

where $dl_x = dx$ is a differential length along $\hat{\mathbf{x}}$, and similar interpretations apply to $dl_y = dy$ and $dl_z = dz$.

Differential area vectors

$$d\mathbf{s}_x = \hat{\mathbf{x}}\, dl_y\, dl_z = \hat{\mathbf{x}}\, dy\, dz \qquad (y\text{–}z \text{ plane}), \qquad (3.35a)$$

with the subscript on $d\mathbf{s}$ denoting its direction. Similarly,

$$d\mathbf{s}_y = \hat{\mathbf{y}}\, dx\, dz \qquad (x\text{–}z \text{ plane}), \qquad (3.35b)$$

$$d\mathbf{s}_z = \hat{\mathbf{z}}\, dx\, dy \qquad (x\text{–}y \text{ plane}). \qquad (3.35c)$$

A ***differential volume*** equals the product of all three differential lengths:

$$dV = dx\, dy\, dz. \qquad (3.36)$$

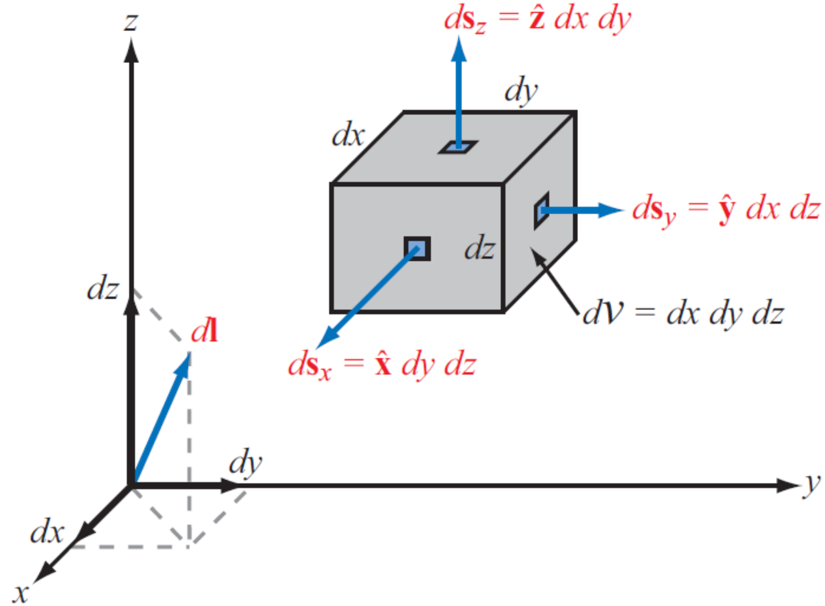
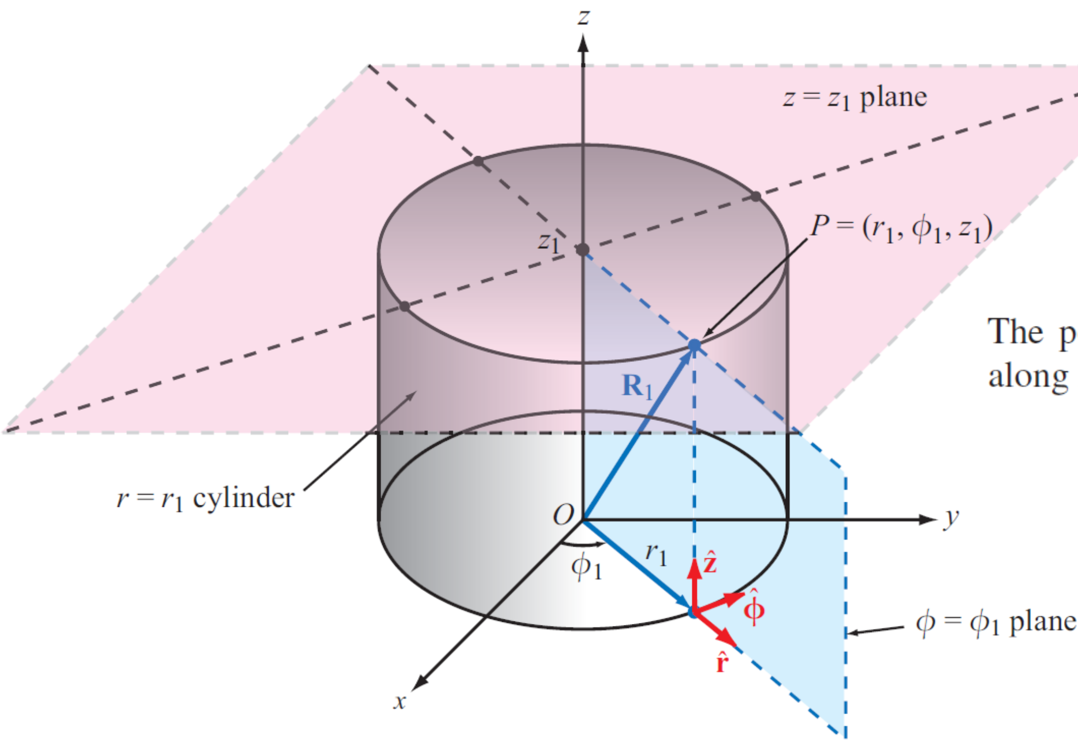


Table 3-1: Summary of vector relations.

	Cartesian Coordinates	Cylindrical Coordinates	Spherical Coordinates
Coordinate variables	x, y, z	r, ϕ, z	R, θ, ϕ
Vector representation $\mathbf{A} =$	$\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$	$\hat{\mathbf{r}}A_r + \hat{\boldsymbol{\phi}}A_\phi + \hat{\mathbf{z}}A_z$	$\hat{\mathbf{R}}A_R + \hat{\boldsymbol{\theta}}A_\theta + \hat{\boldsymbol{\phi}}A_\phi$
Magnitude of A $\lvert\mathbf{A}\rvert =$	$\sqrt[+]{A_x^2 + A_y^2 + A_z^2}$	$\sqrt[+]{A_r^2 + A_\phi^2 + A_z^2}$	$\sqrt[+]{A_R^2 + A_\theta^2 + A_\phi^2}$
Position vector $\overrightarrow{OP_1} =$	$\hat{\mathbf{x}}x_1 + \hat{\mathbf{y}}y_1 + \hat{\mathbf{z}}z_1$, for $P = (x_1, y_1, z_1)$	$\hat{\mathbf{r}}r_1 + \hat{\mathbf{z}}z_1$, for $P = (r_1, \phi_1, z_1)$	$\hat{\mathbf{R}}R_1$, for $P = (R_1, \theta_1, \phi_1)$
Base vectors properties	$\hat{\mathbf{x}} \cdot \hat{\mathbf{x}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{z}} = 1$ $\hat{\mathbf{x}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{z}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{x}} = 0$ $\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}}$ $\hat{\mathbf{y}} \times \hat{\mathbf{z}} = \hat{\mathbf{x}}$ $\hat{\mathbf{z}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}}$	$\hat{\mathbf{r}} \cdot \hat{\mathbf{r}} = \hat{\boldsymbol{\phi}} \cdot \hat{\boldsymbol{\phi}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{z}} = 1$ $\hat{\mathbf{r}} \cdot \hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}} \cdot \hat{\mathbf{z}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{r}} = 0$ $\hat{\mathbf{r}} \times \hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}$ $\hat{\boldsymbol{\phi}} \times \hat{\mathbf{z}} = \hat{\mathbf{r}}$ $\hat{\mathbf{z}} \times \hat{\mathbf{r}} = \hat{\boldsymbol{\phi}}$	$\hat{\mathbf{R}} \cdot \hat{\mathbf{R}} = \hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}} \cdot \hat{\boldsymbol{\phi}} = 1$ $\hat{\mathbf{R}} \cdot \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}} \cdot \hat{\mathbf{R}} = 0$ $\hat{\mathbf{R}} \times \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}$ $\hat{\boldsymbol{\theta}} \times \hat{\boldsymbol{\phi}} = \hat{\mathbf{R}}$ $\hat{\boldsymbol{\phi}} \times \hat{\mathbf{R}} = \hat{\boldsymbol{\theta}}$
Dot product $\mathbf{A} \cdot \mathbf{B} =$	$A_x B_x + A_y B_y + A_z B_z$	$A_r B_r + A_\phi B_\phi + A_z B_z$	$A_R B_R + A_\theta B_\theta + A_\phi B_\phi$
Cross product $\mathbf{A} \times \mathbf{B} =$	$\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\phi}} & \hat{\mathbf{z}} \\ A_r & A_\phi & A_z \\ B_r & B_\phi & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{R}} & \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\phi}} \\ A_R & A_\theta & A_\phi \\ B_R & B_\theta & B_\phi \end{vmatrix}$
Differential length $d\mathbf{l} =$	$\hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz$	$\hat{\mathbf{r}}\,dr + \hat{\boldsymbol{\phi}}r\,d\phi + \hat{\mathbf{z}}\,dz$	$\hat{\mathbf{R}}\,dR + \hat{\boldsymbol{\theta}}R\,d\theta + \hat{\boldsymbol{\phi}}R\sin\theta\,d\phi$
Differential surface areas	$d\mathbf{s}_x = \hat{\mathbf{x}}\,dy\,dz$ $d\mathbf{s}_y = \hat{\mathbf{y}}\,dx\,dz$ $d\mathbf{s}_z = \hat{\mathbf{z}}\,dx\,dy$	$d\mathbf{s}_r = \hat{\mathbf{r}}r\,d\phi\,dz$ $d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}\,dr\,dz$ $d\mathbf{s}_z = \hat{\mathbf{z}}r\,dr\,d\phi$	$d\mathbf{s}_R = \hat{\mathbf{R}}R^2\sin\theta\,d\theta\,d\phi$ $d\mathbf{s}_\theta = \hat{\boldsymbol{\theta}}R\sin\theta\,dR\,d\phi$ $d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}R\,dR\,d\theta$
Differential volume $dV =$	$dx\,dy\,dz$	$r\,dr\,d\phi\,dz$	$R^2\sin\theta\,dR\,d\theta\,d\phi$

Cylindrical Coordinate System



The position vector $\overrightarrow{OP}$ shown in Fig. 3-9 has components along r and z only. Thus,

$$\mathbf{R}_1 = \overrightarrow{OP} = \hat{\mathbf{r}} r_1 + \hat{\mathbf{z}} z_1. \tag{3.40}$$

The mutually perpendicular base vectors are $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\phi}}$, and $\hat{\mathbf{z}}$, with $\hat{\mathbf{r}}$ pointing away from the origin along r, $\hat{\boldsymbol{\phi}}$ pointing in a direction tangential to the cylindrical surface, and $\hat{\mathbf{z}}$ pointing along the vertical. Unlike the Cartesian system, in which the base vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ are independent of the location of P, in the cylindrical system both $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\phi}}$ are functions of ϕ.

Cylindrical Coordinate System

The base unit vectors obey the following right-hand cyclic relations:

$$\hat{\mathbf{r}} \times \hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}, \qquad \hat{\boldsymbol{\phi}} \times \hat{\mathbf{z}} = \hat{\mathbf{r}}, \qquad \hat{\mathbf{z}} \times \hat{\mathbf{r}} = \hat{\boldsymbol{\phi}}, \qquad (3.37)$$

and like all unit vectors, $\hat{\mathbf{r}} \cdot \hat{\mathbf{r}} = \hat{\boldsymbol{\phi}} \cdot \hat{\boldsymbol{\phi}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{z}} = 1$, and $\hat{\mathbf{r}} \times \hat{\mathbf{r}} = \hat{\boldsymbol{\phi}} \times \hat{\boldsymbol{\phi}} = \hat{\mathbf{z}} \times \hat{\mathbf{z}} = 0$.

In cylindrical coordinates, a vector is expressed as

$$\mathbf{A} = \hat{\mathbf{a}}|\mathbf{A}| = \hat{\mathbf{r}}A_r + \hat{\boldsymbol{\phi}}A_\phi + \hat{\mathbf{z}}A_z, \qquad (3.38)$$

$$dl_r = dr, \qquad dl_\phi = r\,d\phi, \qquad dl_z = dz. \qquad (3.41)$$

Note that the differential length along $\hat{\boldsymbol{\phi}}$ is $r\,d\phi$, not just $d\phi$. The differential length $d\mathbf{l}$ in cylindrical coordinates is given by

$$d\mathbf{l} = \hat{\mathbf{r}}\,dl_r + \hat{\boldsymbol{\phi}}\,dl_\phi + \hat{\mathbf{z}}\,dl_z = \hat{\mathbf{r}}\,dr + \hat{\boldsymbol{\phi}}r\,d\phi + \hat{\mathbf{z}}\,dz. \qquad (3.42)$$

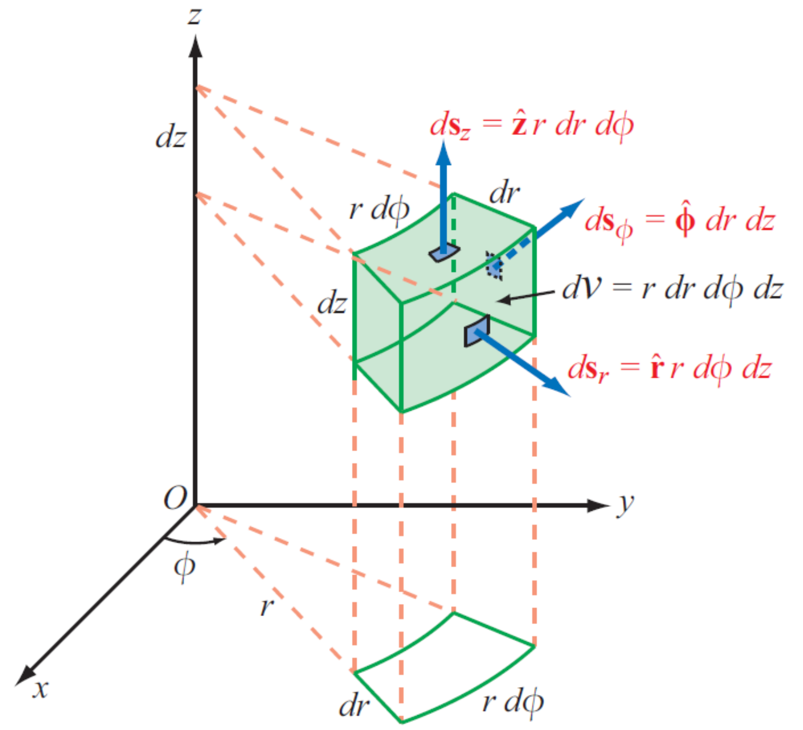


Figure 3-10: Differential areas and volume in cylindrical coordinates.

Example 3-3: Distance Vector in Cylindrical Coordinates

Find an expression for the unit vector of vector **A** shown in Fig. 3-11 in cylindrical coordinates.

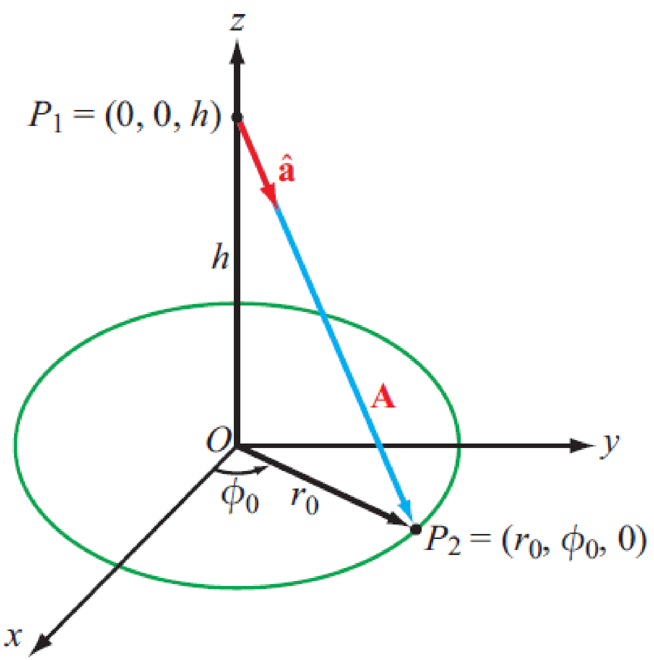


Figure 3-11: Geometry of Example 3-3.

Solution: In triangle OP_1P_2,

$$\overrightarrow{OP_2} = \overrightarrow{OP_1} + \mathbf{A}.$$

Hence,

$$\mathbf{A} = \overrightarrow{OP_2} - \overrightarrow{OP_1} = \hat{\mathbf{r}}r_0 - \hat{\mathbf{z}}h,$$

and

$$\hat{\mathbf{a}} = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{\hat{\mathbf{r}}r_0 - \hat{\mathbf{z}}h}{\sqrt{r_0^2 + h^2}}.$$

We note that the expression for **A** is independent of ϕ_0. That is, all vectors from point P_1 to any point on the circle defined by $r = r_0$ in the x–y plane are equal in the cylindrical coordinate system. The ambiguity can be eliminated by specifying that **A** passes through a point whose $\phi = \phi_0$.

Example 3-4: Cylindrical Area

Find the area of a cylindrical surface described by $r = 5$, $30° \leq \phi \leq 60°$, and $0 \leq z \leq 3$ (Fig. 3-12).

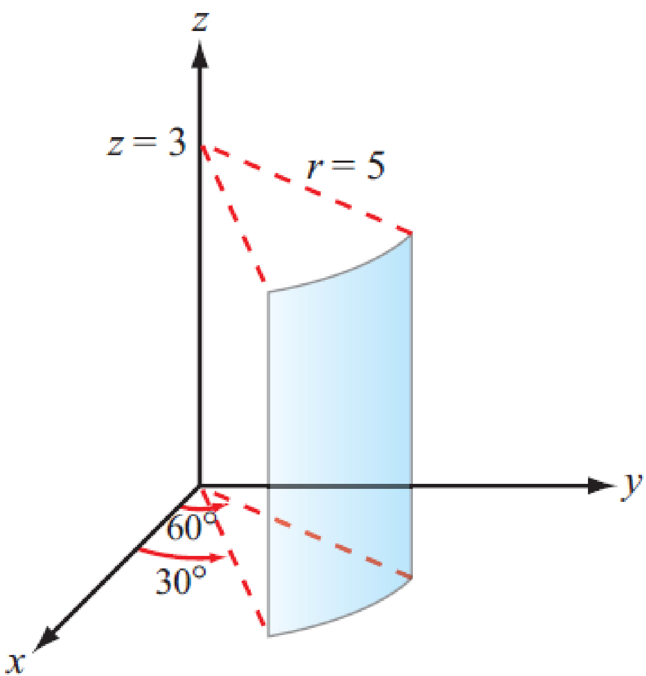


Figure 3-12: Cylindrical surface of Example 3-4.

Solution: The prescribed surface is shown in Fig. 3-12. Use of Eq. (3.43a) for a surface element with constant r gives

$$S = r \int_{\phi=30°}^{60°} d\phi \int_{z=0}^{3} dz$$

$$= 5\phi \Big|_{\pi/6}^{\pi/3} \; z \Big|_{0}^{3}$$

$$= \frac{5\pi}{2} .$$

Note that ϕ had to be converted to radians before evaluating the integration limits.

Spherical Coordinate System

$$\hat{\mathbf{R}} \times \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}, \quad \hat{\boldsymbol{\theta}} \times \hat{\boldsymbol{\phi}} = \hat{\mathbf{R}}, \quad \hat{\boldsymbol{\phi}} \times \hat{\mathbf{R}} = \hat{\boldsymbol{\theta}}. \qquad (3.45)$$

A vector with components A_R, A_θ, and A_ϕ is written as

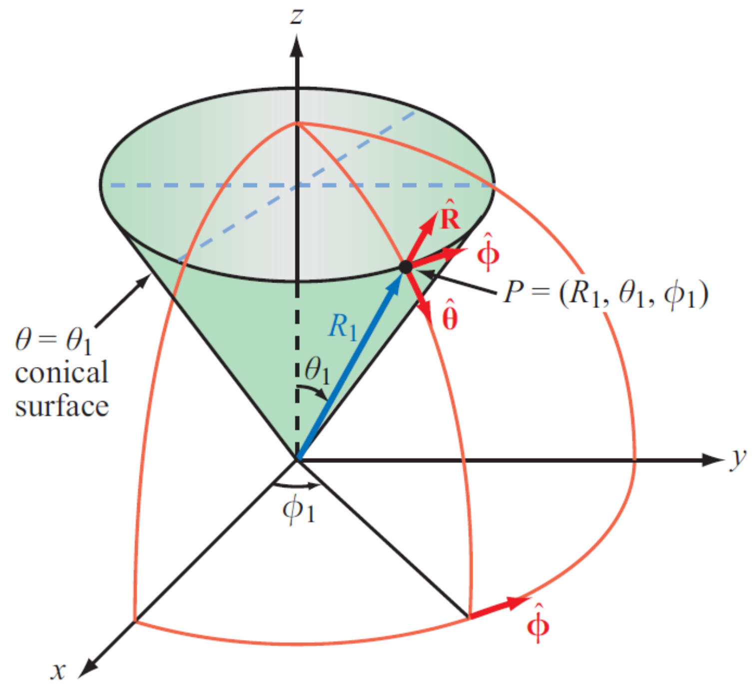
$$\mathbf{A} = \hat{\mathbf{a}}|\mathbf{A}| = \hat{\mathbf{R}}A_R + \hat{\boldsymbol{\theta}}A_\theta + \hat{\boldsymbol{\phi}}A_\phi, \qquad (3.46)$$

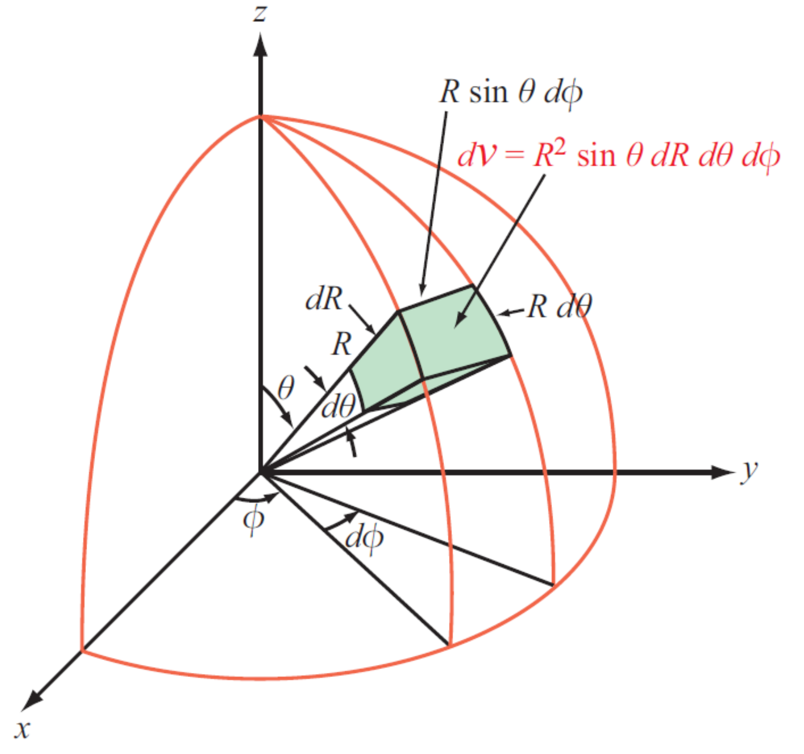
and its magnitude is

$$|\mathbf{A}| = \sqrt[+]{\mathbf{A} \cdot \mathbf{A}} = \sqrt[+]{A_R^2 + A_\theta^2 + A_\phi^2}. \qquad (3.47)$$

The position vector of point $P = (R_1, \theta_1, \phi_1)$ is simply

$$\mathbf{R}_1 = \overrightarrow{OP} = \hat{\mathbf{R}}R_1, \qquad (3.48)$$





Example 3-5: Surface Area in Spherical Coordinates

The spherical strip shown in Fig. 3-15 is a section of a sphere of radius 3 cm. Find the area of the strip.

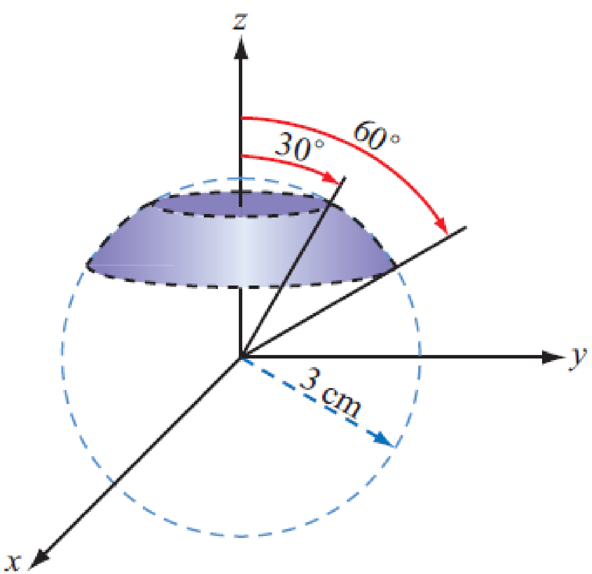


Figure 3-15: Spherical strip of Example 3-5.

Solution: Use of Eq. (3.50b) for the area of an elemental spherical area with constant radius R gives

$$S = R^2 \int_{\theta=30^\circ}^{60^\circ} \sin\theta \; d\theta \int_{\phi=0}^{2\pi} d\phi$$

$$= 9(-\cos\theta)\Big|_{30^\circ}^{60^\circ} \; \phi\Big|_0^{2\pi} \qquad (\text{cm}^2)$$

$$= 18\pi(\cos 30^\circ - \cos 60^\circ) = 20.7 \text{ cm}^2.$$

Example 3-6: Charge in a Sphere

A sphere of radius 2 cm contains a volume charge density ρ_v given by

$$\rho_v = 4\cos^2\theta \qquad (\text{C/m}^3).$$

Find the total charge Q contained in the sphere.

Solution:

$$Q = \int_{\mathcal{V}} \rho_v \; d\mathcal{V}$$

$$= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{R=0}^{2\times10^{-2}} (4\cos^2\theta) R^2 \sin\theta \; dR \; d\theta \; d\phi$$

$$= 4 \int_0^{2\pi} \int_0^{\pi} \left(\frac{R^3}{3}\right)\Big|_0^{2\times10^{-2}} \sin\theta \cos^2\theta \; d\theta \; d\phi$$

$$= \frac{32}{3} \times 10^{-6} \int_0^{2\pi} \left(-\frac{\cos^3\theta}{3}\right)\Big|_0^{\pi} d\phi$$

$$= \frac{64}{9} \times 10^{-6} \int_0^{2\pi} d\phi$$

$$= \frac{128\pi}{9} \times 10^{-6} = 44.68 \qquad (\mu\text{C}).$$

Technology Brief 5: GPS

Figure TF5-1: iPhone map feature.

Figure TF5-2: GPS nominal satellite constellation. Four satellites in each plane, 20,200 km altitudes, 55° inclination.

How does a GPS receiver determine its location?

GPS: Minimum of 4 Satellites Needed

Unknown: location of receiver (x_0, y_0, z_0)
Also unknown: time offset of receiver clock t_0

Quantities known with high precision: locations of satellites and their atomic clocks (satellites use expensive high precision clocks, whereas receivers do not)

Solving for 4 unknowns requires at least 4 equations (four satellites)

$$d_1^2 = (x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2 = c\,[(t_1 + t_0)]^2\,,$$
$$d_2^2 = (x_2 - x_0)^2 + (y_2 - y_0)^2 + (z_2 - z_0)^2 = c\,[(t_2 + t_0)]^2\,,$$
$$d_3^2 = (x_3 - x_0)^2 + (y_3 - y_0)^2 + (z_3 - z_0)^2 = c\,[(t_3 + t_0)]^2\,,$$
$$d_4^2 = (x_4 - x_0)^2 + (y_4 - y_0)^2 + (z_4 - z_0)^2 = c\,[(t_4 + t_0)]^2\,.$$

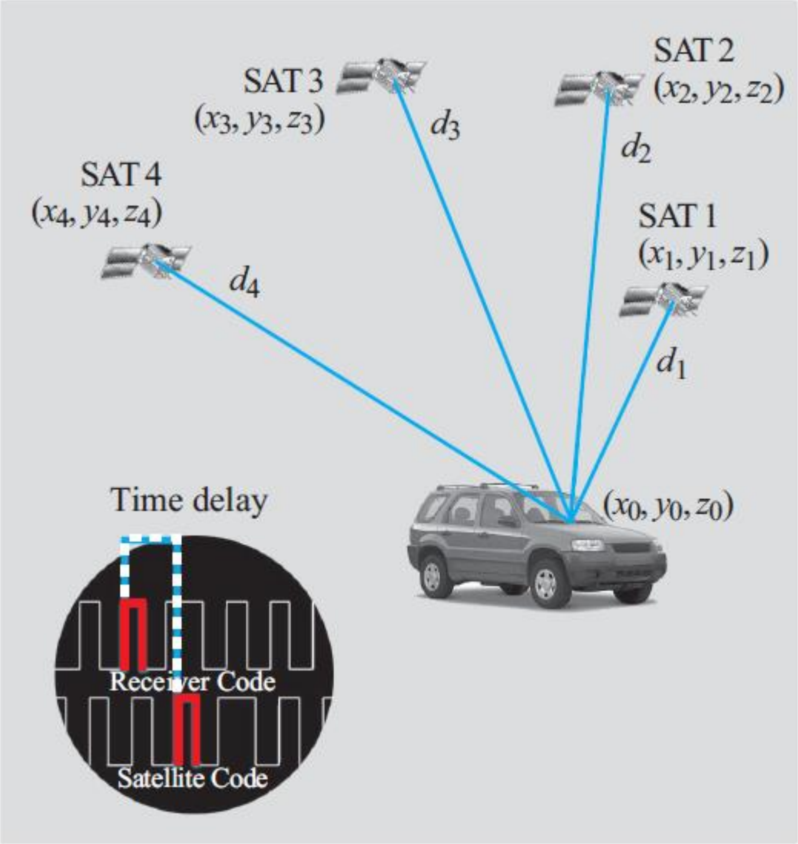
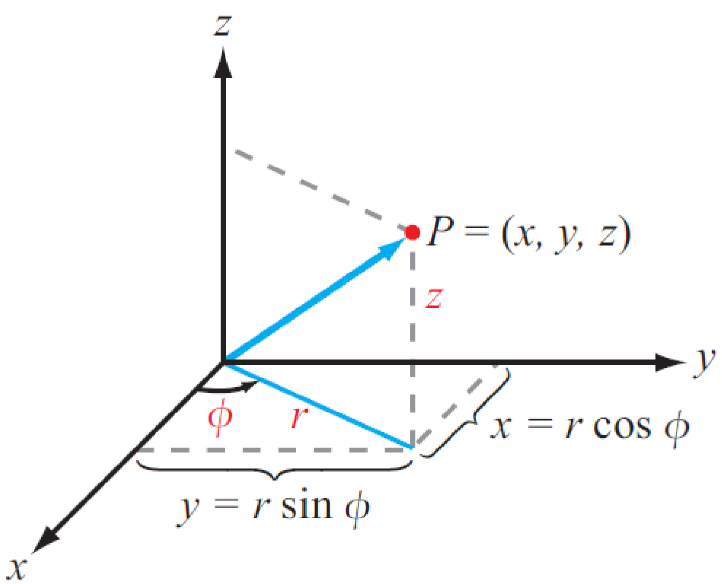


Figure TF5-3: Automobile GPS receiver at location (x_0, y_0, z_0).

Coordinate Transformations: Coordinates

- To solve a problem, we select the coordinate system that best fits its geometry
- Sometimes we need to transform between coordinate systems



$$r = \sqrt[+]{x^2 + y^2}, \qquad \phi = \tan^{-1}\left(\frac{y}{x}\right),$$

and the inverse relations are

$$x = r\cos\phi, \qquad y = r\sin\phi.$$

Figure 3-16: Interrelationships between Cartesian coordinates (x, y, z) and cylindrical coordinates (r, ϕ, z).

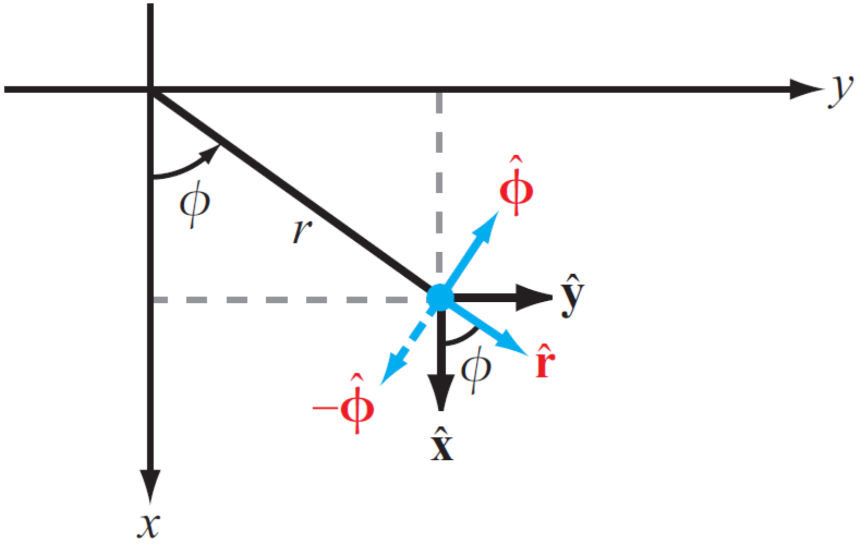
Coordinate Transformations: Unit Vectors

$$\hat{\mathbf{r}} \cdot \hat{\mathbf{x}} = \cos\phi, \qquad \hat{\mathbf{r}} \cdot \hat{\mathbf{y}} = \sin\phi,$$

$$\hat{\boldsymbol{\phi}} \cdot \hat{\mathbf{x}} = -\sin\phi, \qquad \hat{\boldsymbol{\phi}} \cdot \hat{\mathbf{y}} = \cos\phi.$$

$$\hat{\mathbf{r}} = \hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi.$$

$$\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi.$$



$$\hat{\mathbf{x}} = \hat{\mathbf{r}}\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi,$$

$$\hat{\mathbf{y}} = \hat{\mathbf{r}}\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi.$$

Table 3-2: Coordinate transformation relations.

Transformation	Coordinate Variables	Unit Vectors	Vector Components
Cartesian to cylindrical	$r = \sqrt[+]{x^2 + y^2}$ $\phi = \tan^{-1}(y/x)$ $z = z$	$\hat{\mathbf{r}} = \hat{\mathbf{x}} \cos\phi + \hat{\mathbf{y}} \sin\phi$ $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}} \sin\phi + \hat{\mathbf{y}} \cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$A_r = A_x \cos\phi + A_y \sin\phi$ $A_\phi = -A_x \sin\phi + A_y \cos\phi$ $A_z = A_z$
Cylindrical to Cartesian	$x = r\cos\phi$ $y = r\sin\phi$ $z = z$	$\hat{\mathbf{x}} = \hat{\mathbf{r}} \cos\phi - \hat{\boldsymbol{\phi}} \sin\phi$ $\hat{\mathbf{y}} = \hat{\mathbf{r}} \sin\phi + \hat{\boldsymbol{\phi}} \cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$A_x = A_r \cos\phi - A_\phi \sin\phi$ $A_y = A_r \sin\phi + A_\phi \cos\phi$ $A_z = A_z$
Cartesian to spherical	$R = \sqrt[+]{x^2 + y^2 + z^2}$ $\theta = \tan^{-1}[\sqrt[+]{x^2 + y^2}/z]$ $\phi = \tan^{-1}(y/x)$	$\hat{\mathbf{R}} = \hat{\mathbf{x}} \sin\theta \cos\phi + \hat{\mathbf{y}} \sin\theta \sin\phi + \hat{\mathbf{z}} \cos\theta$ $\hat{\boldsymbol{\theta}} = \hat{\mathbf{x}} \cos\theta \cos\phi + \hat{\mathbf{y}} \cos\theta \sin\phi - \hat{\mathbf{z}} \sin\theta$ $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}} \sin\phi + \hat{\mathbf{y}} \cos\phi$	$A_R = A_x \sin\theta \cos\phi + A_y \sin\theta \sin\phi + A_z \cos\theta$ $A_\theta = A_x \cos\theta \cos\phi + A_y \cos\theta \sin\phi - A_z \sin\theta$ $A_\phi = -A_x \sin\phi + A_y \cos\phi$
Spherical to Cartesian	$x = R\sin\theta\cos\phi$ $y = R\sin\theta\sin\phi$ $z = R\cos\theta$	$\hat{\mathbf{x}} = \hat{\mathbf{R}} \sin\theta \cos\phi + \hat{\boldsymbol{\theta}} \cos\theta \cos\phi - \hat{\boldsymbol{\phi}} \sin\phi$ $\hat{\mathbf{y}} = \hat{\mathbf{R}} \sin\theta \sin\phi + \hat{\boldsymbol{\theta}} \cos\theta \sin\phi + \hat{\boldsymbol{\phi}} \cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{R}} \cos\theta - \hat{\boldsymbol{\theta}} \sin\theta$	$A_x = A_R \sin\theta \cos\phi + A_\theta \cos\theta \cos\phi - A_\phi \sin\phi$ $A_y = A_R \sin\theta \sin\phi + A_\theta \cos\theta \sin\phi + A_\phi \cos\phi$ $A_z = A_R \cos\theta - A_\theta \sin\theta$
Cylindrical to spherical	$R = \sqrt[+]{r^2 + z^2}$ $\theta = \tan^{-1}(r/z)$ $\phi = \phi$	$\hat{\mathbf{R}} = \hat{\mathbf{r}} \sin\theta + \hat{\mathbf{z}} \cos\theta$ $\hat{\boldsymbol{\theta}} = \hat{\mathbf{r}} \cos\theta - \hat{\mathbf{z}} \sin\theta$ $\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}$	$A_R = A_r \sin\theta + A_z \cos\theta$ $A_\theta = A_r \cos\theta - A_z \sin\theta$ $A_\phi = A_\phi$
Spherical to cylindrical	$r = R\sin\theta$ $\phi = \phi$ $z = R\cos\theta$	$\hat{\mathbf{r}} = \hat{\mathbf{R}} \sin\theta + \hat{\boldsymbol{\theta}} \cos\theta$ $\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}$ $\hat{\mathbf{z}} = \hat{\mathbf{R}} \cos\theta - \hat{\boldsymbol{\theta}} \sin\theta$	$A_r = A_R \sin\theta + A_\theta \cos\theta$ $A_\phi = A_\phi$ $A_z = A_R \cos\theta - A_\theta \sin\theta$

Example 3-7: Cartesian to Cylindrical Transformations

Given point $P_1 = (3, -4, 3)$ and vector $\mathbf{A} = \hat{\mathbf{x}}2 - \hat{\mathbf{y}}3 + \hat{\mathbf{z}}4$, defined in Cartesian coordinates, express P_1 and $\mathbf{A}$ in cylindrical coordinates and evaluate $\mathbf{A}$ at P_1.

Solution: For point P_1, $x = 3$, $y = -4$, and $z = 3$. Using Eq. (3.51), we have

$$r = \sqrt[+]{x^2 + y^2} = 5, \quad \phi = \tan^{-1}\frac{y}{x} = -53.1^\circ = 306.9^\circ,$$

and z remains unchanged. Hence, $P_1 = (5, 306.9^\circ, 3)$ in cylindrical coordinates.

The cylindrical components of vector $\mathbf{A} = \hat{\mathbf{r}}A_r + \hat{\boldsymbol{\phi}}A_\phi + \hat{\mathbf{z}}A_z$ can be determined by applying Eqs. (3.58a) and (3.58b):

$$A_r = A_x \cos\phi + A_y \sin\phi = 2\cos\phi - 3\sin\phi,$$

$$A_\phi = -A_x \sin\phi + A_y \cos\phi = -2\sin\phi - 3\cos\phi,$$

$$A_z = 4.$$

Hence,

$$\mathbf{A} = \hat{\mathbf{r}}(2\cos\phi - 3\sin\phi) - \hat{\boldsymbol{\phi}}(2\sin\phi + 3\cos\phi) + \hat{\mathbf{z}}4.$$

At point P, $\phi = 306.9^\circ$, which gives

$$\mathbf{A} = \hat{\mathbf{r}}3.60 - \hat{\boldsymbol{\phi}}0.20 + \hat{\mathbf{z}}4.$$

Example 3-8: Cartesian to Spherical Transformation

Express vector $\mathbf{A} = \hat{\mathbf{x}}(x + y) + \hat{\mathbf{y}}(y - x) + \hat{\mathbf{z}}z$ in spherical coordinates.

Solution: Using the transformation relation for A_R given in Table 3-2, we have

$$\begin{aligned} A_R &= A_x \sin\theta\cos\phi + A_y \sin\theta\sin\phi + A_z\cos\theta \\ &= (x + y)\sin\theta\cos\phi + (y - x)\sin\theta\sin\phi + z\cos\theta. \end{aligned}$$

Using the expressions for x, y, and z given by Eq. (3.61c), we have

$$\begin{aligned} A_R &= (R\sin\theta\cos\phi + R\sin\theta\sin\phi)\sin\theta\cos\phi \\ &\quad + (R\sin\theta\sin\phi - R\sin\theta\cos\phi)\sin\theta\sin\phi + R\cos^2\theta \\ &= R\sin^2\theta\,(\cos^2\phi + \sin^2\phi) + R\cos^2\theta \\ &= R\sin^2\theta + R\cos^2\theta = R. \end{aligned}$$

Similarly,

$$\begin{aligned} A_\theta &= (x + y)\cos\theta\cos\phi + (y - x)\cos\theta\sin\phi - z\sin\theta, \\ A_\phi &= -(x + y)\sin\phi + (y - x)\cos\phi, \end{aligned}$$

Using the relations:

$$\begin{aligned} x &= R\sin\theta\cos\phi, \\ y &= R\sin\theta\sin\phi, \\ z &= R\cos\theta. \end{aligned}$$

leads to:

$$\begin{aligned} A_\theta &= 0, \\ A_\phi &= -R\sin\theta. \end{aligned}$$

$$\mathbf{A} = \hat{\mathbf{R}}A_R + \hat{\boldsymbol{\theta}}A_\theta + \hat{\boldsymbol{\phi}}A_\phi = \hat{\mathbf{R}}R - \hat{\boldsymbol{\phi}}R\sin\theta.$$

Distance Between 2 Points

$$d = |\mathbf{R}_{12}| = [(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]^{1/2}. \quad (3.66)$$

$$d = \Big[(r_2 \cos\phi_2 - r_1 \cos\phi_1)^2 + (r_2 \sin\phi_2 - r_1 \sin\phi_1)^2 + (z_2 - z_1)^2\Big]^{1/2}$$
$$= \Big[r_2^2 + r_1^2 - 2r_1 r_2 \cos(\phi_2 - \phi_1) + (z_2 - z_1)^2\Big]^{1/2} \quad \text{(cylindrical).} \quad (3.67)$$

$$d = \Big\{R_2^2 + R_1^2 - 2R_1 R_2[\cos\theta_2 \cos\theta_1 + \sin\theta_1 \sin\theta_2 \cos(\phi_2 - \phi_1)]\Big\}^{1/2} \quad \text{(spherical).} \quad (3.68)$$

Gradient of A Scalar Field

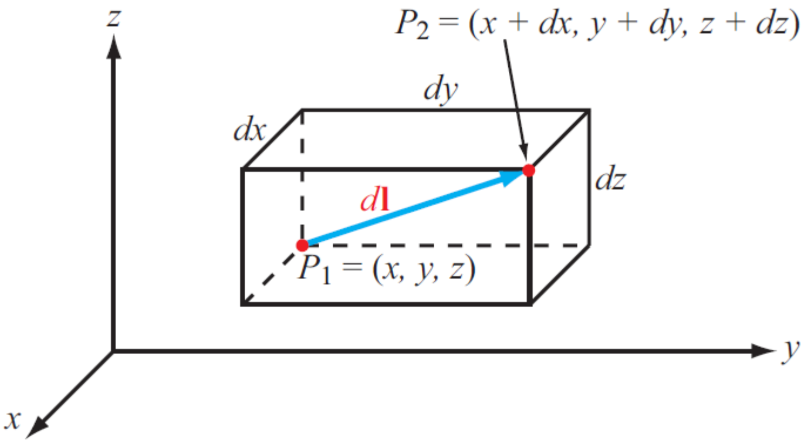


Figure 3-19: Differential distance vector $d\mathbf{l}$ between points P_1 and P_2.

From differential calculus, the temperature difference between points P_1 and P_2, $dT = T_2 - T_1$, is

$$dT = \frac{\partial T}{\partial x}\,dx + \frac{\partial T}{\partial y}\,dy + \frac{\partial T}{\partial z}\,dz. \quad (3.70)$$

Because $dx = \hat{\mathbf{x}} \cdot d\mathbf{l}$, $dy = \hat{\mathbf{y}} \cdot d\mathbf{l}$, and $dz = \hat{\mathbf{z}} \cdot d\mathbf{l}$, Eq. (3.70) can be rewritten as

$$\begin{aligned} dT &= \hat{\mathbf{x}}\frac{\partial T}{\partial x} \cdot d\mathbf{l} + \hat{\mathbf{y}}\frac{\partial T}{\partial y} \cdot d\mathbf{l} + \hat{\mathbf{z}}\frac{\partial T}{\partial z} \cdot d\mathbf{l} \\ &= \left[\hat{\mathbf{x}}\frac{\partial T}{\partial x} + \hat{\mathbf{y}}\frac{\partial T}{\partial y} + \hat{\mathbf{z}}\frac{\partial T}{\partial z}\right] \cdot d\mathbf{l}. \quad (3.71) \end{aligned}$$

$$\nabla T = \text{grad } T = \hat{\mathbf{x}}\frac{\partial T}{\partial x} + \hat{\mathbf{y}}\frac{\partial T}{\partial y} + \hat{\mathbf{z}}\frac{\partial T}{\partial z}. \quad (3.72)$$

Equation (3.71) can then be expressed as

$$dT = \nabla T \cdot d\mathbf{l}. \quad (3.73)$$

The symbol ∇ is called the ***del*** or ***gradient operator*** and is defined as

$$\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z} \quad \text{(Cartesian).} \quad (3.74)$$

Gradient (cont.)

With $d\mathbf{l} = \hat{\mathbf{a}}_l \, dl$, where $\hat{\mathbf{a}}_l$ is the unit vector of $d\mathbf{l}$, the ***directional derivative*** of T along $\hat{\mathbf{a}}_l$ is

$$\frac{dT}{dl} = \nabla T \cdot \hat{\mathbf{a}}_l. \qquad (3.75)$$

We can find the difference $(T_2 - T_1)$, where $T_1 = T(x_1, y_1, z_1)$ and $T_2 = T(x_2, y_2, z_2)$ are the values of T at points $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ not necessarily infinitesimally close to one another, by integrating both sides of Eq. (3.73). Thus,

$$T_2 - T_1 = \int_{P_1}^{P_2} \nabla T \cdot d\mathbf{l}. \qquad (3.76)$$

Example 3-9: Directional Derivative

Find the directional derivative of $T = x^2 + y^2 z$ along direction $\hat{\mathbf{x}}2 + \hat{\mathbf{y}}3 - \hat{\mathbf{z}}2$ and evaluate it at $(1, -1, 2)$.

Solution: First, we find the gradient of T:

$$\nabla T = \left(\hat{\mathbf{x}} \frac{\partial}{\partial x} + \hat{\mathbf{y}} \frac{\partial}{\partial y} + \hat{\mathbf{z}} \frac{\partial}{\partial z} \right) (x^2 + y^2 z)$$
$$= \hat{\mathbf{x}}2x + \hat{\mathbf{y}}2yz + \hat{\mathbf{z}}y^2.$$

We denote $\mathbf{l}$ as the given direction,

$$\mathbf{l} = \hat{\mathbf{x}}2 + \hat{\mathbf{y}}3 - \hat{\mathbf{z}}2.$$

Its unit vector is

$$\hat{\mathbf{a}}_l = \frac{\mathbf{l}}{|\mathbf{l}|} = \frac{\hat{\mathbf{x}}2 + \hat{\mathbf{y}}3 - \hat{\mathbf{z}}2}{\sqrt{2^2 + 3^2 + 2^2}} = \frac{\hat{\mathbf{x}}2 + \hat{\mathbf{y}}3 - \hat{\mathbf{z}}2}{\sqrt{17}}.$$

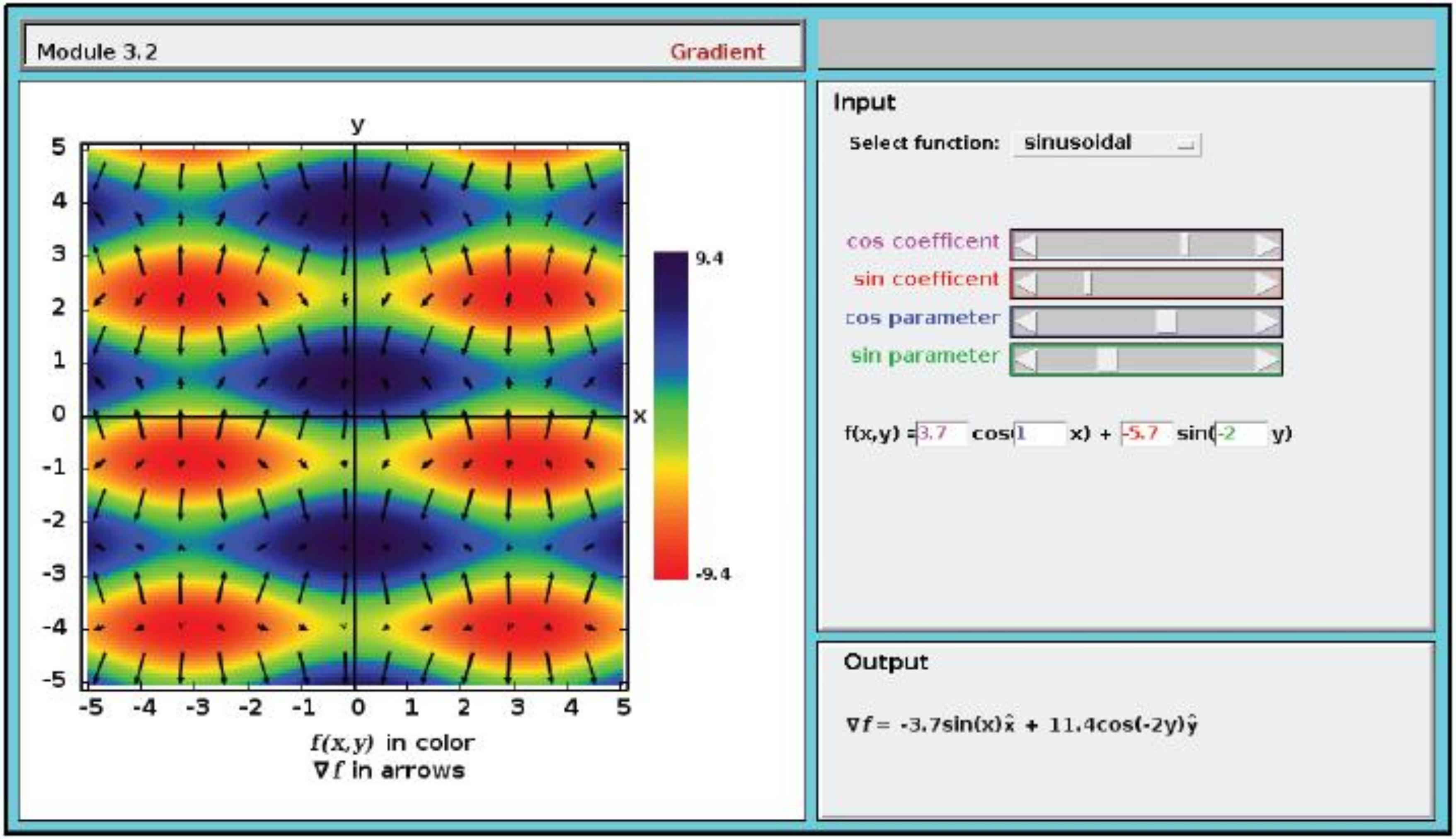
Application of Eq. (3.75) gives

$$\frac{dT}{dl} = \nabla T \cdot \hat{\mathbf{a}}_l = (\hat{\mathbf{x}}2x + \hat{\mathbf{y}}2yz + \hat{\mathbf{z}}y^2) \cdot \left(\frac{\hat{\mathbf{x}}2 + \hat{\mathbf{y}}3 - \hat{\mathbf{z}}2}{\sqrt{17}} \right)$$
$$= \frac{4x + 6yz - 2y^2}{\sqrt{17}}.$$

At $(1, -1, 2)$,

$$\left. \frac{dT}{dl} \right|_{(1,-1,2)} = \frac{4 - 12 - 2}{\sqrt{17}} = \frac{-10}{\sqrt{17}}.$$

Module 3.2 Gradient Select a scalar function $f(x, y, z)$, evaluate its gradient, and display both in an appropriate 2-D plane.



Divergence of a Vector Field

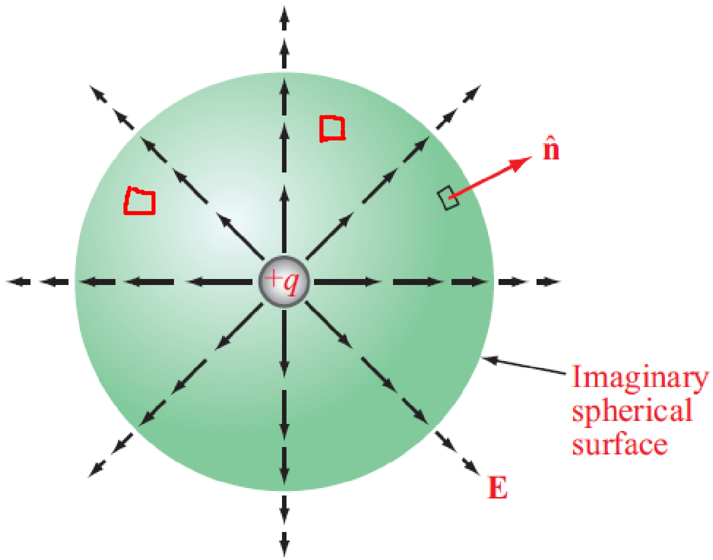


Figure 3-20: Flux lines of the electric field **E** due to a positive charge q.

At a surface boundary, ***flux density*** is defined as the amount of outward flux crossing a unit surface ds:

$$\text{Flux density of } \mathbf{E} = \frac{\mathbf{E} \cdot d\mathbf{s}}{|d\mathbf{s}|} = \frac{\mathbf{E} \cdot \hat{\mathbf{n}}\, ds}{ds} = \mathbf{E} \cdot \hat{\mathbf{n}}, \qquad (3.85)$$

where $\hat{\mathbf{n}}$ is the normal to $d\mathbf{s}$. The ***total flux*** outwardly crossing a closed surface S, such as the enclosed surface of the imaginary sphere outlined in Fig. 3-20, is

$$\text{Total flux} = \oint_S \mathbf{E} \cdot d\mathbf{s}. \qquad (3.86)$$

$$\text{div}\,\mathbf{E} \triangleq \lim_{\Delta V \to 0} \frac{\oint_S \mathbf{E} \cdot d\mathbf{s}}{\Delta V}, \qquad (3.95)$$

where S encloses the elemental volume ΔV. Instead of denoting the divergence of **E** by div **E**, it is common practice to denote it as $\nabla \cdot \mathbf{E}$. That is,

$$\nabla \cdot \mathbf{E} = \text{div}\,\mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \qquad (3.96)$$

for a vector **E** in Cartesian coordinates.

From the definition of the divergence of **E** *given by Eq. (3.95), field* **E** *has positive divergence if the net flux out of surface S is positive, which may be "viewed" as if volume ΔV contains a* ***source*** *of field lines. If the divergence is negative, ΔV may be viewed as containing a* ***sink*** *of field lines because the net flux is into ΔV. For a uniform field* **E***, the same amount of flux enters ΔV as leaves it; hence, its divergence is zero and the field is said to be* ***divergenceless***.

Divergence Theorem

$$\int_{V} \nabla \cdot \mathbf{E} \, dV = \oint_{S} \mathbf{E} \cdot d\mathbf{s} \qquad \text{(divergence theorem).} \qquad (3.98)$$

Useful tool for converting integration over a volume to one over the surface enclosing that volume, and vice versa

Example 3-11: Calculating the Divergence

Determine the divergence of each of the following vector fields and then evaluate them at the indicated points:

(a) $\mathbf{E} = \hat{\mathbf{x}}3x^2 + \hat{\mathbf{y}}2z + \hat{\mathbf{z}}x^2z$ at $(2, -2, 0)$;

(b) $\mathbf{E} = \hat{\mathbf{R}}(a^3 \cos\theta / R^2) - \hat{\boldsymbol{\theta}}(a^3 \sin\theta / R^2)$ at $(a/2, 0, \pi)$.

Solution:

(a)
$$\begin{aligned}\nabla \cdot \mathbf{E} &= \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \\ &= \frac{\partial}{\partial x}(3x^2) + \frac{\partial}{\partial y}(2z) + \frac{\partial}{\partial z}(x^2 z) \\ &= 6x + 0 + x^2 \\ &= x^2 + 6x.\end{aligned}$$

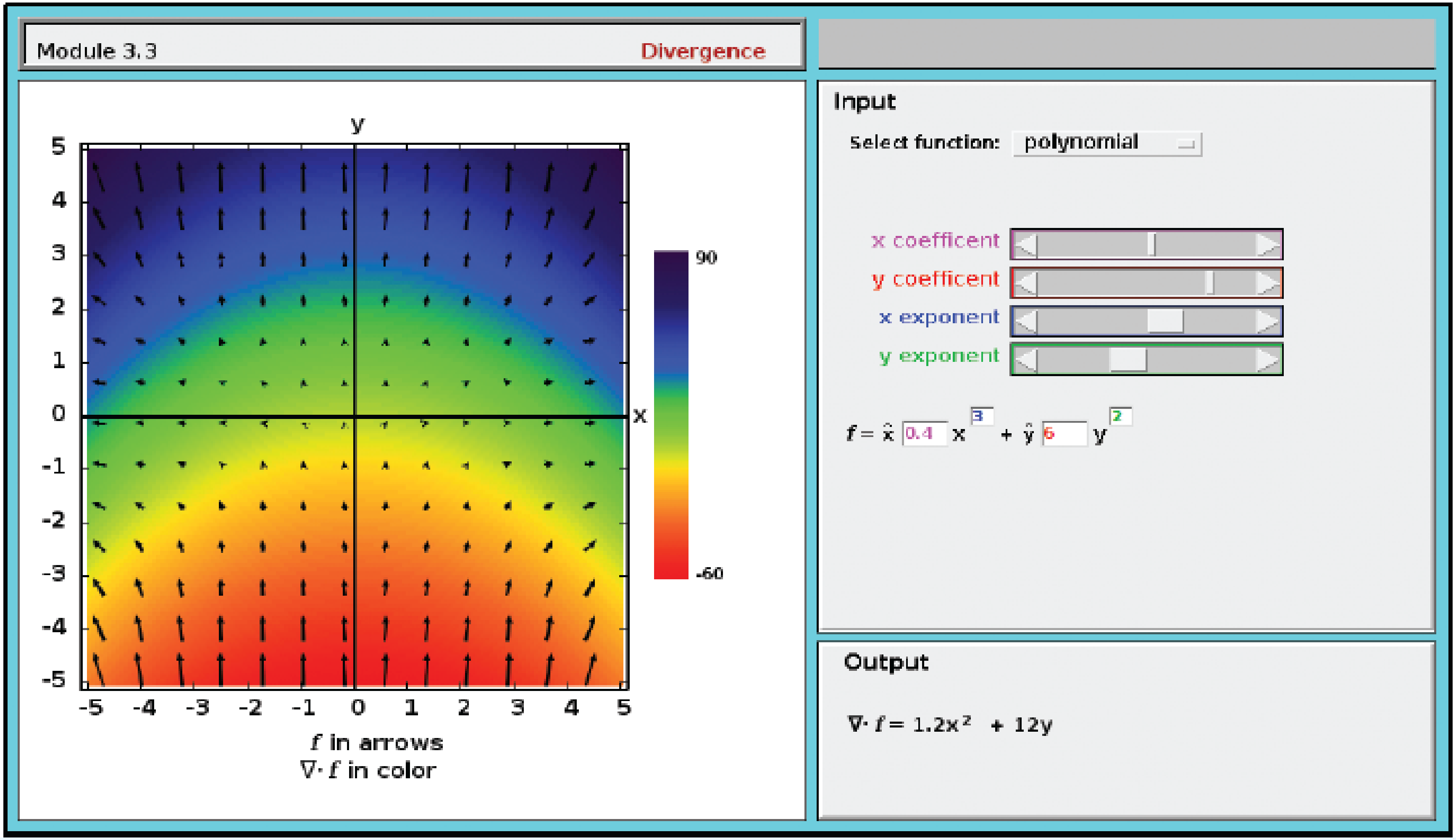
At $(2, -2, 0)$, $\nabla \cdot \mathbf{E}\Big|_{(2,-2,0)} = 16.$

(b) From the expression given on the inside of the back cover of the book for the divergence of a vector in spherical coordinates, it follows that

$$\begin{aligned}\nabla \cdot \mathbf{E} &= \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 E_R) + \frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}(E_\theta \sin\theta) \\ &\quad + \frac{1}{R\sin\theta}\frac{\partial E_\phi}{\partial\phi} \\ &= \frac{1}{R^2}\frac{\partial}{\partial R}(a^3\cos\theta) + \frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}\left(-\frac{a^3\sin^2\theta}{R^2}\right) \\ &= 0 - \frac{2a^3\cos\theta}{R^3} \\ &= -\frac{2a^3\cos\theta}{R^3}.\end{aligned}$$

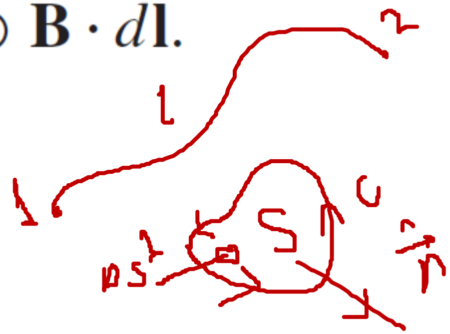
At $R = a/2$ and $\theta = 0$, $\nabla \cdot \mathbf{E}\Big|_{(a/2,0,\pi)} = -16.$

Module 3.3 Divergence Select a vector function $\mathbf{f}(x, y, z)$, evaluate its divergence, and display both in an appropriate 2-D plane.



Curl of a Vector Field

$$\text{Circulation} = \oint_C \mathbf{B} \cdot d\mathbf{l}.$$



$$\nabla \times \mathbf{B} = \text{curl } \mathbf{B} = \lim_{\Delta s \to 0} \frac{1}{\Delta s} \left[\hat{\mathbf{n}} \oint_C \mathbf{B} \cdot d\mathbf{l} \right]_{\max} . \quad (3.103)$$

Thus, curl **B** *is the circulation of* **B** *per unit area, with the area* Δs *of the contour C being oriented such that the circulation is maximum.*

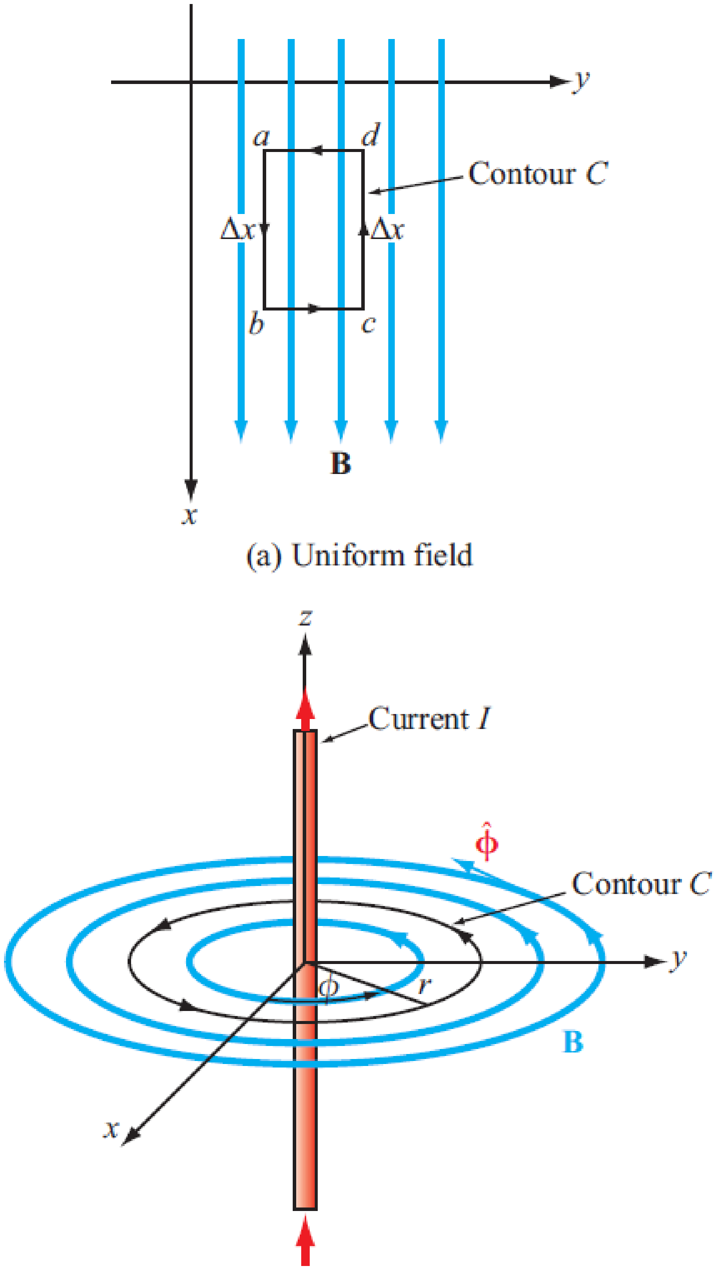


Figure 3-22: Circulation is zero for the uniform field in (a), but it is not zero for the azimuthal field in (b).

Stokes's Theorem

***Stokes's theorem** converts the surface integral of the curl of a vector over an open surface S into a line integral of the vector along the contour C bounding the surface S.*

For the geometry shown in Fig. 3-23, ***Stokes's theorem*** states

$$\int_S (\nabla \times \mathbf{B}) \cdot d\mathbf{s} = \oint_C \mathbf{B} \cdot d\mathbf{l} \qquad \text{(Stokes's theorem)}, \tag{3.107}$$

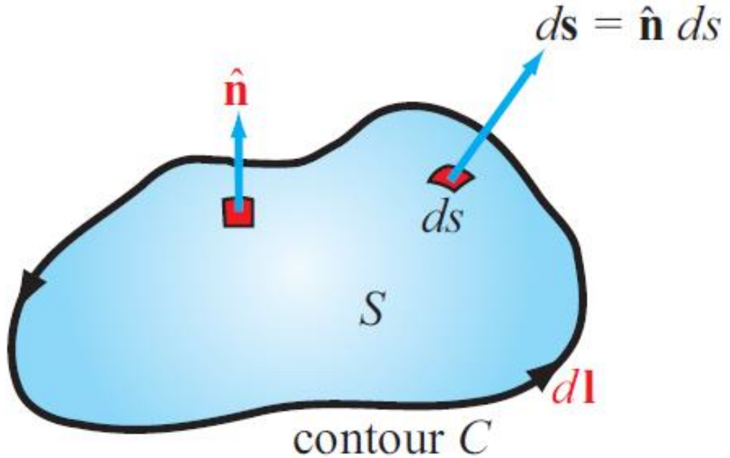
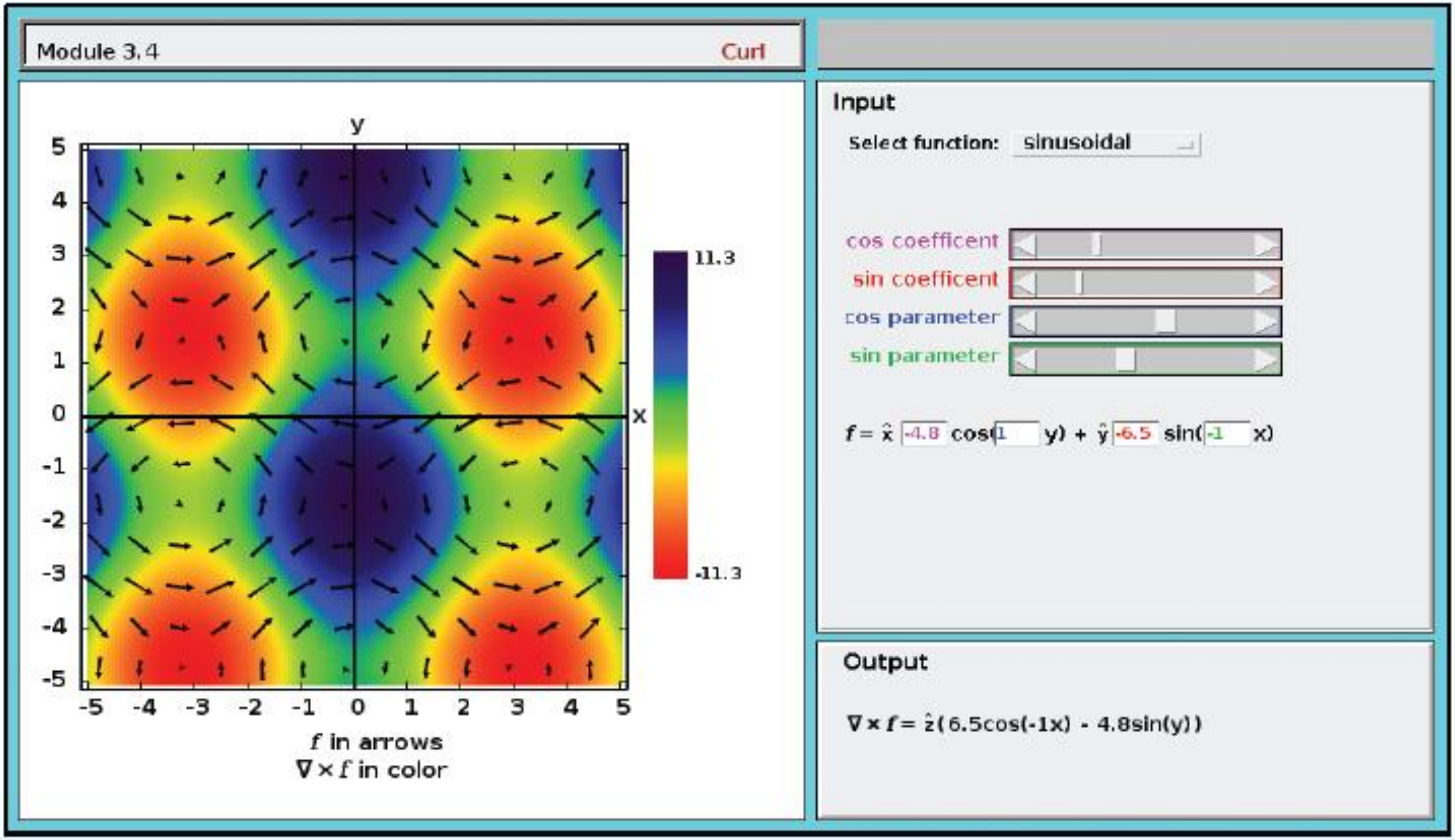


Figure 3-23: The direction of the unit vector $\hat{\mathbf{n}}$ is along the thumb when the other four fingers of the right hand follow $d\mathbf{l}$.

Module 3.4 Curl Select a vector $\mathbf{f}(x, y)$, evaluate its curl, and display both in the x-y plane.



Laplacian Operator

Laplacian of a Scalar Field

$$\nabla^2 V = \nabla \cdot (\nabla V) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} . \qquad (3.110)$$

Laplacian of a Vector Field

$$\nabla^2 \mathbf{E} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \mathbf{E}$$

$$= \hat{\mathbf{x}}\, \nabla^2 E_x + \hat{\mathbf{y}}\, \nabla^2 E_y + \hat{\mathbf{z}}\, \nabla^2 E_z$$

Useful Relation

$$\nabla^2 \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \nabla \times (\nabla \times \mathbf{E}). \qquad (3.113)$$

Tech Brief 6: X-Ray Computed Tomography

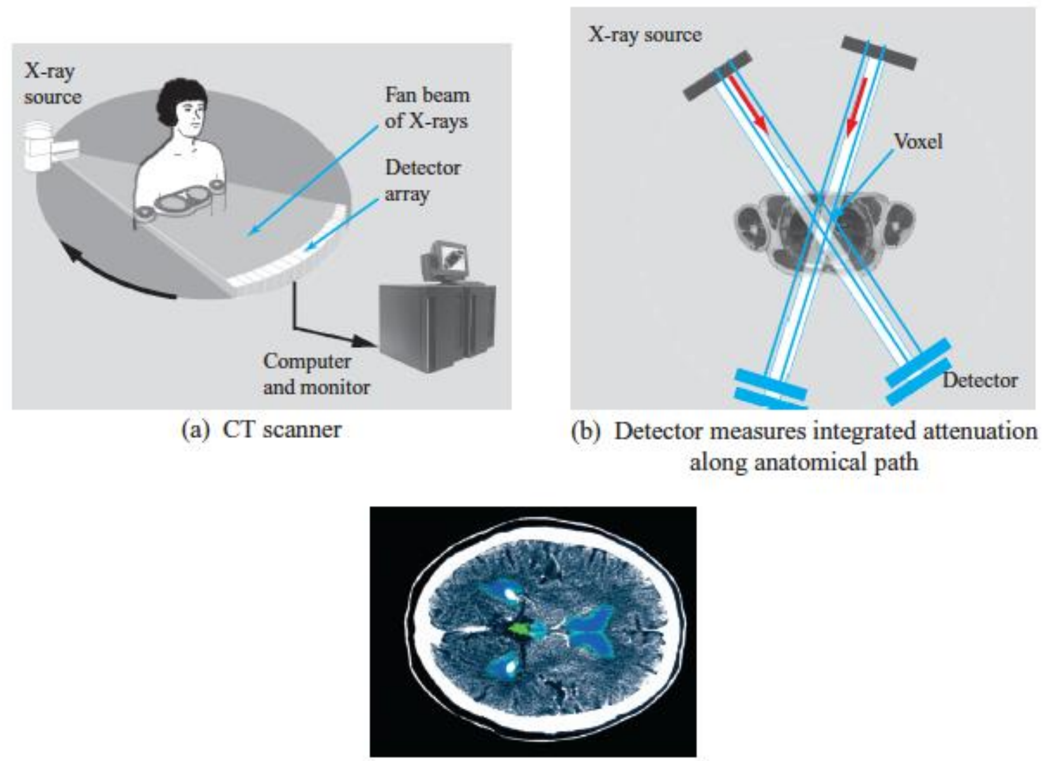
How does a CT scanner generate a 3-D image?

Figure TF6-1 2-D X-ray image.

Figure TF6-2 CT scanner.

Tech Brief 6: X-Ray Computed Tomography

- For each anatomical slice, the CT scanner generates on the order of 7×10^5 measurements (1,000 angular orientations x 700 detector channels)
- Use of vector calculus allows the extraction of the 2-D image of a slice
- Combining multiple slices generates a 3-D scan



(a) CT scanner

(b) Detector measures integrated attenuation along anatomical path

(c) CT image of a normal brain

Figure TF6-3 Basic elements of a CT scanner.

Chapter 3 Relationships

Distance Between Two Points

$$d = [(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]^{1/2}$$

$$d = \left[r_2^2 + r_1^2 - 2r_1 r_2 \cos(\phi_2 - \phi_1) + (z_2 - z_1)^2\right]^{1/2}$$

$$d = \{R_2^2 + R_1^2 - 2R_1 R_2[\cos\theta_2 \cos\theta_1 + \sin\theta_1 \sin\theta_2 \cos(\phi_2 - \phi_1)]\}^{1/2}$$

Coordinate Systems

Table 3-1

Coordinate Transformations

Table 3-2

Vector Products

$$\mathbf{A} \cdot \mathbf{B} = AB \cos\theta_{AB}$$

$$\mathbf{A} \times \mathbf{B} = \hat{\mathbf{n}}\, AB \sin\theta_{AB}$$

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

Divergence Theorem

$$\int_V \nabla \cdot \mathbf{E}\, dV = \oint_S \mathbf{E} \cdot d\mathbf{s}$$

Vector Operators

$$\nabla T = \hat{\mathbf{x}} \frac{\partial T}{\partial x} + \hat{\mathbf{y}} \frac{\partial T}{\partial y} + \hat{\mathbf{z}} \frac{\partial T}{\partial z}$$

$$\nabla \cdot \mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

$$\nabla \times \mathbf{B} = \hat{\mathbf{x}} \left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} \right) + \hat{\mathbf{y}} \left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} \right) + \hat{\mathbf{z}} \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} \right)$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

(see back cover for cylindrical and spherical coordinates)

Stokes's Theorem

$$\int_S (\nabla \times \mathbf{B}) \cdot d\mathbf{s} = \oint_C \mathbf{B} \cdot d\mathbf{l}$$