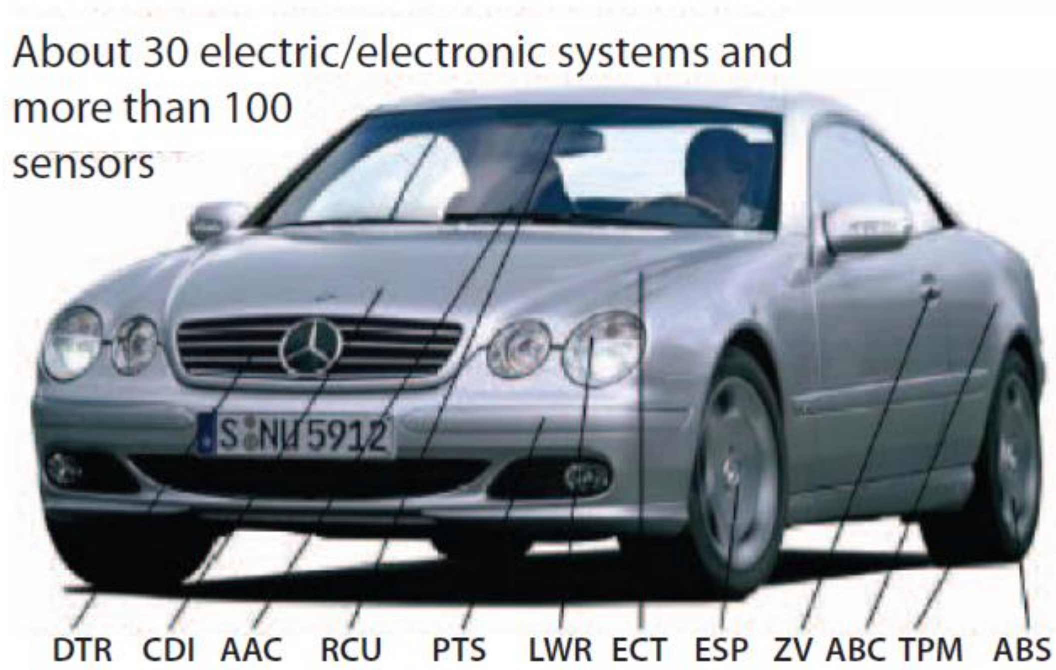


System	Abbrev.	Sensors	System	Abbrev.	Sensors
Distronic	DTR	3	Common-rail diesel injection	CDI	11
Electronic controlled transmission	ECT	9	Automatic air condition	AAC	13
Roof control unit	RCU	7	Active body control	ABC	12
Antilock braking system	ABS	4	Tire pressure monitoring	TPM	11
Central locking system	ZV	3	Elektron. stability program	ESP	14
Dyn. beam levelling	LWR	6	Parktronic system	PTS	12

Figure TF7-1: Most cars use on the order of 100 sensors. (Courtesy Mercedes-Benz.)

4. ELECTROSTATICS

7e Applied EM by Ulaby and Ravaioli

Chapter 4 Overview

Chapter Contents

Objectives

Upon learning the material presented in this chapter, you should be able to:

1. Evaluate the electric field and electric potential due to any distribution of electric charges.
2. Apply Gauss's law.
3. Calculate the resistance R of any shaped object, given the electric field at every point in its volume.
4. Describe the operational principles of resistive and capacitive sensors.
5. Calculate the capacitance of two-conductor configurations.

Maxwell's Equations

God said:

$$\nabla \cdot \mathbf{D} = \rho_v,$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

$$\nabla \cdot \mathbf{B} = 0,$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}.$$

And there was light!

Under ***static*** conditions, none of the quantities appearing in Maxwell's equations are functions of time (i.e., $\partial/\partial t = 0$). ***This happens when all charges are permanently fixed in space, or, if they move, they do so at a steady rate so that ρ_v and J are constant in time.*** Under these circumstances, the time derivatives of **B** and **D** in Eqs. (4.1b) and (4.1d) vanish, and Maxwell's equations reduce to

Electrostatics

$$\nabla \cdot \mathbf{D} = \rho_v, \tag{4.2a}$$

$$\nabla \times \mathbf{E} = 0. \tag{4.2b}$$

Magnetostatics

$$\nabla \cdot \mathbf{B} = 0, \tag{4.3a}$$

$$\nabla \times \mathbf{H} = \mathbf{J}. \tag{4.3b}$$

Electric and magnetic fields become decoupled under static conditions.

Charge Distributions

Volume charge density:

$$\rho_v = \lim_{\Delta \mathcal{V} \to 0} \frac{\Delta q}{\Delta \mathcal{V}} = \frac{dq}{d\mathcal{V}} \qquad (\text{C/m}^3)$$

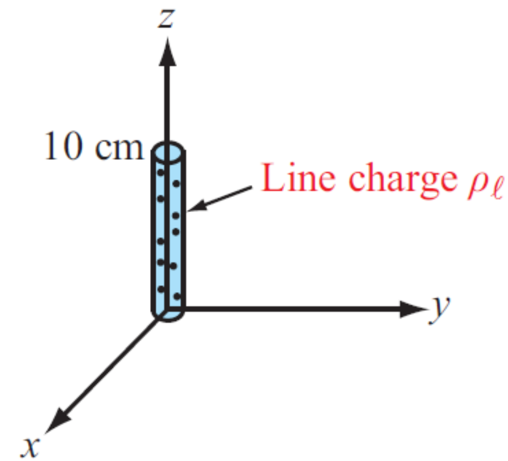
Total Charge in a Volume

$$Q = \int_{\mathcal{V}} \rho_v \, d\mathcal{V} \qquad (\text{C})$$

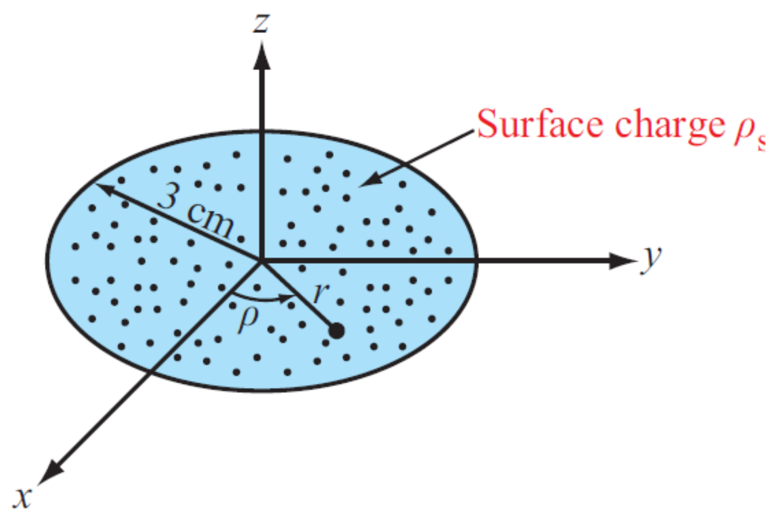
Surface and Line Charge Densities

$$\rho_s = \lim_{\Delta s \to 0} \frac{\Delta q}{\Delta s} = \frac{dq}{ds} \qquad (\text{C/m}^2)$$

$$\rho_\ell = \lim_{\Delta l \to 0} \frac{\Delta q}{\Delta l} = \frac{dq}{dl} \qquad (\text{C/m})$$



(a) Line charge distribution



Current Density

The amount of charge that crosses the tube's cross-sectional surface $\Delta s'$ in time Δt is therefore

$$\Delta q' = \rho_v \, \Delta \mathcal{V} = \rho_v \, \Delta l \, \Delta s' = \rho_v u \, \Delta s' \, \Delta t. \qquad (4.8)$$

For a surface with any orientation:

$$\Delta q = \rho_v \mathbf{u} \cdot \Delta \mathbf{s} \, \Delta t, \qquad (4.9)$$

where $\Delta \mathbf{s} = \hat{\mathbf{n}} \, \Delta s$ and the corresponding total current flowing in the tube is

$$\Delta I = \frac{\Delta q}{\Delta t} = \rho_v \mathbf{u} \cdot \Delta \mathbf{s} = \mathbf{J} \cdot \Delta \mathbf{s}, \qquad (4.10)$$

where

$$\mathbf{J} = \rho_v \mathbf{u} \qquad (\text{A/m}^2) \qquad (4.11)$$

J is called the current density

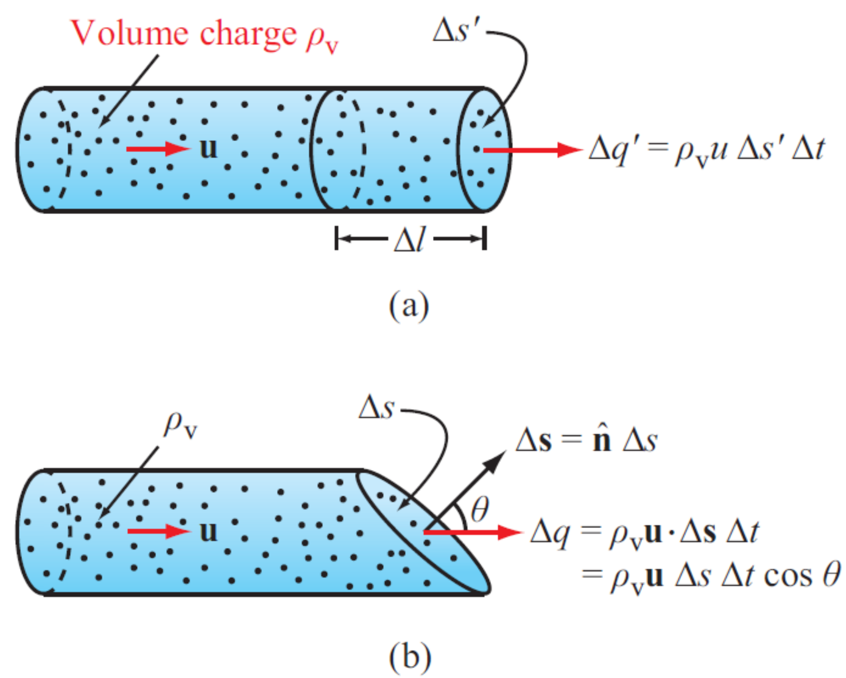


Figure 4-2: Charges with velocity **u** moving through a cross section $\Delta s'$ in (a) and Δs in (b).

$$I = \int_S \mathbf{J} \cdot d\mathbf{s} \qquad (\text{A}). \qquad (4.12)$$

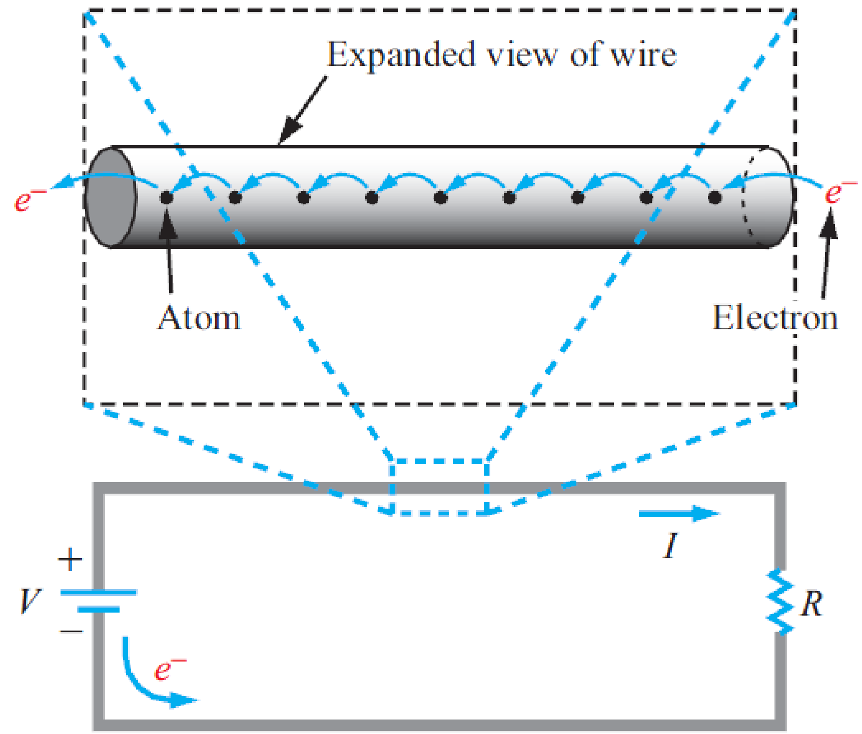
When a current is due to the actual movement of electrically charged matter, it is called a ***convection current****, and* **J** *is called a* ***convection current density****.*

Convection vs. Conduction

When a current is due to the movement of charged particles relative to their host material, **J** is called a ***conduction current density***.

*This movement of electrons from atom to atom constitutes a **conduction current**. The electrons that emerge from the wire are not necessarily the same electrons that entered the wire at the other end.*

Conduction current, which is discussed in more detail in Section 4-6, obeys Ohm's law, whereas convection current does not.



Coulomb's Law

Electric field at point P due to single charge

$$\mathbf{E} = \hat{\mathbf{R}} \frac{q}{4\pi\varepsilon R^2} \qquad \text{(V/m)}$$

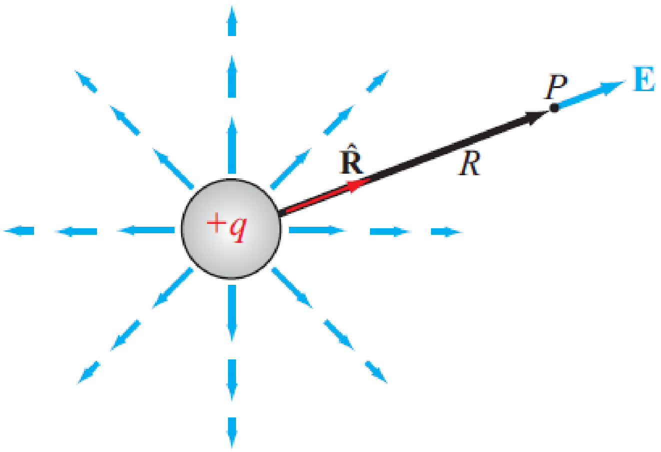


Figure 4-3: Electric-field lines due to a charge q.

Electric force on a test charge placed at *P*

$$\mathbf{F} = q'\mathbf{E} \qquad \text{(N)}$$

Electric flux density **D**

$$\mathbf{D} = \varepsilon \mathbf{E}$$

If ε is independent of the magnitude of **E**, *then the material is said to be* ***linear*** *because* **D** *and* **E** *are related linearly, and if it is independent of the direction of* **E**, *the material is said to be* ***isotropic.***

$$\varepsilon = \varepsilon_r \varepsilon_0,$$

$$\varepsilon_0 = 8.85 \times 10^{-12} \simeq (1/36\pi) \times 10^{-9} \qquad \text{(F/m)}$$

Electric Field Due to 2 Charges

with R, the distance between q_1 and P, replaced with $|\mathbf{R} - \mathbf{R}_1|$ and the unit vector $\hat{\mathbf{R}}$ replaced with $(\mathbf{R} - \mathbf{R}_1)/|\mathbf{R} - \mathbf{R}_1|$. Thus,

$$\mathbf{E}_1 = \frac{q_1(\mathbf{R} - \mathbf{R}_1)}{4\pi\varepsilon|\mathbf{R} - \mathbf{R}_1|^3} \quad \text{(V/m)}. \qquad (4.17a)$$

Similarly, the electric field at P due to q_2 alone is

$$\mathbf{E}_2 = \frac{q_2(\mathbf{R} - \mathbf{R}_2)}{4\pi\varepsilon|\mathbf{R} - \mathbf{R}_2|^3} \quad \text{(V/m)}. \qquad (4.17b)$$

The electric field obeys the principle of linear superposition.

Hence, the total electric field $\mathbf{E}$ at P due to q_1 and q_2 is

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 = \frac{1}{4\pi\varepsilon}\left[\frac{q_1(\mathbf{R} - \mathbf{R}_1)}{|\mathbf{R} - \mathbf{R}_1|^3} + \frac{q_2(\mathbf{R} - \mathbf{R}_2)}{|\mathbf{R} - \mathbf{R}_2|^3}\right]. \qquad (4.18)$$

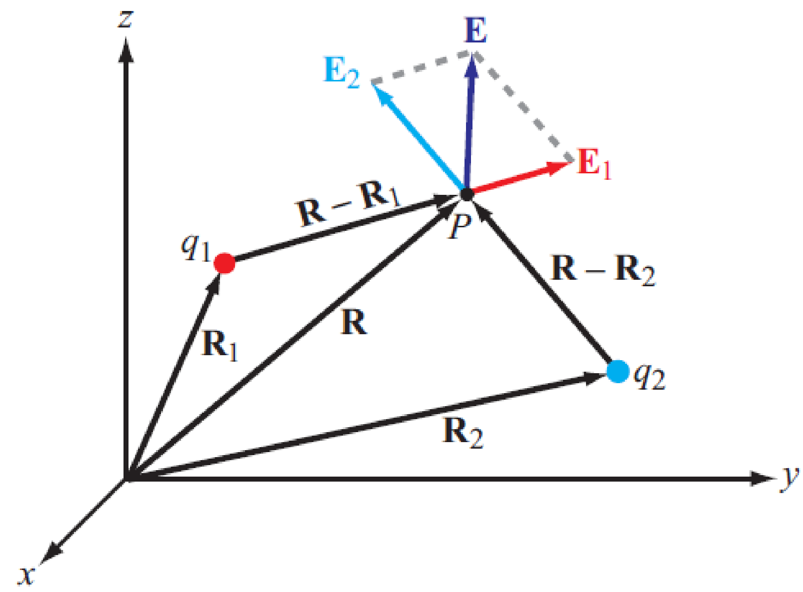


Figure 4-4: The electric field $\mathbf{E}$ at P due to two charges is equal to the vector sum of $\mathbf{E}_1$ and $\mathbf{E}_2$.

Electric Field due to Multiple Charges

$$\mathbf{E} = \frac{1}{4\pi\varepsilon} \sum_{i=1}^{N} \frac{q_i(\mathbf{R} - \mathbf{R}_i)}{|\mathbf{R} - \mathbf{R}_i|^3} \qquad \text{(V/m)}.$$

Example 4-3: Electric Field Due to Two Point Charges

Two point charges with $q_1 = 2 \times 10^{-5}$ C and $q_2 = -4 \times 10^{-5}$ C are located in free space at points with Cartesian coordinates $(1, 3, -1)$ and $(-3, 1, -2)$, respectively. Find (a) the electric field $\mathbf{E}$ at $(3, 1, -2)$ and (b) the force on a 8×10^{-5} C charge located at that point. All distances are in meters.

Solution: (a) From Eq. (4.18), the electric field $\mathbf{E}$ with $\varepsilon = \varepsilon_0$ (free space) is

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\left[q_1 \frac{(\mathbf{R} - \mathbf{R}_1)}{|\mathbf{R} - \mathbf{R}_1|^3} + q_2 \frac{(\mathbf{R} - \mathbf{R}_2)}{|\mathbf{R} - \mathbf{R}_2|^3}\right] \qquad \text{(V/m)}.$$

The vectors $\mathbf{R}_1$, $\mathbf{R}_2$, and $\mathbf{R}$ are

$$\mathbf{R}_1 = \hat{\mathbf{x}} + \hat{\mathbf{y}}3 - \hat{\mathbf{z}},$$

$$\mathbf{R}_2 = -\hat{\mathbf{x}}3 + \hat{\mathbf{y}} - \hat{\mathbf{z}}2,$$

$$\mathbf{R} = \hat{\mathbf{x}}3 + \hat{\mathbf{y}} - \hat{\mathbf{z}}2.$$

Hence,

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\left[\frac{2(\hat{\mathbf{x}}2 - \hat{\mathbf{y}}2 - \hat{\mathbf{z}})}{27} - \frac{4(\hat{\mathbf{x}}6)}{216}\right] \times 10^{-5}$$

$$= \frac{\hat{\mathbf{x}} - \hat{\mathbf{y}}4 - \hat{\mathbf{z}}2}{108\pi\varepsilon_0} \times 10^{-5} \qquad \text{(V/m)}.$$

(b)

$$\mathbf{F} = q_3\mathbf{E} = 8 \times 10^{-5} \times \frac{\hat{\mathbf{x}} - \hat{\mathbf{y}}4 - \hat{\mathbf{z}}2}{108\pi\varepsilon_0} \times 10^{-5}$$

$$= \frac{\hat{\mathbf{x}}2 - \hat{\mathbf{y}}8 - \hat{\mathbf{z}}4}{27\pi\varepsilon_0} \times 10^{-10} \qquad \text{(N)}.$$

Electric Field Due to Charge Distributions

Field due to:

a differential amount of charge $dq = \rho_v \, d\mathcal{V}'$ contained in a differential volume $d\mathcal{V}'$ is

$$d\mathbf{E} = \hat{\mathbf{R}}' \frac{dq}{4\pi\varepsilon R'^2} = \hat{\mathbf{R}}' \frac{\rho_v \, d\mathcal{V}'}{4\pi\varepsilon R'^2}, \qquad (4.20)$$

$$\mathbf{E} = \int_{\mathcal{V}'} d\mathbf{E} = \frac{1}{4\pi\varepsilon} \int_{\mathcal{V}'} \hat{\mathbf{R}}' \frac{\rho_v \, d\mathcal{V}'}{R'^2}$$

(volume distribution). (4.21a)

$$\mathbf{E} = \frac{1}{4\pi\varepsilon} \int_{S'} \hat{\mathbf{R}}' \frac{\rho_s \, ds'}{R'^2} \quad \text{(surface distribution)}, \qquad (4.21b)$$

$$\mathbf{E} = \frac{1}{4\pi\varepsilon} \int_{l'} \hat{\mathbf{R}}' \frac{\rho_\ell \, dl'}{R'^2} \quad \text{(line distribution)}. \qquad (4.21c)$$

Example 4-4: Electric Field of a Ring of Charge

A ring of charge of radius b is characterized by a uniform line charge density of positive polarity ρ_ℓ. The ring resides in free space and is positioned in the x–y plane as shown in Fig. 4-6. Determine the electric field intensity $\mathbf{E}$ at a point $P = (0, 0, h)$ along the axis of the ring at a distance h from its center.

Solution: We start by considering the electric field generated by a differential ring segment with cylindrical coordinates $(b, \phi, 0)$ in Fig. 4-6(a). The segment has length $dl = b\,d\phi$ and contains charge $dq = \rho_\ell\, dl = \rho_\ell b\, d\phi$. The distance vector $\mathbf{R}'_1$ from segment 1 to point $P = (0, 0, h)$ is

$$\mathbf{R}'_1 = -\hat{\mathbf{r}}b + \hat{\mathbf{z}}h,$$

from which it follows that

$$R'_1 = |\mathbf{R}'_1| = \sqrt{b^2 + h^2}\,, \qquad \hat{\mathbf{R}}'_1 = \frac{\mathbf{R}'_1}{|\mathbf{R}'_1|} = \frac{-\hat{\mathbf{r}}b + \hat{\mathbf{z}}h}{\sqrt{b^2 + h^2}}\,.$$

The electric field at $P = (0, 0, h)$ due to the charge in segment 1 therefore is

$$d\mathbf{E}_1 = \frac{1}{4\pi\varepsilon_0}\,\hat{\mathbf{R}}'_1\,\frac{\rho_\ell\, dl}{{R'_1}^2} = \frac{\rho_\ell b}{4\pi\varepsilon_0}\,\frac{(-\hat{\mathbf{r}}b + \hat{\mathbf{z}}h)}{(b^2 + h^2)^{3/2}}\,d\phi.$$

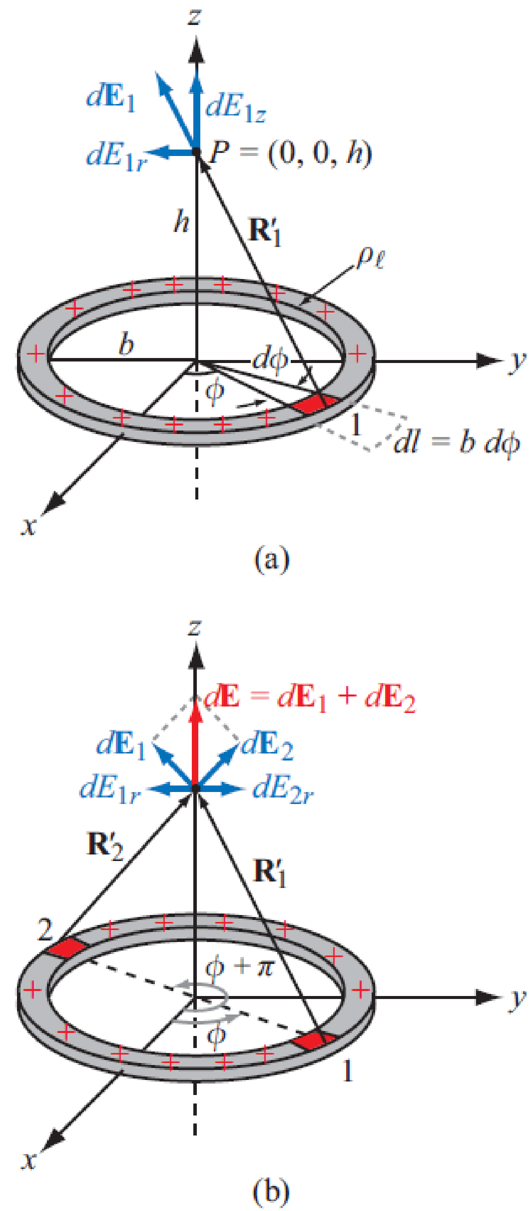


Figure 4-6: Ring of charge with line density ρ_ℓ. (a) The field $d\mathbf{E}_1$ due to infinitesimal segment 1 and (b) the fields $d\mathbf{E}_1$ and $d\mathbf{E}_2$ due to segments at diametrically opposite locations (Example 4-4).

Cont.

$$dE_1 = \frac{1}{4\pi\varepsilon_0}\,\hat{\mathbf{R}}'_1\,\frac{\rho_\ell\,dl}{R_1'^2} = \frac{\rho_\ell b}{4\pi\varepsilon_0}\,\frac{(-\hat{\mathbf{r}}b + \hat{\mathbf{z}}h)}{(b^2+h^2)^{3/2}}\,d\phi.$$

The field $d\mathbf{E}_1$ has component dE_{1r} along $-\hat{\mathbf{r}}$ and component dE_{1z} along $\hat{\mathbf{z}}$. From symmetry considerations, the field $d\mathbf{E}_2$ generated by differential segment 2 in Fig. 4-6(b), which is located diametrically opposite to segment 1, is identical to $d\mathbf{E}_1$ except that the $\hat{\mathbf{r}}$-component of $d\mathbf{E}_2$ is opposite that of $d\mathbf{E}_1$. Hence, the $\hat{\mathbf{r}}$-components in the sum cancel and the $\hat{\mathbf{z}}$-contributions add. The sum of the two contributions is

$$d\mathbf{E} = d\mathbf{E}_1 + d\mathbf{E}_2 = \hat{\mathbf{z}}\,\frac{\rho_\ell bh}{2\pi\varepsilon_0}\,\frac{d\phi}{(b^2+h^2)^{3/2}}\,. \qquad (4.22)$$

Since for every ring segment in the semicircle defined over the azimuthal range $0 \le \phi \le \pi$ (the right-hand half of the circular ring) there is a corresponding segment located diametrically opposite at $(\phi + \pi)$, we can obtain the total field generated by the ring by integrating Eq. (4.22) over a semicircle as

$$\begin{aligned}\mathbf{E} &= \hat{\mathbf{z}}\,\frac{\rho_\ell bh}{2\pi\varepsilon_0(b^2+h^2)^{3/2}}\int_0^\pi d\phi \\ &= \hat{\mathbf{z}}\,\frac{\rho_\ell bh}{2\varepsilon_0(b^2+h^2)^{3/2}} \\ &= \hat{\mathbf{z}}\,\frac{h}{4\pi\varepsilon_0(b^2+h^2)^{3/2}}\,Q, \qquad (4.23)\end{aligned}$$

where $Q = 2\pi b\rho_\ell$ is the total charge on the ring.

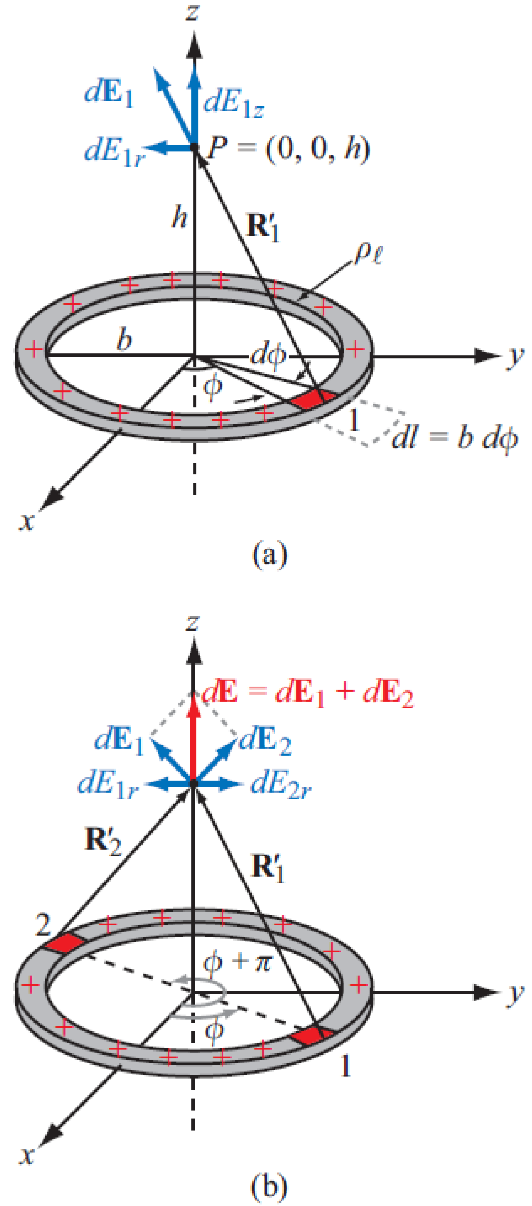


Figure 4-6: Ring of charge with line density ρ_ℓ. (a) The field $d\mathbf{E}_1$ due to infinitesimal segment 1 and (b) the fields $d\mathbf{E}_1$ and $d\mathbf{E}_2$ due to segments at diametrically opposite locations (Example 4-4).

Example 4-5: Electric Field of a Circular Disk of Charge

Find the electric field at point P with Cartesian coordinates $(0, 0, h)$ due to a circular disk of radius a and uniform charge density ρ_s residing in the x–y plane (Fig. 4-7). Also, evaluate $\mathbf{E}$ due to an infinite sheet of charge density ρ_s by letting $a \to \infty$.

Solution: Building on the expression obtained in Example 4-4 for the on-axis electric field due to a circular ring of charge, we can determine the field due to the circular disk by treating the disk as a set of concentric rings. A ring of radius r and width dr has an area $ds = 2\pi r\, dr$ and contains charge $dq = \rho_s\, ds = 2\pi \rho_s r\, dr$. Upon using this expression in Eq. (4.23) and also replacing b with r, we obtain the following expression for the field due to the ring:

$$d\mathbf{E} = \hat{\mathbf{z}}\, \frac{h}{4\pi \varepsilon_0 (r^2 + h^2)^{3/2}}\, (2\pi \rho_s r\, dr).$$

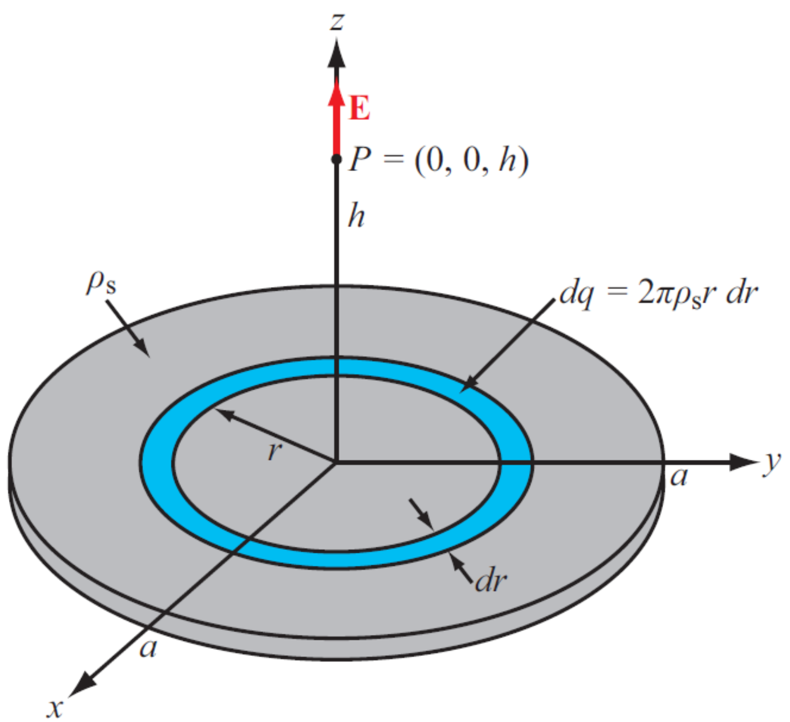


Figure 4-7: Circular disk of charge with surface charge density ρ_s. The electric field at $P = (0, 0, h)$ points along the z-direction (Example 4-5).

Cont.

Example 4-5 cont.

The total field at P is obtained by integrating the expression over the limits $r = 0$ to $r = a$:

$$\mathbf{E} = \hat{\mathbf{z}} \frac{\rho_s h}{2\varepsilon_0} \int_0^a \frac{r\,dr}{(r^2 + h^2)^{3/2}}$$

$$= \pm\hat{\mathbf{z}} \frac{\rho_s}{2\varepsilon_0} \left[1 - \frac{|h|}{\sqrt{a^2 + h^2}}\right], \qquad (4.24)$$

with the plus sign for $h > 0$ (P above the disk) and the minus sign when $h < 0$ (P below the disk).

For an infinite sheet of charge with $a = \infty$,

$$\mathbf{E} = \pm\hat{\mathbf{z}} \frac{\rho_s}{2\varepsilon_0} \quad \text{(infinite sheet of charge).} \qquad (4.25)$$

We note that for an infinite sheet of charge $\mathbf{E}$ is the same at all points above the x–y plane, and a similar statement applies for points below the x-y plane.

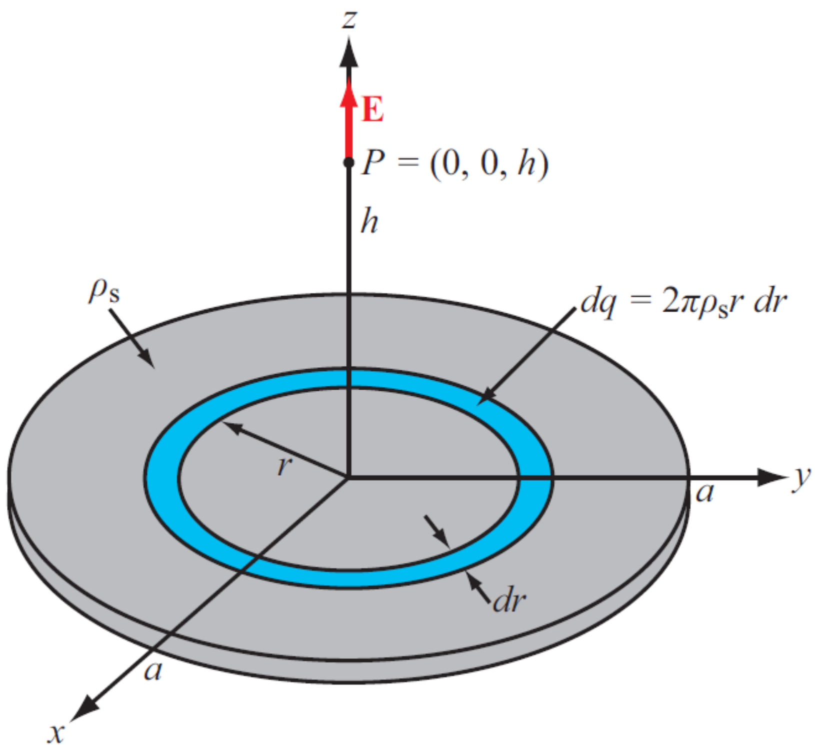


Figure 4-7: Circular disk of charge with surface charge density ρ_s. The electric field at $P = (0, 0, h)$ points along the z-direction (Example 4-5).

Gauss's Law

$$\nabla \cdot \mathbf{D} = \rho_v$$

(Differential form of Gauss's law),

$$\int_{\mathcal{V}} \nabla \cdot \mathbf{D}\, d\mathcal{V} = \int_{\mathcal{V}} \rho_v\, d\mathcal{V} = Q$$

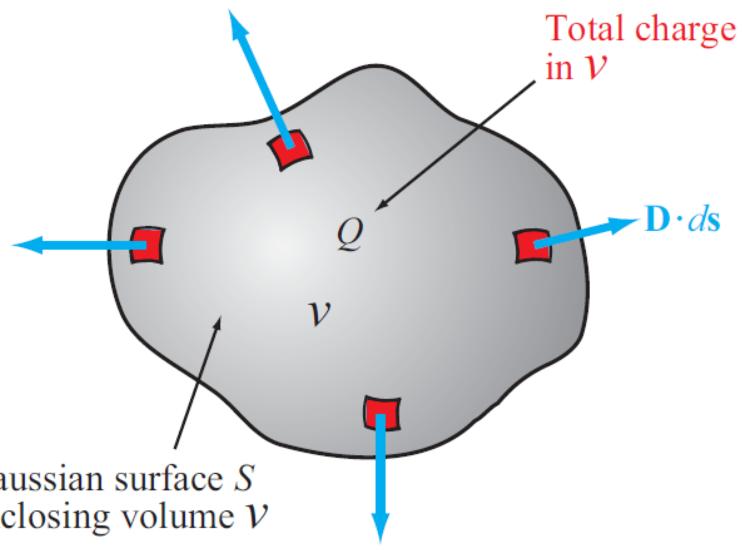


Figure 4-8: The integral form of Gauss's law states that the outward flux of **D** through a surface is proportional to the enclosed charge Q.

Application of the divergence theorem gives:

$$\int_{\mathcal{V}} \nabla \cdot \mathbf{D}\, d\mathcal{V} = \oint_S \mathbf{D} \cdot d\mathbf{s}. \qquad (4.28)$$

Comparison of Eq. (4.27) with Eq. (4.28) leads to

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q \qquad (4.29)$$

(Integral form of Gauss's law).

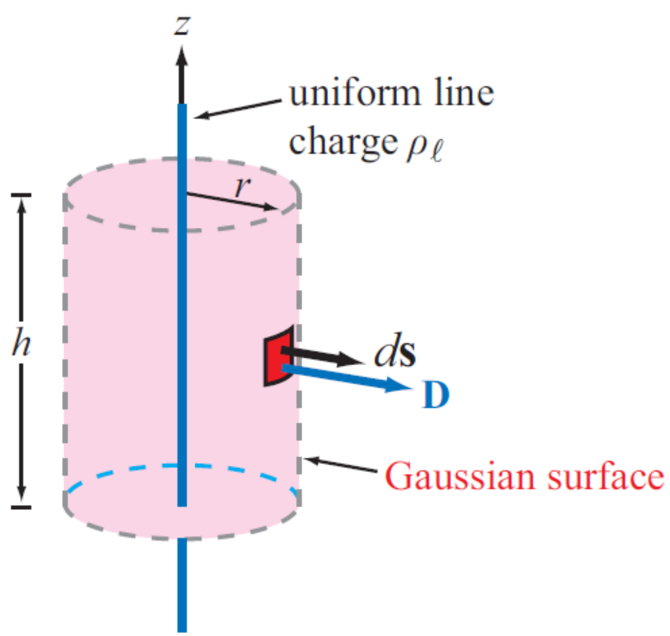
The integral form of Gauss's law is illustrated diagrammatically in Fig. 4-8; for each differential surface element $d\mathbf{s}$, $\mathbf{D} \cdot d\mathbf{s}$ *is the electric field flux flowing outward of* $\mathcal{V}$ *through* $d\mathbf{s}$, *and the total flux through surface* S *equals the enclosed charge* Q. *The surface* S *is called a* ***Gaussian surface***.

Applying Gauss's Law

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q \qquad (4.29)$$

(Integral form of Gauss's law).

Gauss's law, as given by Eq. (4.29), provides a convenient method for determining the flux density **D** *when the charge distribution possesses symmetry properties that allow us to infer the variations of the magnitude and direction of* **D** *as a function of spatial location, thereby facilitating the integration of* **D** *over a cleverly chosen Gaussian surface.*



Example 4-6: Electric Field of an Infinite Line Charge

Use Gauss's law to obtain an expression for **E** due to an infinitely long line with uniform charge density ρ_ℓ that resides along the z-axis in free space.

Construct an imaginary Gaussian cylinder of radius *r* and height *h*:

$$\int_{z=0}^{h} \int_{\phi=0}^{2\pi} \hat{\mathbf{r}} D_r \cdot \hat{\mathbf{r}} r \; d\phi \; dz = \rho_\ell h$$

or

$$2\pi h D_r r = \rho_\ell h,$$

which yields

$$\mathbf{E} = \frac{\mathbf{D}}{\varepsilon_0} = \hat{\mathbf{r}} \frac{D_r}{\varepsilon_0} = \hat{\mathbf{r}} \frac{\rho_\ell}{2\pi \varepsilon_0 r} \qquad (4.33)$$

(infinite line charge).

Electric Scalar Potential

The term "voltage" is short for "voltage potential" and synonymous with ***electric potential****.*

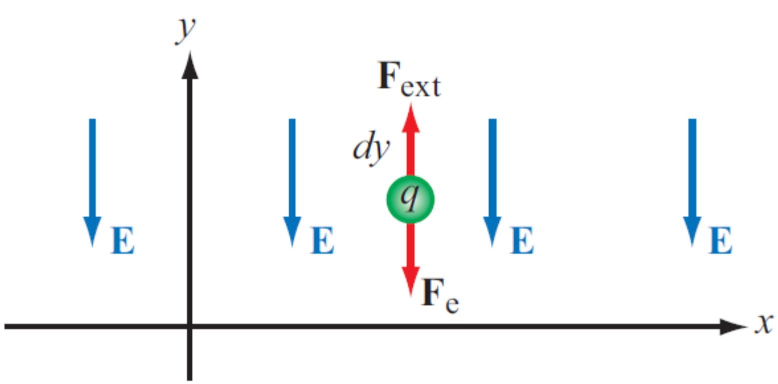


Figure 4-11: Work done in moving a charge q a distance dy against the electric field $\mathbf{E}$ is $dW = qE\,dy$.

Minimum force needed to move charge against **E** field:

$$\mathbf{F}_{\text{ext}} = -\mathbf{F}_{\text{e}} = -q\mathbf{E}. \qquad (4.34)$$

The work done, or energy expended, in moving any object a vector differential distance $d\mathbf{l}$ while exerting a force $\mathbf{F}_{\text{ext}}$ is

$$dW = \mathbf{F}_{\text{ext}} \cdot d\mathbf{l} = -q\mathbf{E} \cdot d\mathbf{l} \qquad \text{(J)}. \qquad (4.35)$$

Work, or energy, is measured in joules (J). If the charge is moved a distance dy along $\hat{\mathbf{y}}$, then

$$dW = -q(-\hat{\mathbf{y}}E) \cdot \hat{\mathbf{y}}\,dy = qE\,dy. \qquad (4.36)$$

The differential electric potential energy dW per unit charge is called the ***differential electric potential*** (or differential voltage) dV. That is,

$$dV = \frac{dW}{q} = -\mathbf{E} \cdot d\mathbf{l} \qquad \text{(J/C or V)}. \qquad (4.37)$$

Electric Scalar Potential

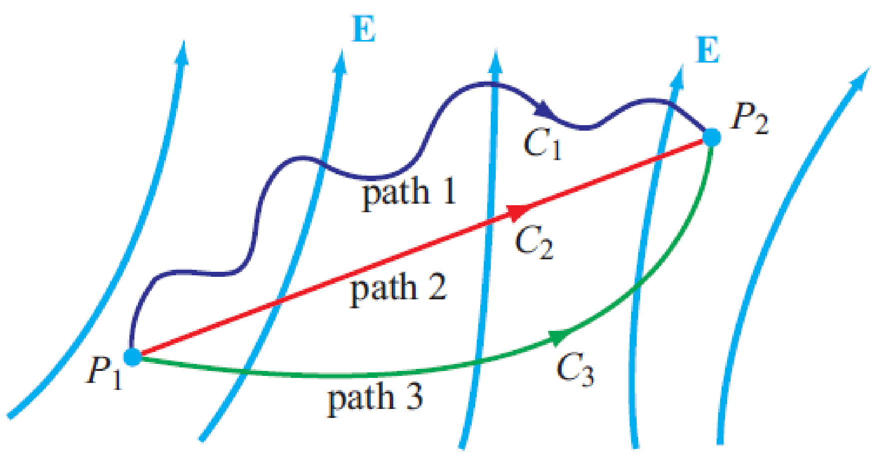


Figure 4-12: In electrostatics, the potential difference between P_2 and P_1 is the same irrespective of the path used for calculating the line integral of the electric field between them.

$$\int_{P_1}^{P_2} dV = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\mathbf{l},$$

$$V_{21} = V_2 - V_1 = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\mathbf{l}, \qquad (4.39)$$

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0 \qquad \text{(Electrostatics).} \qquad (4.40)$$

*A vector field whose line integral along any closed path is zero is called a **conservative** or an **irrotational** field. Hence, the electrostatic field* **E** *is conservative.*

Electric Potential Due to Charges

$$\int_{P_1}^{P_2} dV = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\mathbf{l},$$

$$V_{21} = V_2 - V_1 = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\mathbf{l}, \qquad (4.39)$$

In electric circuits, we usually select a convenient node that we call ground and assign it zero reference voltage. In free space and material media, we choose infinity as reference with $V = 0$. Hence, at a point P

$$V = -\int_{\infty}^{P} \mathbf{E} \cdot d\mathbf{l} \qquad \text{(V)}. \qquad (4.43)$$

For a point charge, V at range R is:

$$V = -\int_{\infty}^{R} \left(\hat{\mathbf{R}} \frac{q}{4\pi\varepsilon R^2} \right) \cdot \hat{\mathbf{R}}\, dR = \frac{q}{4\pi\varepsilon R} \qquad \text{(V)}. \qquad (4.45)$$

For continuous charge distributions:

$$V = \frac{1}{4\pi\varepsilon} \int_{\mathcal{V}'} \frac{\rho_{\mathrm{v}}}{R'}\, d\mathcal{V}' \quad \text{(volume distribution)}, \qquad (4.48a)$$

$$V = \frac{1}{4\pi\varepsilon} \int_{S'} \frac{\rho_{\mathrm{s}}}{R'}\, ds' \quad \text{(surface distribution)}, \qquad (4.48b)$$

$$V = \frac{1}{4\pi\varepsilon} \int_{l'} \frac{\rho_{\ell}}{R'}\, dl' \quad \text{(line distribution)}. \qquad (4.48c)$$

Relating **E** to V

$$dV = -\mathbf{E} \cdot d\mathbf{l}. \qquad (4.49)$$

For a scalar function V, Eq. (3.73) gives

$$dV = \nabla V \cdot d\mathbf{l}, \qquad (4.50)$$

where ∇V is the gradient of V. Comparison of Eq. (4.49) with Eq. (4.50) leads to

$$\mathbf{E} = -\nabla V. \qquad (4.51)$$

*This differential relationship between V and **E** allows us to determine **E** for any charge distribution by first calculating V and then taking the negative gradient of V to find **E**.*

Example 4-7: Electric Field of an Electric Dipole

Solution: To simplify the derivation, we align the dipole along the z-axis and center it at the origin [Fig. 4-13(a)]. For the two charges shown in Fig. 4-13(a), application of Eq. (4.47) gives

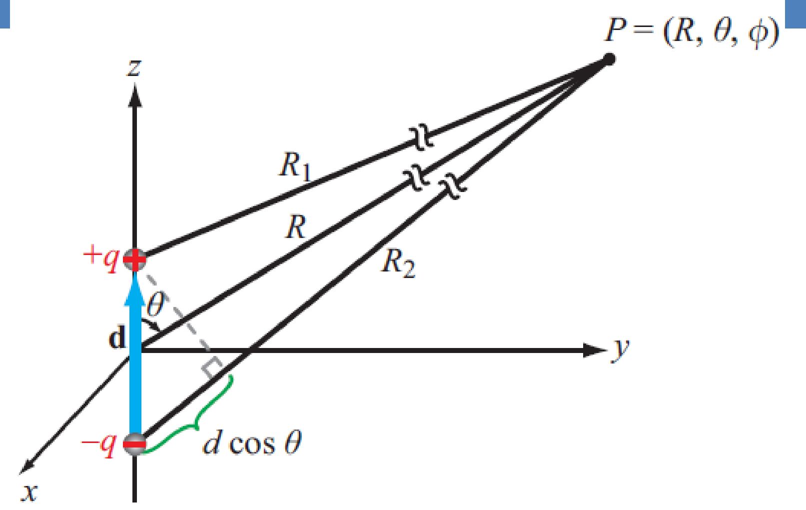
$$V = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{R_1} + \frac{-q}{R_2}\right) = \frac{q}{4\pi\varepsilon_0}\left(\frac{R_2 - R_1}{R_1 R_2}\right).$$

Since $d \ll R$, the lines labeled R_1 and R_2 in Fig. 4-13(a) are approximately parallel to each other, in which case the following approximations apply:

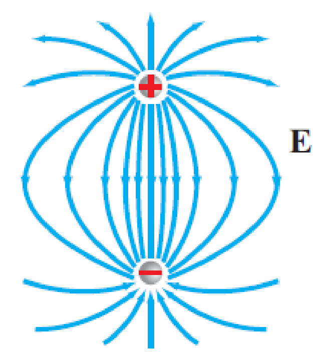
$$R_2 - R_1 \simeq d\cos\theta, \qquad R_1 R_2 \simeq R^2.$$

Hence,

$$V = \frac{qd\cos\theta}{4\pi\varepsilon_0 R^2}. \qquad (4.52)$$



(a) Electric dipole



(b) Electric-field pattern

https://www.youtube.com/watch?v=LB8Rhcb4eQM&t=68s

Cont.

Example 4-7: Electric Field of an Electric Dipole (cont.)

$$qd\cos\theta = q\mathbf{d}\cdot\hat{\mathbf{R}} = \mathbf{p}\cdot\hat{\mathbf{R}},$$

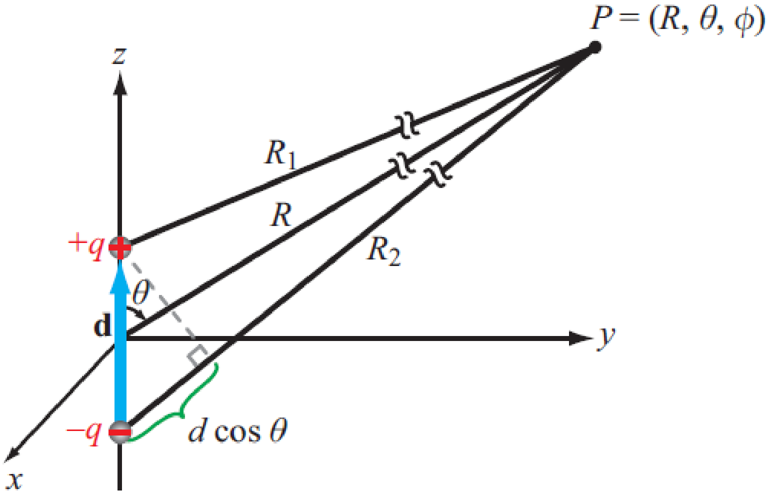
where $\mathbf{p} = q\mathbf{d}$ is called the ***dipole moment***. Using Eq. (4.53) in Eq. (4.52) then gives

$$V = \frac{\mathbf{p}\cdot\hat{\mathbf{R}}}{4\pi\varepsilon_0 R^2} \qquad \text{(electric dipole).} \qquad (4.54)$$

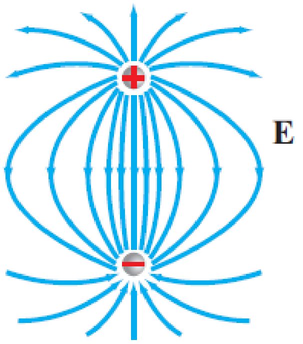
In spherical coordinates, Eq. (4.51) is given by

$$\mathbf{E} = -\nabla V = -\left(\hat{\mathbf{R}}\frac{\partial V}{\partial R} + \hat{\boldsymbol{\theta}}\frac{1}{R}\frac{\partial V}{\partial \theta} + \hat{\boldsymbol{\phi}}\frac{1}{R\sin\theta}\frac{\partial V}{\partial \phi}\right), \qquad (4.55)$$

$$\mathbf{E} = \frac{qd}{4\pi\varepsilon_0 R^3}(\hat{\mathbf{R}}\,2\cos\theta + \hat{\boldsymbol{\theta}}\sin\theta) \quad \text{(V/m).}$$



(a) Electric dipole



(b) Electric-field pattern

Poisson's & Laplace's Equations

With $\mathbf{D} = \varepsilon\mathbf{E}$, the differential form of Gauss's law given by Eq. (4.26) may be cast as

$$\nabla \cdot \mathbf{E} = \frac{\rho_v}{\varepsilon} . \qquad (4.57)$$

Inserting Eq. (4.51) in Eq. (4.57) gives

$$\nabla \cdot (\nabla V) = -\frac{\rho_v}{\varepsilon} . \qquad (4.58)$$

Given Eq. (3.110) for the Laplacian of a scalar function V,

$$\nabla^2 V = \nabla \cdot (\nabla V) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} , \qquad (4.59)$$

Eq. (4.58) can be cast in the abbreviated form

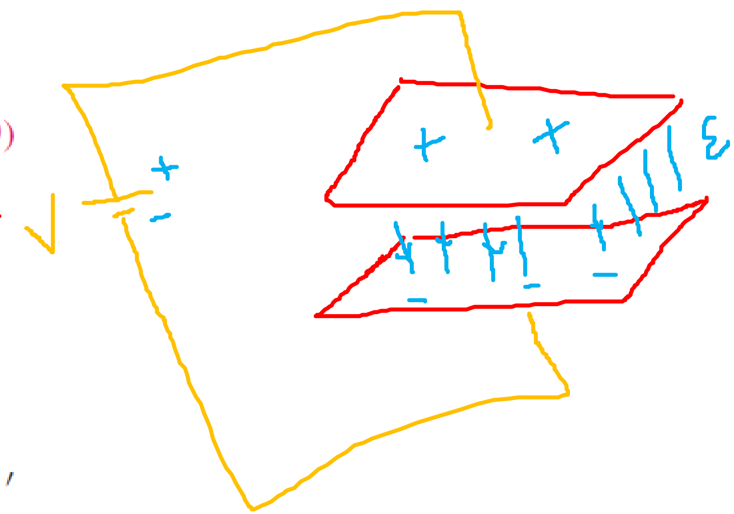
$$\nabla^2 V = -\frac{\rho_v}{\varepsilon} \qquad \text{(Poisson's equation).} \qquad (4.60)$$

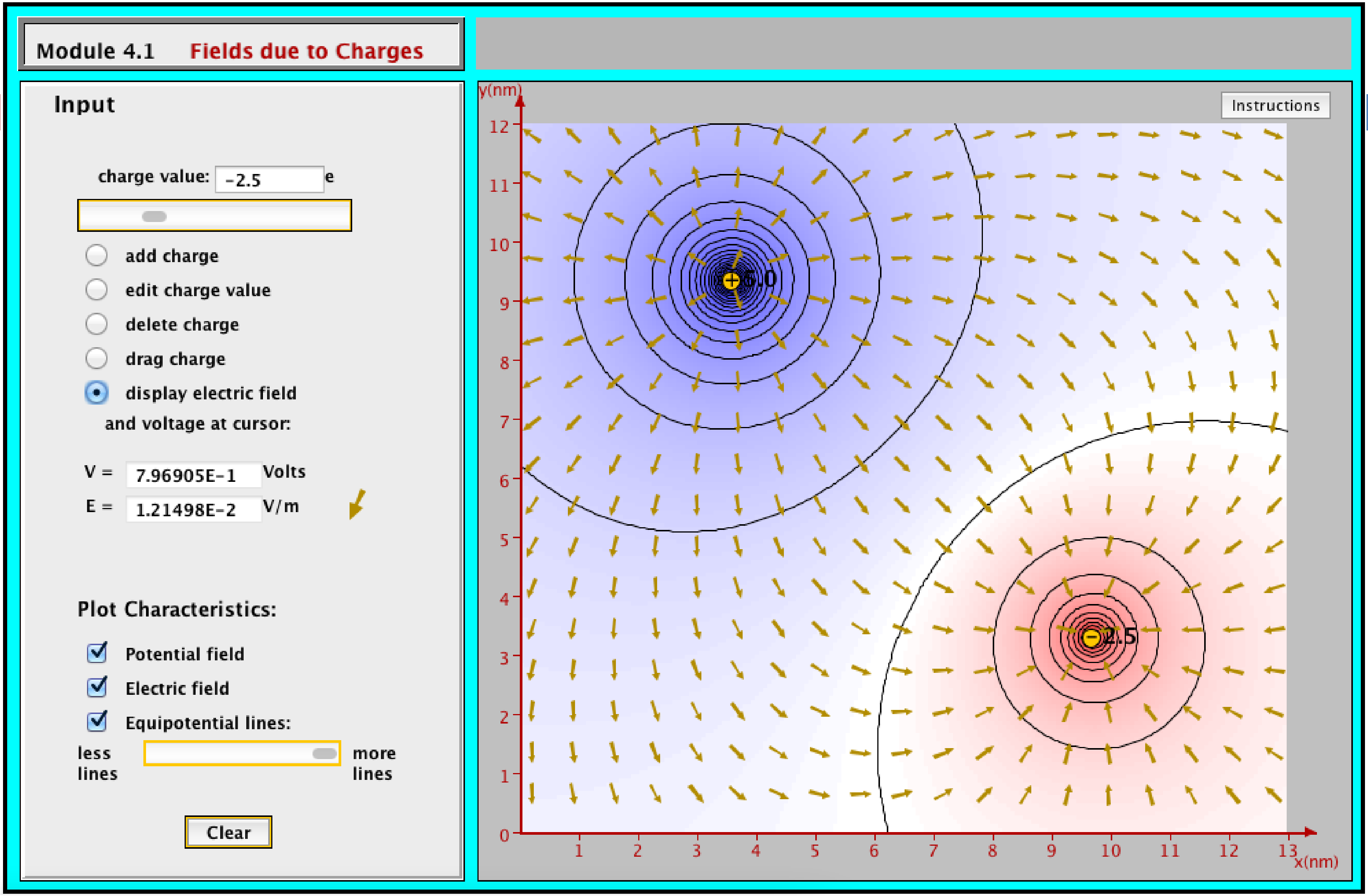
This is known as ***Poisson's equation***. For a volume $\mathcal{V}'$ containing a volume charge density distribution ρ_v, the solution for V derived previously and expressed by Eq. (4.48a) as

$$V = \frac{1}{4\pi\varepsilon} \int_{\mathcal{V}'} \frac{\rho_v}{R'} \, d\mathcal{V}' \qquad (4.61)$$

In the absence of charges:

$$\nabla^2 V = 0 \qquad \text{(Laplace's equation),}$$





Module 4.1 Fields due to Charges
Input
charge value: -2.5 e
add charge
edit charge value
delete charge
drag charge
display electric field
and voltage at cursor:
V = 7.96905E-1 Volts
E = 1.21498E-2 V/m
Plot Characteristics:
Potential field
Electric field
Equipotential lines:
less lines
more lines
Clear
Instructions
y(nm)
x(nm)
5.0
2.5

Conduction Current

The conductivity of a material is a measure of how easily electrons can travel through the material under the influence of an externally applied electric field.

Conduction current density:

$$\mathbf{J} = \sigma \mathbf{E} \qquad (\text{A/m}^2) \qquad (\text{Ohm's law}),$$

original

A ***perfect dielectric*** *is a material with $\sigma = 0$. In contrast, a* ***perfect conductor*** *is a material with $\sigma = \infty$. Some materials, called superconductors, exhibit such a behavior.*

Table 4-1: Conductivity of some common materials at 20°C.

Material	**Conductivity**, σ (S/m)
Conductors	
Silver	6.2×10^7
Copper	5.8×10^7
Gold	4.1×10^7
Aluminum	3.5×10^7
Iron	10^7
Mercury	10^6
Carbon	3×10^4
Semiconductors	
Pure germanium	2.2
Pure silicon	4.4×10^{-4}
Insulators	
Glass	10^{-12}
Paraffin	10^{-15}
Mica	10^{-15}
Fused quartz	10^{-17}

Note how wide the range is, over 24 orders of magnitude

Conductivity

$$\sigma = -\rho_{ve}\mu_e + \rho_{vh}\mu_h$$
$$= (N_e\mu_e + N_h\mu_h)e \qquad \text{(S/m)} \quad \text{(semiconductor)}, \qquad (4.67a)$$

and its unit is siemens per meter (S/m). For a good conductor, $N_h\mu_h \ll N_e\mu_e$, and Eq. (4.67a) reduces to

$$\sigma = -\rho_{ve}\mu_e = N_e\mu_e e \qquad \text{(S/m)} \quad \text{(conductor)}. \qquad (4.67b)$$

In view of Eq. (4.66), in a perfect dielectric with $\sigma = 0$, $\mathbf{J} = 0$ *regardless of* $\mathbf{E}$, *and in a perfect conductor with* $\sigma = \infty$, $\mathbf{E} = \mathbf{J}/\sigma = 0$ *regardless of* $\mathbf{J}$.

$$\mathbf{J} = \sigma\mathbf{E} \qquad (\text{A/m}^2) \quad \text{(Ohm's law)},$$

That is,

Perfect dielectric:	$\mathbf{J} = 0$,
Perfect conductor:	$\mathbf{E} = 0$.

ρ_{ve} = volume charge density of electrons

ρ_{he} = volume charge density of holes

μ_e = electron mobility

μ_h = hole mobility

N_e = number of electrons per unit volume

N_h = number of holes per unit volume

Example 4-8: Conduction Current in a Copper Wire

A 2-mm-diameter copper wire with conductivity of 5.8×10^7 S/m and electron mobility of 0.0032 ($\text{m}^2/\text{V}\cdot\text{s}$) is subjected to an electric field of 20 (mV/m). Find (a) the volume charge density of the free electrons, (b) the current density, (c) the current flowing in the wire, (d) the electron drift velocity, and (e) the volume density of the free electrons.

Solution:

(a)

$$\rho_{\text{ve}} = -\frac{\sigma}{\mu_{\text{e}}} = -\frac{5.8 \times 10^7}{0.0032} = -1.81 \times 10^{10} \ (\text{C/m}^3).$$

(b)

$$J = \sigma E = 5.8 \times 10^7 \times 20 \times 10^{-3} = 1.16 \times 10^6 \ (\text{A/m}^2).$$

(c)

$$I = JA$$
$$= J\left(\frac{\pi d^2}{4}\right) = 1.16 \times 10^6 \left(\frac{\pi \times 4 \times 10^{-6}}{4}\right) = 3.64 \text{ A}.$$

(d)

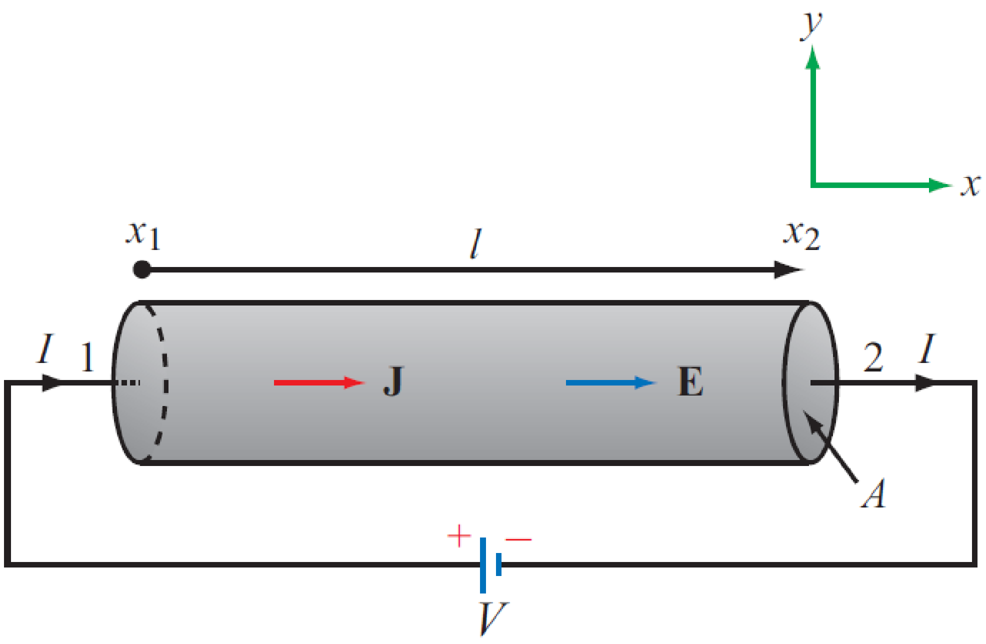
$$u_{\text{e}} = -\mu_{\text{e}} E = -0.0032 \times 20 \times 10^{-3} = -6.4 \times 10^{-5} \text{ m/s}.$$

The minus sign indicates that $\mathbf{u}_{\text{e}}$ is in the opposite direction of $\mathbf{E}$.

(e)

$$N_{\text{e}} = -\frac{\rho_{\text{ve}}}{e} = \frac{1.81 \times 10^{10}}{1.6 \times 10^{-19}} = 1.13 \times 10^{29} \text{ electrons/m}^3.$$

Resistance



Longitudinal Resistor

$$V = V_1 - V_2 = -\int_{x_2}^{x_1} \mathbf{E} \cdot d\mathbf{l}$$

$$= -\int_{x_2}^{x_1} \hat{\mathbf{x}} E_x \cdot \hat{\mathbf{x}}\, dl = E_x l \qquad \text{(V)}. \qquad (4.68)$$

Using Eq. (4.63), the current flowing through the cross section A at x_2 is

$$I = \int_A \mathbf{J} \cdot d\mathbf{s} = \int_A \sigma \mathbf{E} \cdot d\mathbf{s} = \sigma E_x A \qquad \text{(A)}. \qquad (4.69)$$

From $R = V/I$, the ratio of Eq. (4.68) to Eq. (4.69) gives

$$R = \frac{l}{\sigma A} \qquad (\Omega). \qquad (4.70)$$

For any conductor:

$$R = \frac{V}{I} = \frac{-\int_l \mathbf{E} \cdot d\mathbf{l}}{\int_S \mathbf{J} \cdot d\mathbf{s}} = \frac{-\int_l \mathbf{E} \cdot d\mathbf{l}}{\int_S \sigma \mathbf{E} \cdot d\mathbf{s}} .$$

Example 4-9: Conductance of Coaxial Cable

The radii of the inner and outer conductors of a coaxial cable of length l are a and b, respectively (Fig. 4-15). The insulation material has conductivity σ. Obtain an expression for G', the conductance per unit length of the insulation layer.

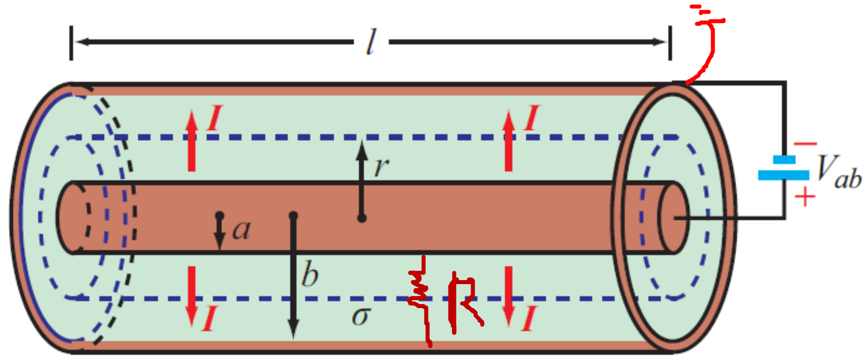
Solution: Let I be the total current flowing radially (along $\hat{\mathbf{r}}$) from the inner conductor to the outer conductor through the insulation material. At any radial distance r from the axis of the center conductor, the area through which the current flows is $A = 2\pi rl$. Hence,

$$\mathbf{J} = \hat{\mathbf{r}}\,\frac{I}{A} = \hat{\mathbf{r}}\,\frac{I}{2\pi rl}\,, \tag{4.73}$$

and from $\mathbf{J} = \sigma\mathbf{E}$,

$$\mathbf{E} = \hat{\mathbf{r}}\,\frac{I}{2\pi\sigma rl}\,. \tag{4.74}$$

In a resistor, the current flows from higher electric potential to lower potential. Hence, if $\mathbf{J}$ is in the $\hat{\mathbf{r}}$-direction, the inner



conductor must be at a higher potential than the outer conductor. Accordingly, the voltage difference between the conductors is

$$V_{ab} = -\int_b^a \mathbf{E}\cdot d\mathbf{l} = -\int_b^a \frac{I}{2\pi\sigma l}\,\frac{\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}\,dr}{r}$$

$$= \frac{I}{2\pi\sigma l}\ln\left(\frac{b}{a}\right). \tag{4.75}$$

The conductance per unit length is then

$$G' = \frac{G}{l} = \frac{1}{Rl} = \frac{I}{V_{ab}l} = \frac{2\pi\sigma}{\ln(b/a)} \quad \text{(S/m)}. \tag{4.76}$$

$G = 1/R$

G'=0 if the insulating material is air or a perfect dielectric with zero conductivity.

Joule's Law

The power dissipated in a volume containing electric field **E** and current density **J** is:

$$P = \int_V \mathbf{E} \cdot \mathbf{J} \, dV \qquad \text{(W)} \qquad \text{(Joule's law)}$$

For a coaxial cable:

$$P = I^2 \ln(b/a)/(2\pi \sigma l)$$

Derive

~~Find~~

For a resistor, Joule's law reduces to:

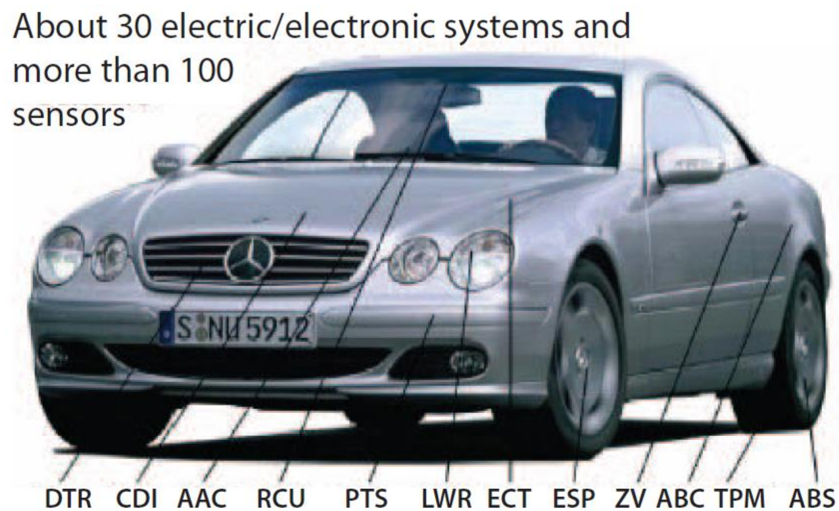
$$P = I^2 R \qquad \text{(W)}$$

Tech Brief 7: Resistive Sensors

An ***electrical sensor*** is a device capable of responding to an applied stimulus by generating an electrical signal whose voltage, current, or some other attribute is related to the intensity of the stimulus.

Typical stimuli : temperature, pressure, position, distance, motion, velocity, acceleration, concentration (of a gas or liquid), blood flow, etc.

Sensing process relies on measuring resistance, capacitance, inductance, induced electromotive force (emf), oscillation frequency or time delay, etc.



System	Abbrev.	Sensors	System	Abbrev.	Sensors
Distronic	DTR	3	Common-rail diesel injection	CDI	11
Electronic controlled transmission	ECT	9	Automatic air condition	AAC	13
Roof control unit	RCU	7	Active body control	ABC	12
Antilock braking system	ABS	4	Tire pressure monitoring	TPM	11
Central locking system	ZV	3	Elektron. stability program	ESP	14
Dyn. beam levelling	LWR	6	Parktronic system	PTS	12

Figure TF7-1: Most cars use on the order of 100 sensors. (Courtesy Mercedes-Benz.)

Piezoresistivity

The Greek word ***piezein*** means to press

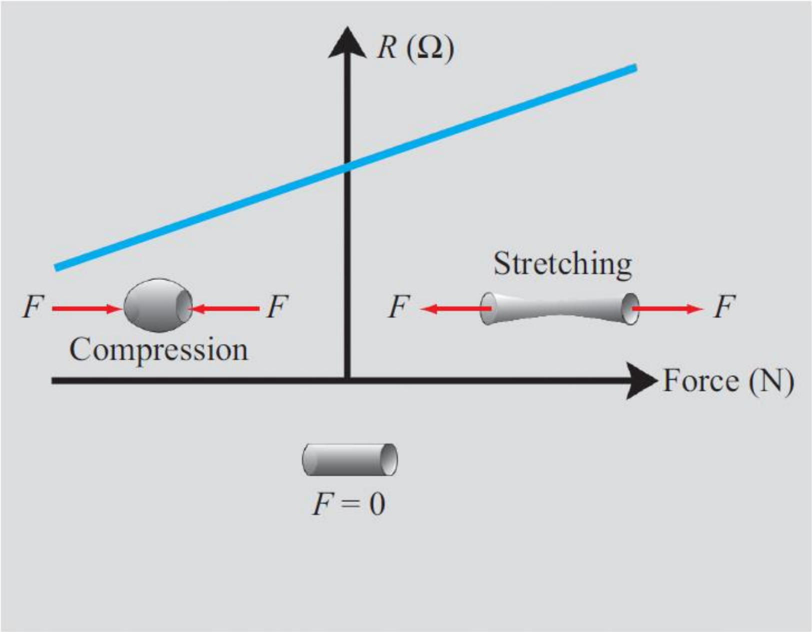


Figure TF7-2: Piezoresistance varies with applied force.

$$R = R_0 \left(1 + \frac{\alpha F}{A_0}\right)$$

R_0 = resistance when $F = 0$
F = applied force
A_0 = cross-section when $F = 0$
α = piezoresistive coefficient of material

Piezoresistors

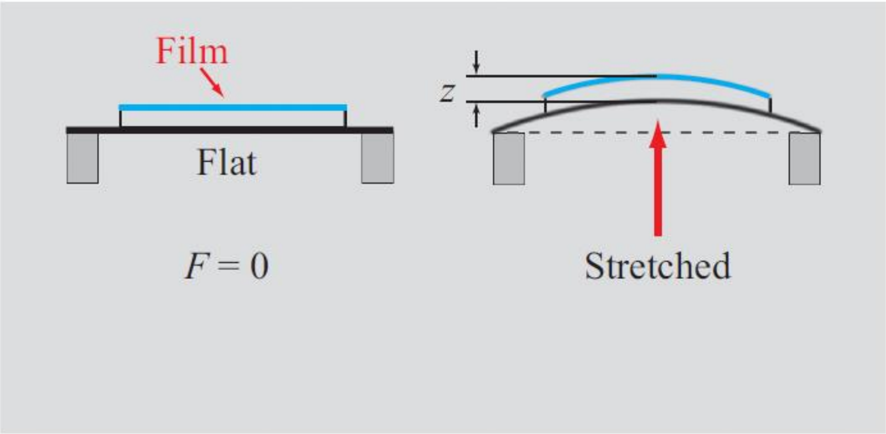


Figure TF7-3: Piezoresistor films.

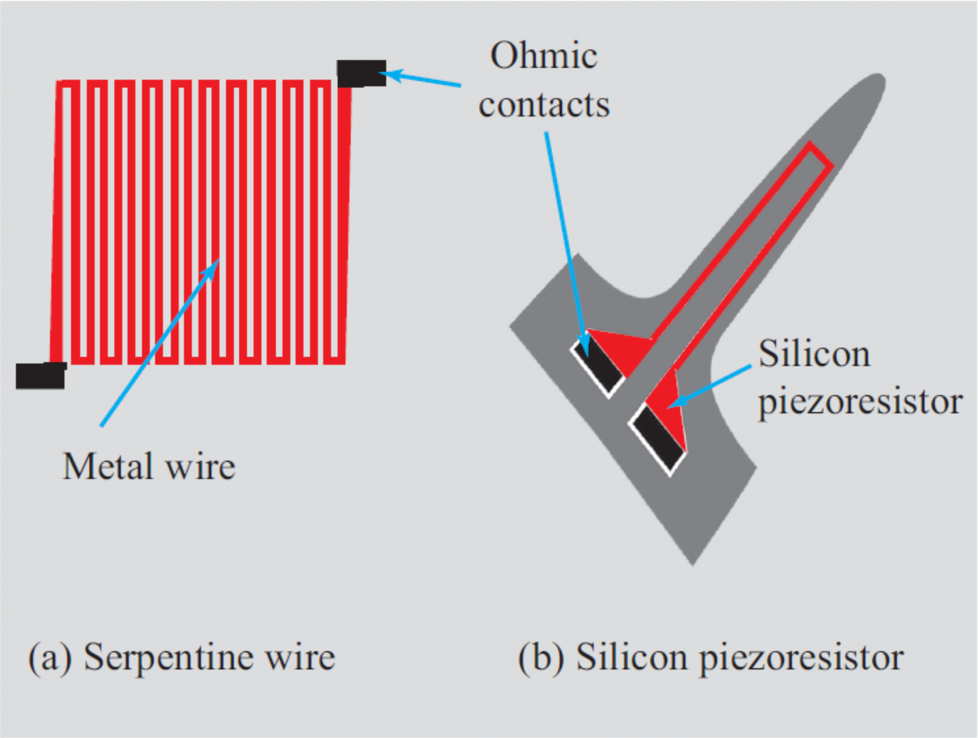
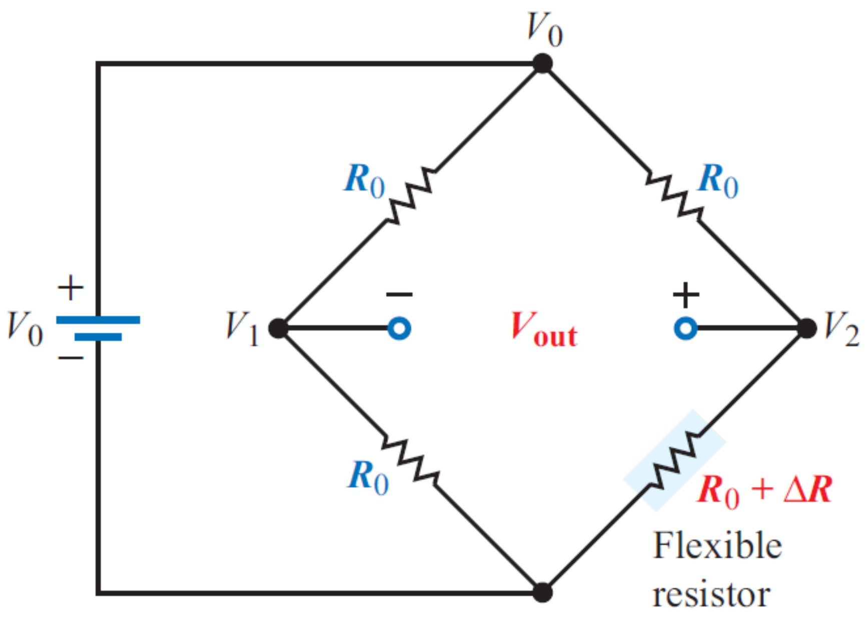


Figure TF7-4: Metal and silicon piezoresistors.

Wheatstone Bridge



Wheatstone bridge is a high sensitivity circuit for measuring small changes in resistance

$$V_{out} = \frac{V_0}{4}\left(\frac{\Delta R}{R_0}\right)$$

Figure TF7-5: Wheatstone bridge circuit with piezoresistor.

Dielectric Materials

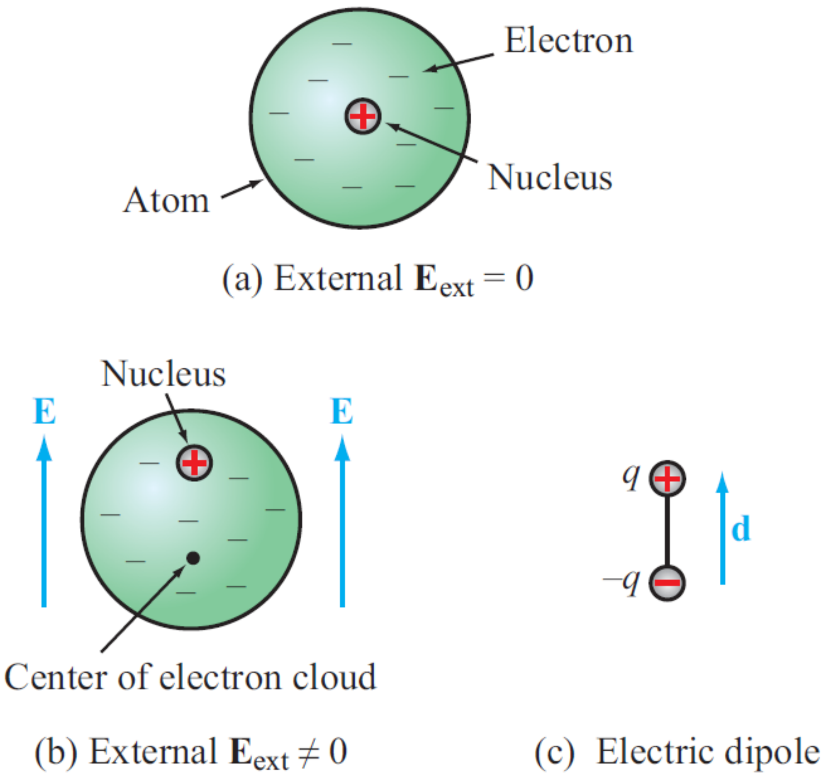


Figure 4-16: In the absence of an external electric field **E**, the center of the electron cloud is co-located with the center of the nucleus, but when a field is applied, the two centers are separated by a distance d.

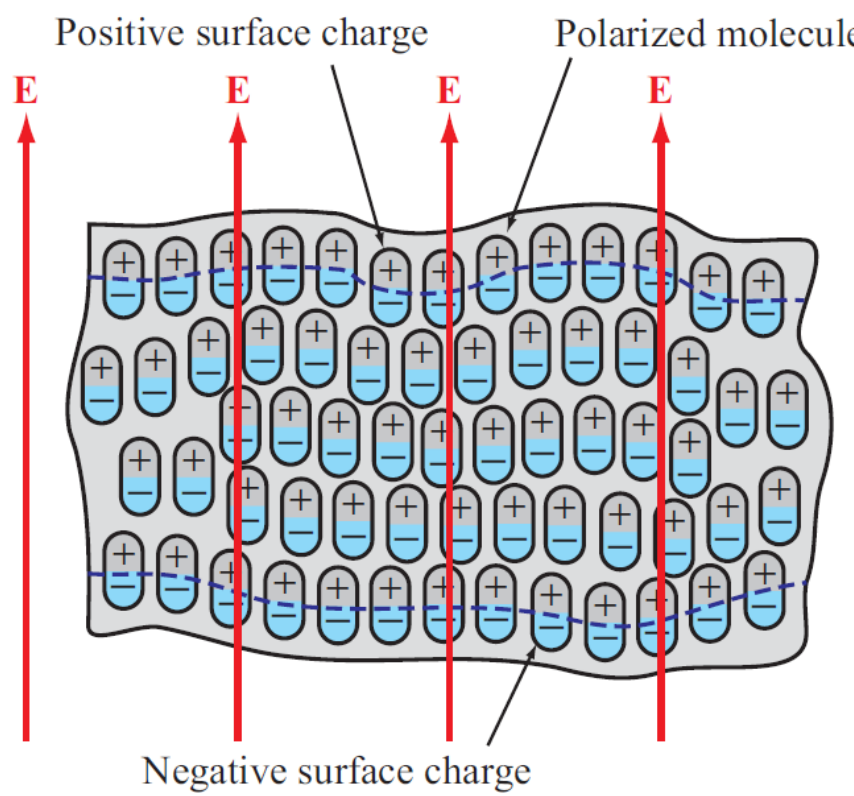


Figure 4-17: A dielectric medium polarized by an external electric field **E**.

Polarization Field

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$$

P = electric flux density induced by **E**

$$\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E}, \tag{4.84}$$

where χ_e is called the ***electric susceptibility*** of the material. Inserting Eq. (4.84) into Eq. (4.83), we have

$$\begin{aligned} \mathbf{D} &= \varepsilon_0 \mathbf{E} + \varepsilon_0 \chi_e \mathbf{E} \\ &= \varepsilon_0 (1 + \chi_e) \mathbf{E} = \varepsilon \mathbf{E}, \end{aligned} \tag{4.85}$$

Electric Breakdown

The dielectric strength E_{ds} is the largest magnitude of **E** *that the material can sustain without breakdown.*

Table 4-2: Relative permittivity (dielectric constant) and dielectric strength of common materials.

Material	Relative Permittivity, ε_r	Dielectric Strength, E_{ds} (MV/m)
Air (at sea level)	1.0006	3
Petroleum oil	2.1	12
Polystyrene	2.6	20
Glass	4.5–10	25–40
Quartz	3.8–5	30
Bakelite	5	20
Mica	5.4–6	200

$\varepsilon = \varepsilon_r \varepsilon_0$ and $\varepsilon_0 = 8.854 \times 10^{-12}$ F/m.

Boundary Conditions

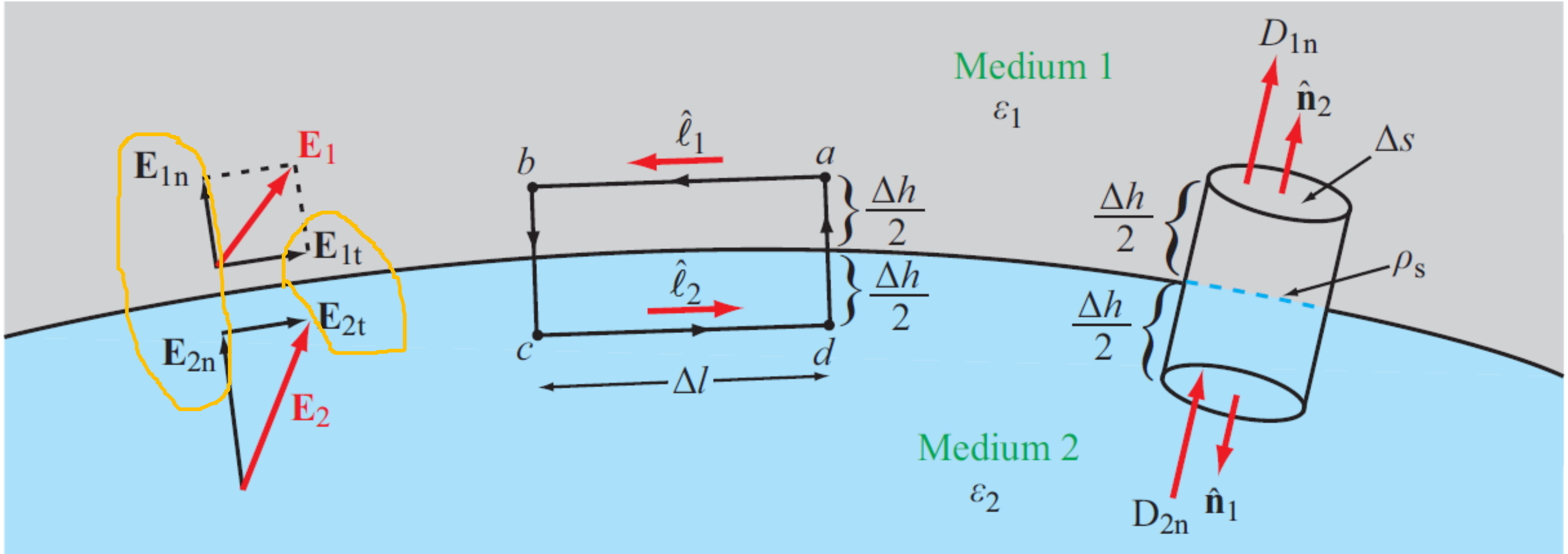


Figure 4-18: Interface between two dielectric media.

$$\mathbf{E}_{1t} = \mathbf{E}_{2t} \qquad \text{(V/m)}. \qquad (4.90)$$

$$\frac{\mathbf{D}_{1t}}{\varepsilon_1} = \frac{\mathbf{D}_{2t}}{\varepsilon_2}. \qquad (4.91)$$

$$\hat{\mathbf{n}}_2 \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s \qquad (\text{C/m}^2).$$

$$D_{1n} - D_{2n} = \rho_s \qquad (\text{C/m}^2). \qquad (4.94)$$

*The normal component of **D** changes abruptly at a charged boundary between two different media in an amount equal to the surface charge density.*

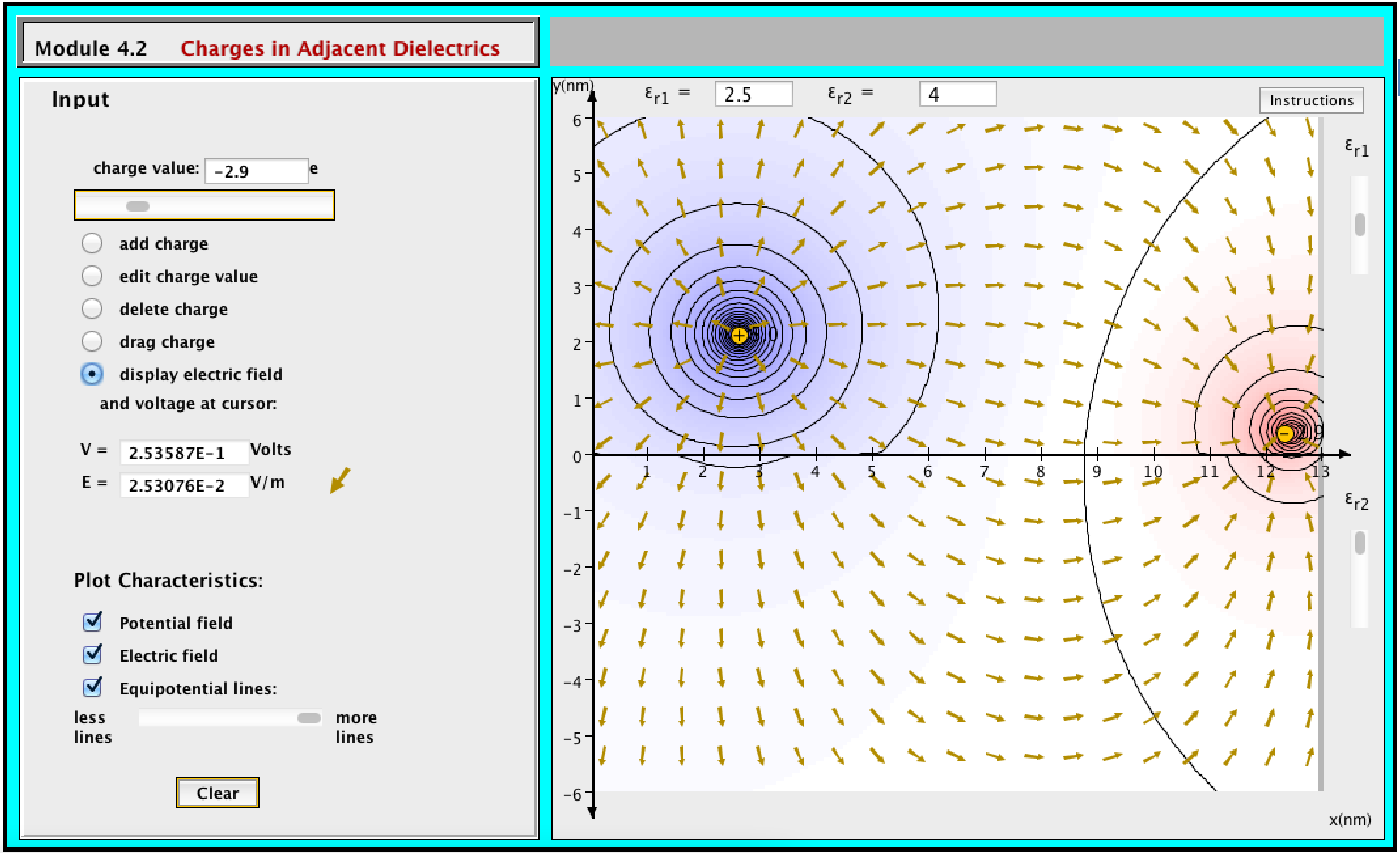
Summary of Boundary Conditions

Table 4-3: Boundary conditions for the electric fields.

Field Component	Any Two Media	Medium 1 Dielectric ε_1	Medium 2 Conductor
Tangential E	$\mathbf{E}_{1t} = \mathbf{E}_{2t}$	$\mathbf{E}_{1t} = \mathbf{E}_{2t} = 0$	
Tangential D	$\mathbf{D}_{1t}/\varepsilon_1 = \mathbf{D}_{2t}/\varepsilon_2$	$\mathbf{D}_{1t} = \mathbf{D}_{2t} = 0$	
Normal E	$\varepsilon_1 E_{1n} - \varepsilon_2 E_{2n} = \rho_s$	$E_{1n} = \rho_s/\varepsilon_1$	$E_{2n} = 0$
Normal D	$D_{1n} - D_{2n} = \rho_s$	$D_{1n} = \rho_s$	$D_{2n} = 0$

Notes: (1) ρ_s is the surface charge density at the boundary; (2) normal components of $\mathbf{E}_1$, $\mathbf{D}_1$, $\mathbf{E}_2$, and $\mathbf{D}_2$ are along $\hat{\mathbf{n}}_2$, the outward normal unit vector of medium 2.

Remember **E** = 0 in a good conductor


Module 4.2 Charges in Adjacent Dielectrics
Input
charge value: -2.9 e
add charge
edit charge value
delete charge
drag charge
display electric field
and voltage at cursor:
V = 2.53587E-1 Volts
E = 2.53076E-2 V/m
Plot Characteristics:
Potential field
Electric field
Equipotential lines:
less lines
more lines
Clear
y(nm)
εr1 = 2.5
εr2 = 4
Instructions
εr1
εr2
x(nm)

Conductors

Net electric field inside a conductor is zero

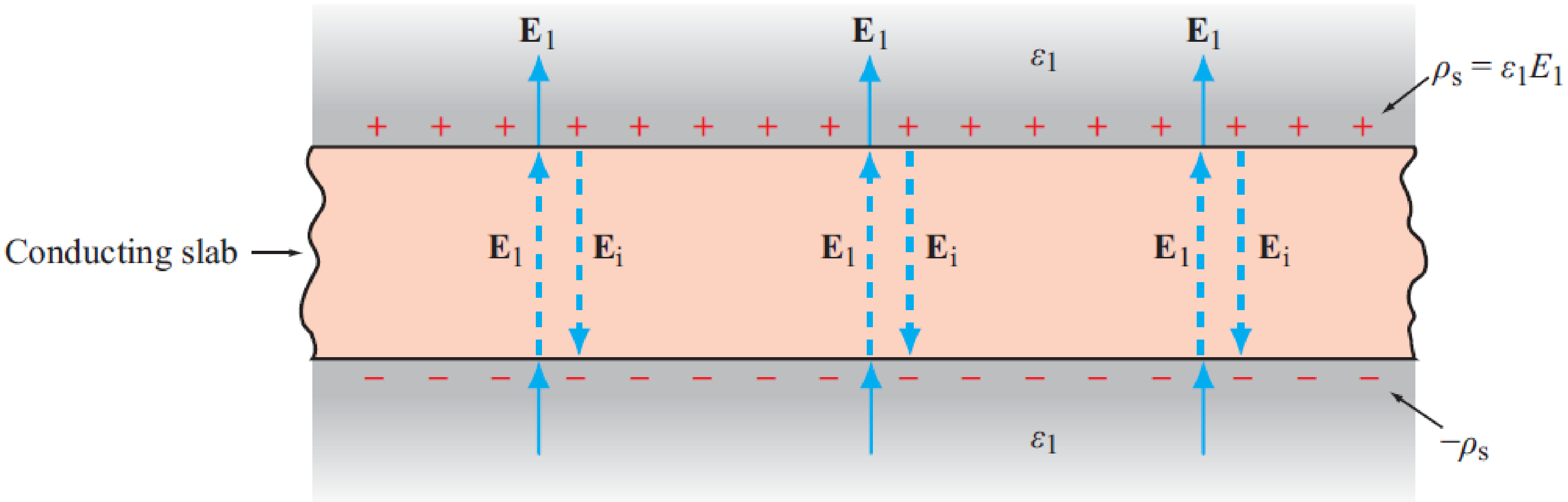


Figure 4-20: When a conducting slab is placed in an external electric field $\mathbf{E}_1$, charges that accumulate on the conductor surfaces induce an internal electric field $\mathbf{E}_i = -\mathbf{E}_1$. Consequently, the total field inside the conductor is zero.

Field Lines at Conductor Boundary

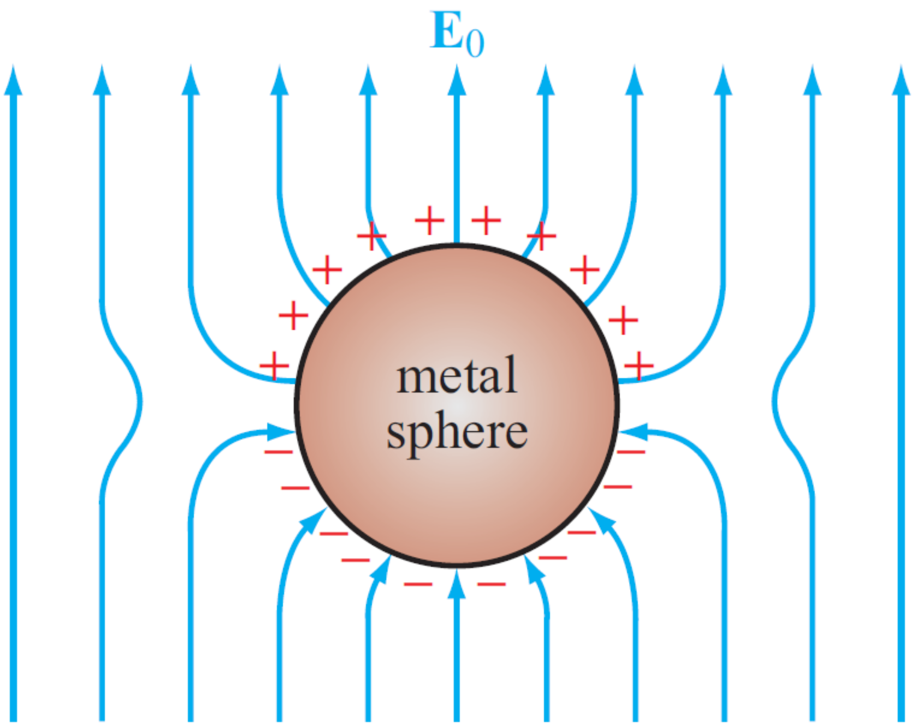
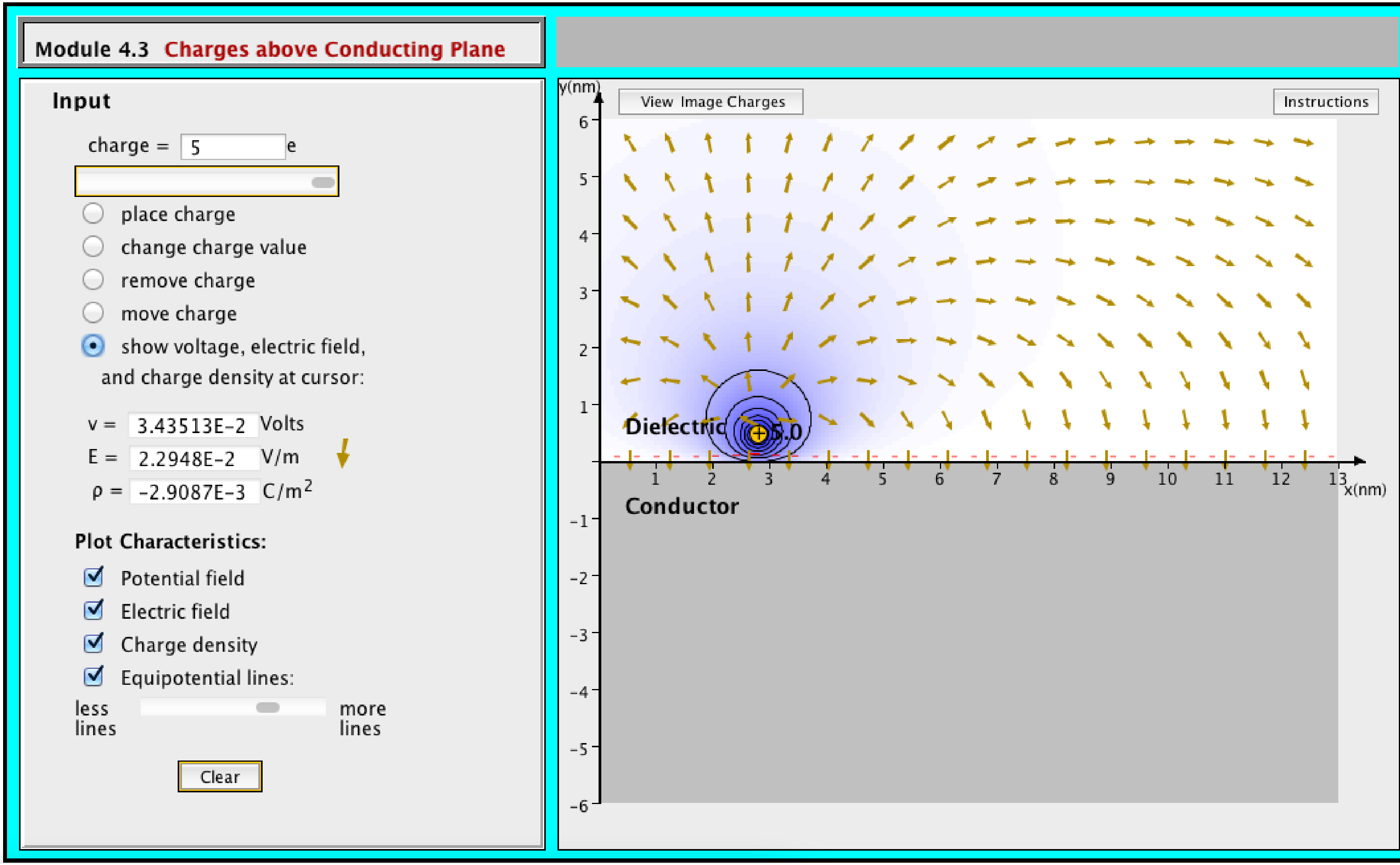
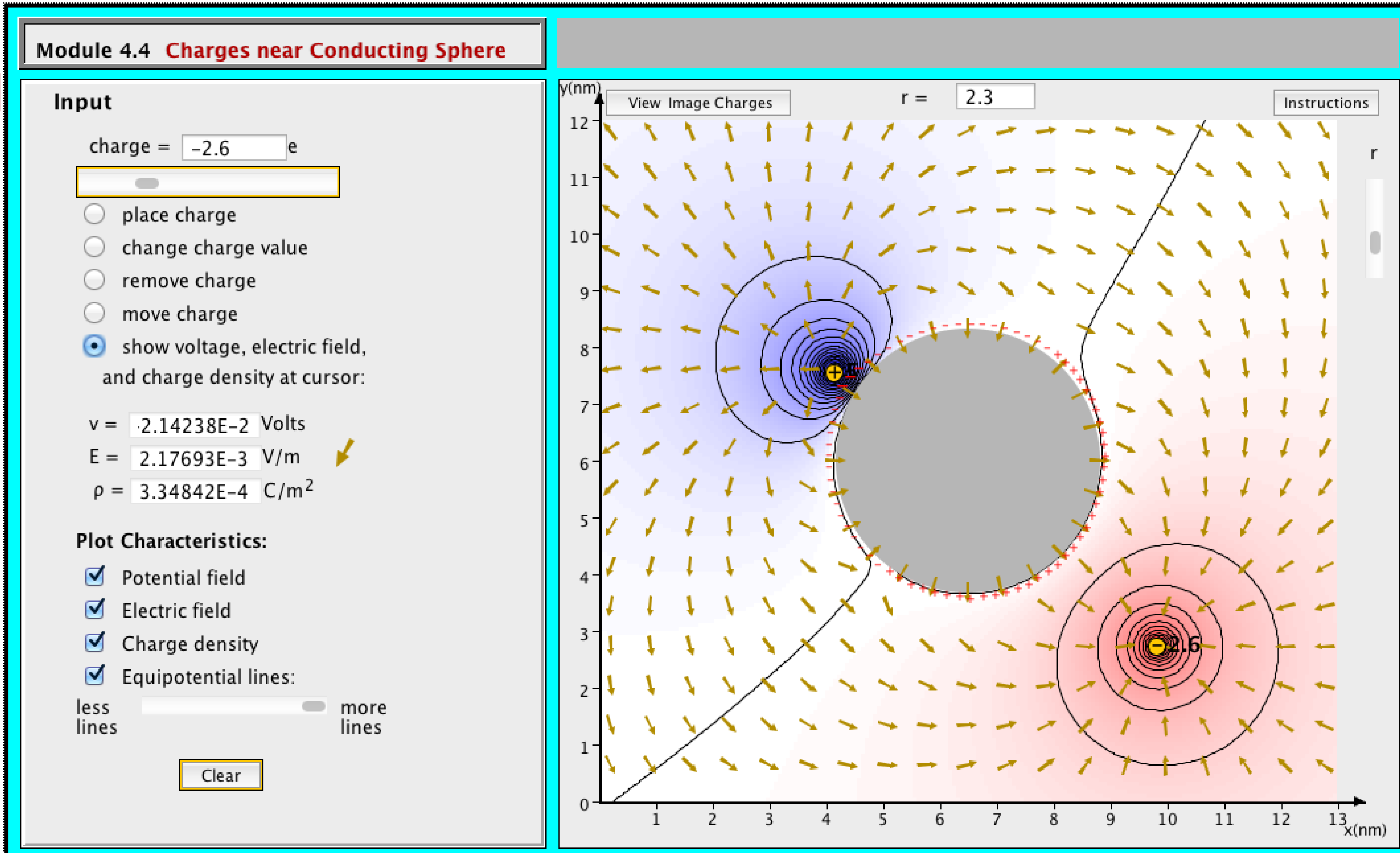


Figure 4-21: Metal sphere placed in an external electric field $\mathbf{E}_0$.

At conductor boundary, **E** field direction is always perpendicular to conductor surface


Module 4.3 Charges above Conducting Plane
Input
charge = 5 e
place charge
change charge value
remove charge
move charge
show voltage, electric field, and charge density at cursor:
v = 3.43513E-2 Volts
E = 2.2948E-2 V/m
ρ = -2.9087E-3 C/m2
Plot Characteristics:
Potential field
Electric field
Charge density
Equipotential lines:
less lines
more lines
Clear
View Image Charges
Instructions
y(nm)
x(nm)
Dielectric
5.0
Conductor



Module 4.4 Charges near Conducting Sphere
Input
charge = -2.6 e
place charge
change charge value
remove charge
move charge
show voltage, electric field, and charge density at cursor:
v = 2.14238E-2 Volts
E = 2.17693E-3 V/m
ρ = 3.34842E-4 C/m2
Plot Characteristics:
Potential field
Electric field
Charge density
Equipotential lines:
less lines
more lines
Clear
View Image Charges
r = 2.3
Instructions
y(nm)
x(nm)
-2.6
r

Capacitance

When a conductor has excess charge, it distributes the charge on its surface in such a manner as to maintain a zero electric field everywhere within the conductor, thereby ensuring that the electric potential is the same at every point in the conductor.

The ***capacitance*** of a two-conductor configuration is defined as

$$C = \frac{Q}{V} \qquad \text{(C/V or F)}, \qquad (4.105)$$

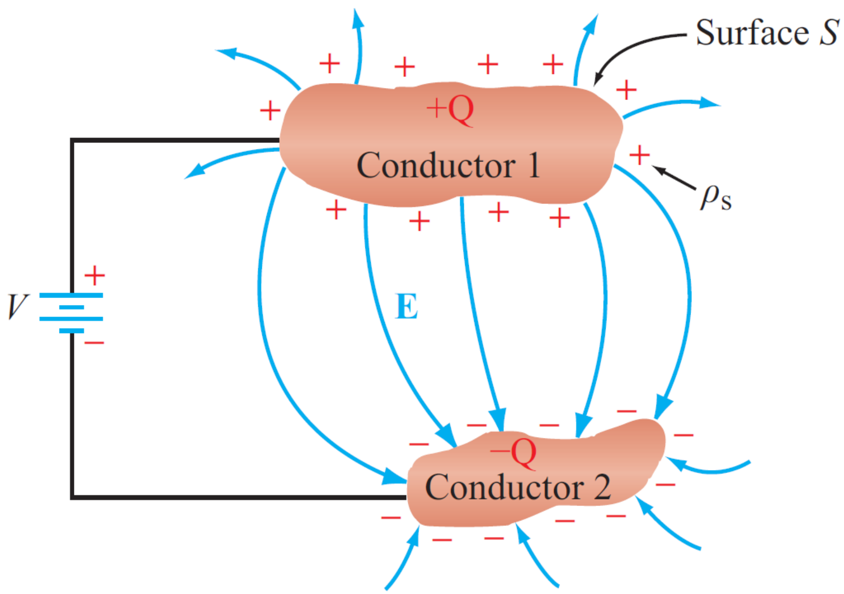


Figure 4-23: A dc voltage source connected to a capacitor composed of two conducting bodies.

Capacitance

For any two-conductor configuration:

$$C = \frac{\int_S \varepsilon \mathbf{E} \cdot d\mathbf{s}}{-\int_l \mathbf{E} \cdot d\mathbf{l}} \qquad \text{(F)},$$

For any resistor:

$$R = \frac{-\int_l \mathbf{E} \cdot d\mathbf{l}}{\int_S \sigma \mathbf{E} \cdot d\mathbf{s}} \qquad (\Omega). \qquad (4.110)$$

For a medium with uniform σ and ε, the product of Eqs. (4.109) and (4.110) gives

$$RC = \frac{\varepsilon}{\sigma}. \qquad (4.111)$$

This simple relation allows us to find R if C is known, or vice versa.

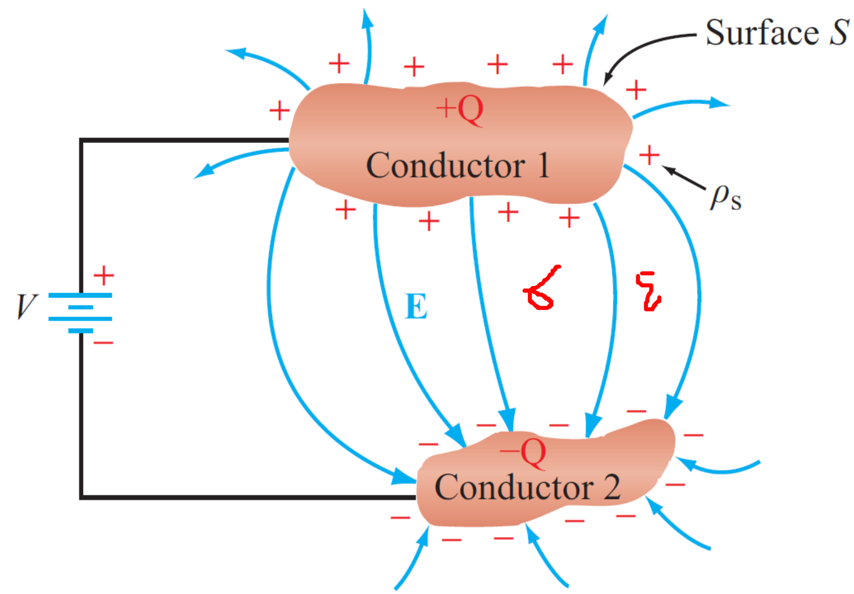


Figure 4-23: A dc voltage source connected to a capacitor composed of two conducting bodies.

Example 4-11: Capacitance and Breakdown Voltage of Parallel-Plate Capacitor

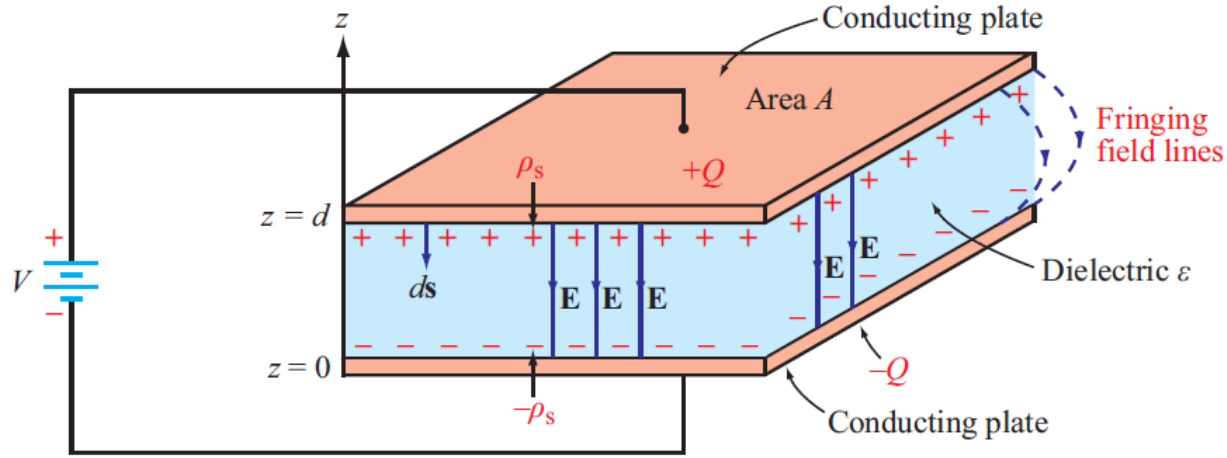


Figure 4-24: A dc voltage source connected to a parallel-plate capacitor (Example 4-11).

$$V = -\int_0^d \mathbf{E} \cdot d\mathbf{l} = -\int_0^d (-\hat{\mathbf{z}}E) \cdot \hat{\mathbf{z}}\, dz = Ed, \qquad (4.112)$$

and the capacitance is

$$C = \frac{Q}{V} = \frac{Q}{Ed} = \frac{\varepsilon A}{d}, \qquad (4.113)$$

where use was made of the relation $E = Q/\varepsilon A$.

From $V = Ed$, as given by Eq. (4.112), $V = V_{\text{br}}$ when $E = E_{\text{ds}}$, the dielectric strength of the material. According to Table 4-2, $E_{\text{ds}} = 30$ (MV/m) for quartz. Hence, the breakdown voltage is

$$V_{\text{br}} = E_{\text{ds}} d = 30 \times 10^6 \times 10^{-2} = 3 \times 10^5 \text{ V}.$$

Example 4-12: Capacitance Per Unit Length of Coaxial Line

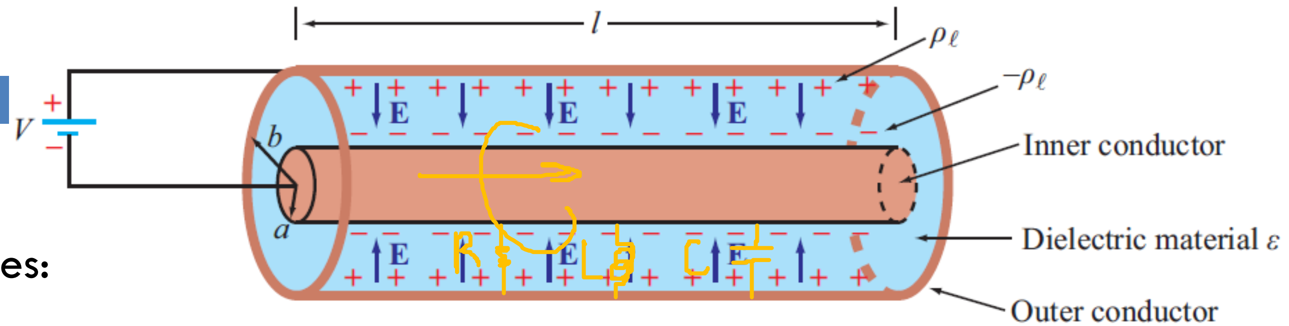


Figure 4-25: Coaxial capacitor filled with insulating material of permittivity ε (Example 4-12).

Application of Gauss's law gives:

$$\mathbf{E} = -\hat{\mathbf{r}}\,\frac{Q}{2\pi\varepsilon r l}\,.$$

The potential difference V between the outer and inner conductors is

$$V = -\int_a^b \mathbf{E}\cdot d\mathbf{l} = -\int_a^b \left(-\hat{\mathbf{r}}\,\frac{Q}{2\pi\varepsilon r l}\right)\cdot(\hat{\mathbf{r}}\,dr)$$

$$= \frac{Q}{2\pi\varepsilon l}\ln\left(\frac{b}{a}\right). \qquad (4.115)$$

Q is total charge on inside of outer cylinder, and –Q is on outside surface of inner cylinder

The capacitance C is then given by

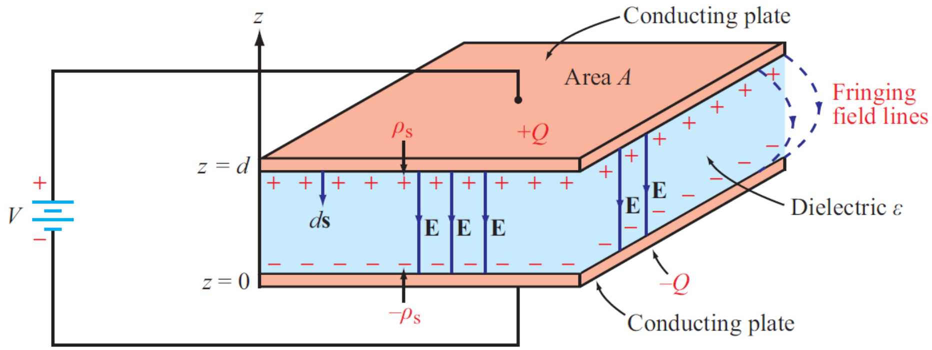
$$C = \frac{Q}{V} = \frac{2\pi\varepsilon l}{\ln(b/a)}\,, \qquad (4.116)$$

and the capacitance per unit length of the coaxial line is

$$C' = \frac{C}{l} = \frac{2\pi\varepsilon}{\ln(b/a)} \qquad \text{(F/m)}. \qquad (4.117)$$

Tech Brief 8: Supercapacitors

For a traditional parallel-plate capacitor, what is the maximum attainable energy density?



Energy density is given by:

$$W' = \frac{\varepsilon V^2}{2\rho d^2} \qquad \text{(J/kg)}$$

ε = permittivity of insulation material
V = applied voltage
ρ = density of insulation material
d = separation between plates

Mica has one of the highest dielectric strengths ~2 x 10**8 V/m.
If we select a voltage rating of 1 V and a breakdown voltage of 2 V (50% safety), this will require that d be no smaller than 10 nm.
For mica, $\varepsilon = 6\varepsilon_0$ and ρ = 3 x 10**3 kg/m^3 .

Hence:

W′ = 90 J/kg = 2.5 x10**–2 Wh/kg.

By comparison, a lithium-ion battery has W′ = 1.5 x 10**2 Wh/kg, almost 4 orders of magnitude greater

A supercapacitor is a “hybrid” battery/capacitor

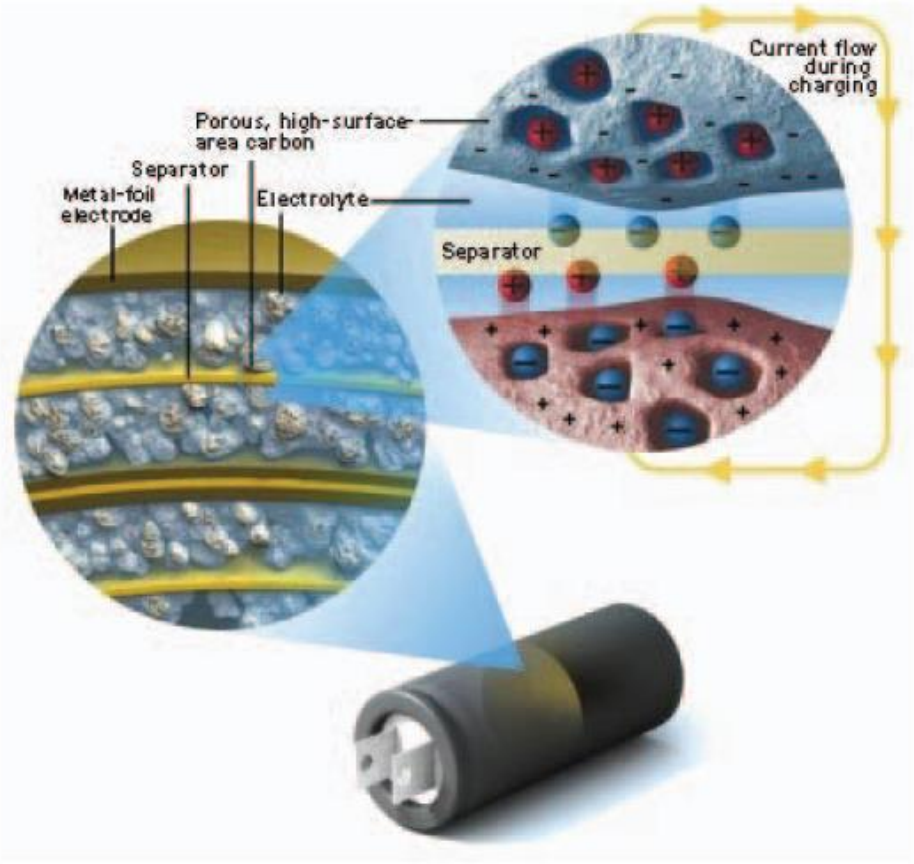


Figure TF8-1: Cross-sectional view of an electrochemical double-layer capacitor (EDLC), otherwise known as a supercapacitor. (Courtesy of Ultracapacitor.org.)

Users of Supercapacitors

Figure TF8-2: Examples of systems that use supercapacitors. (Courtesy of Railway Gazette International; BMW; NASA; Applied Innovative Technologies.)

Energy Comparison

Energy Storage Devices

Feature	Traditional Capacitor	Supercapacitor	Battery
Energy density W' (Wh/kg)	$\sim 10^{-2}$	1 to 10	5 to 150
Power density P' (W/kg)	1,000 to 10,000	1,000 to 5,000	10 to 500
Charge and discharge rate T	10^{-3} sec	$\sim$ 1 sec to 1 min	$\sim$ 1 to 5 hrs
Cycle life N_c	∞	$\sim 10^6$	$\sim 10^3$

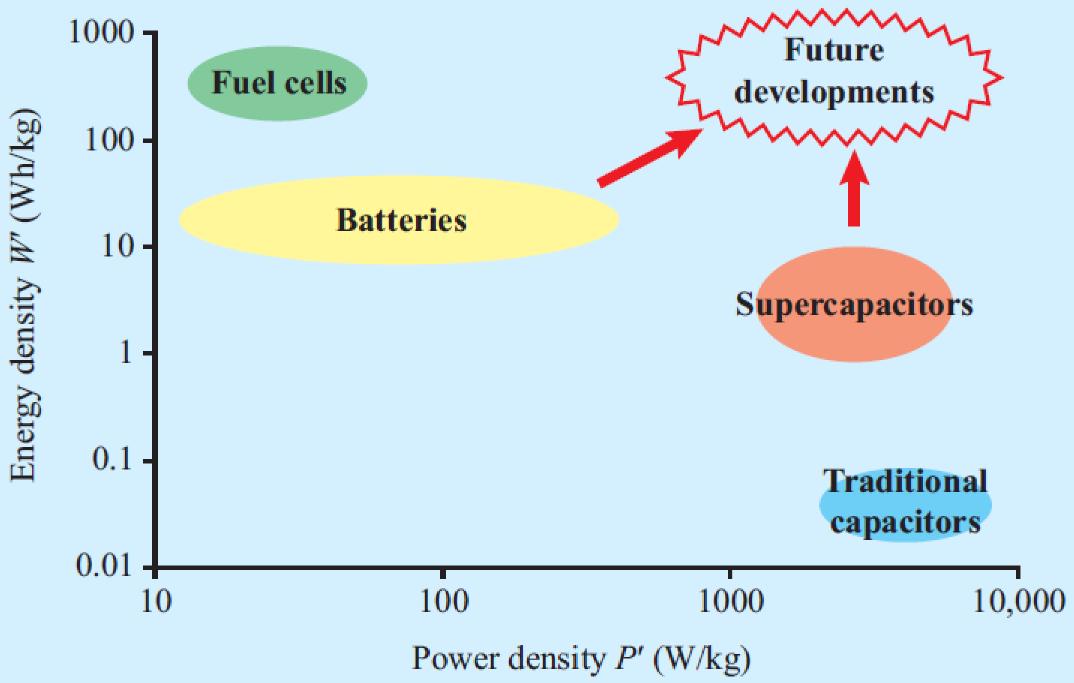


Figure TF8-3: Comparison of energy storage devices.

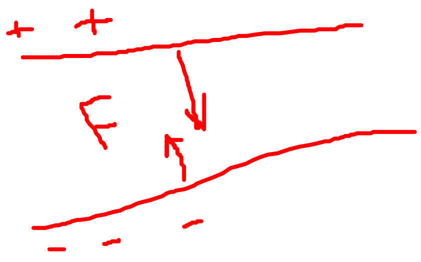
Electrostatic Potential Energy

Electrostatic potential energy density (Joules/volume)

$$w_e = \frac{W_e}{\mathcal{V}} = \frac{1}{2}\,\varepsilon E^2 \qquad (\text{J/m}^3).$$

Energy stored in a capacitor

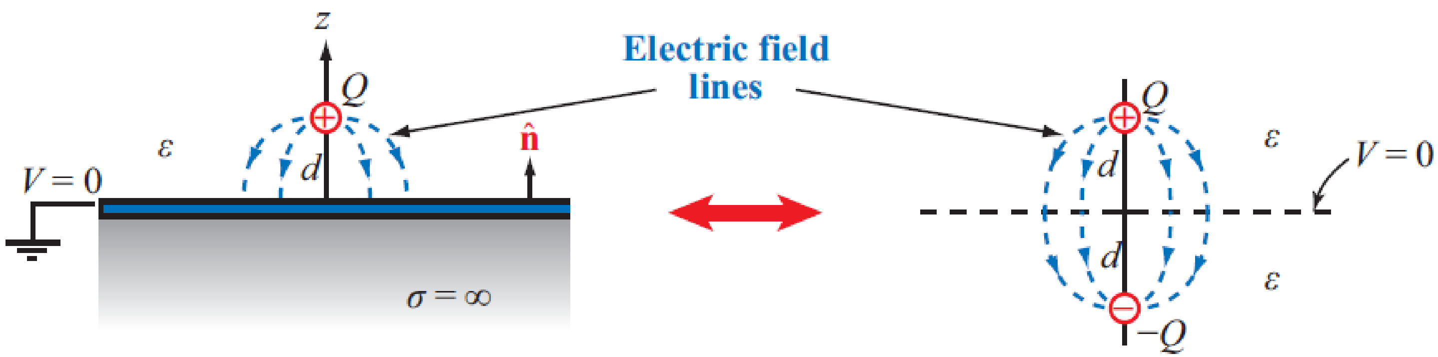
$$W_e = \tfrac{1}{2} C V^2 \qquad (\text{J}).$$



Total electrostatic energy stored in a volume

$$W_e = \frac{1}{2} \int_{\mathcal{V}} \varepsilon E^2 \, d\mathcal{V} \qquad (\text{J})$$

Image Method



(a) Charge Q above grounded plane (b) Equivalent configuration

Figure 4-26: By image theory, a charge Q above a grounded perfectly conducting plane is equivalent to Q and its image $-Q$ with the ground plane removed.

Image method simplifies calculation for **E** and *V* due to charges near conducting planes.

1. For each charge Q, add an image charge –Q
2. Remove conducting plane
3. Calculate field due to all charges

Example 4-13: Image Method for Charge Above Conducting Plane

Use image theory to determine **E** at an arbitrary point $P = (x, y, z)$ in the region $z > 0$ due to a charge Q in free space at a distance d above a grounded conducting plate residing in the $z = 0$ plane.

Solution: In Fig. 4-28, charge Q is at $(0, 0, d)$ and its image $-Q$ is at $(0, 0, -d)$. From Eq. (4.19), the electric field at point $P = (x, y, z)$ due to the two charges is given by

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q\mathbf{R}_1}{R_1^3} + \frac{-Q\mathbf{R}_2}{R_2^3}\right)$$

$$= \frac{Q}{4\pi\varepsilon_0}\left[\frac{\hat{\mathbf{x}}x + \hat{\mathbf{y}}y + \hat{\mathbf{z}}(z-d)}{[x^2+y^2+(z-d)^2]^{3/2}} - \frac{\hat{\mathbf{x}}x + \hat{\mathbf{y}}y + \hat{\mathbf{z}}(z+d)}{[x^2+y^2+(z+d)^2]^{3/2}}\right]$$

for $z \geq 0$.

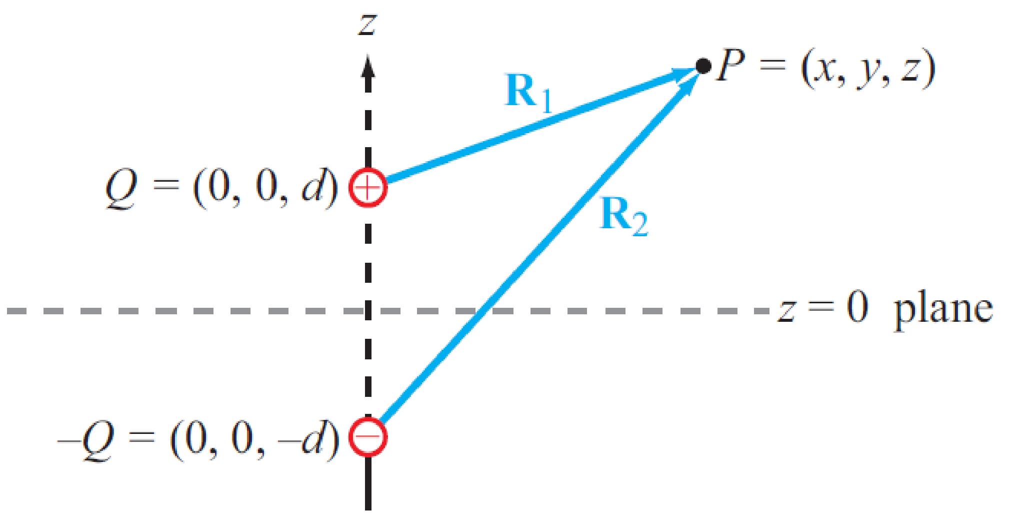


Figure 4-28: Application of the image method for finding **E** a point P (Example 4-13).

Tech Brief 9:
Capacitive Sensors

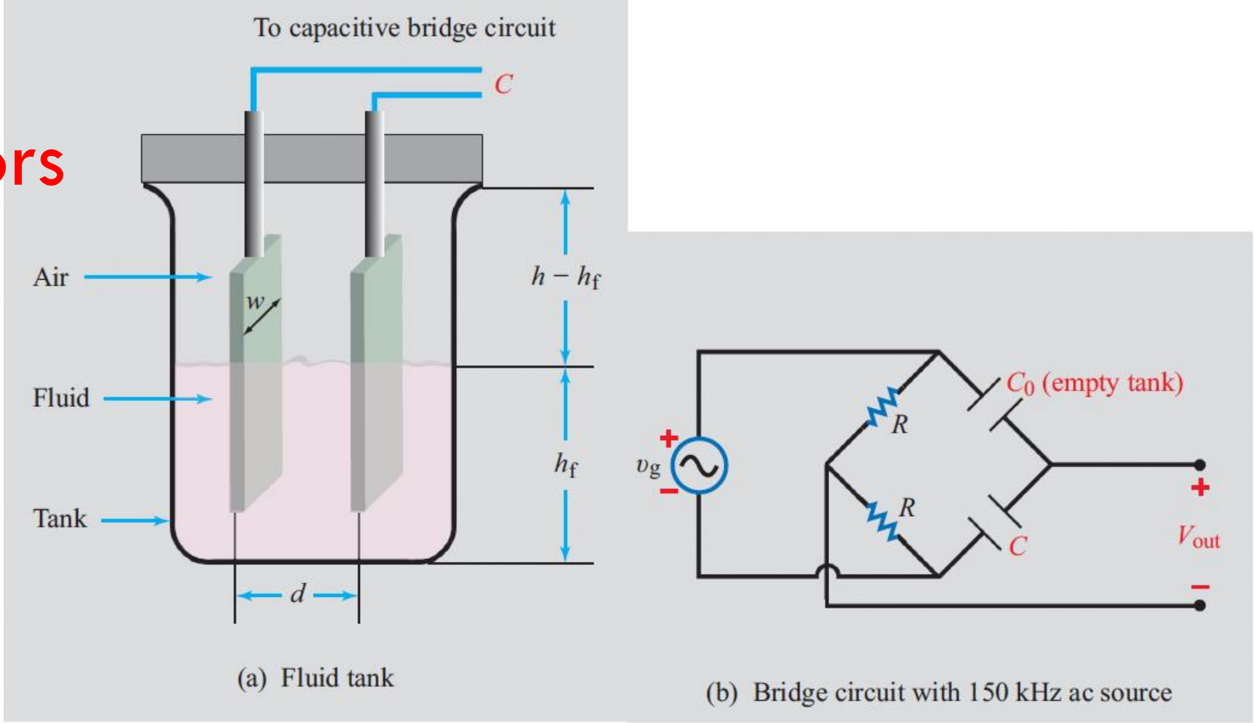


Figure TF9-1: Fluid gauge and associated bridge circuit, with C_0 being the capacitance that an empty tank would have and C the capacitance of the tank under test.

Fluid Gauge

The two metal electrodes in Fig. TF9-1(a), usually rods or plates, form a capacitor whose capacitance is directly proportional to the ***permittivity*** of the material between them. If the fluid section is of height h_f and the height of the empty space above it is $(h - h_f)$, then the overall capacitance is equivalent to two capacitors in parallel, or

$$C = C_f + C_a = \varepsilon_f w \frac{h_f}{d} + \varepsilon_a w \frac{(h - h_f)}{d},$$

where w is the electrode plate width, d is the spacing between electrodes, and ε_f and ε_a are the permittivities of the fluid and air, respectively. Rearranging the expression as a linear equation yields

$$C = kh_f + C_0,$$

Humidity Sensor

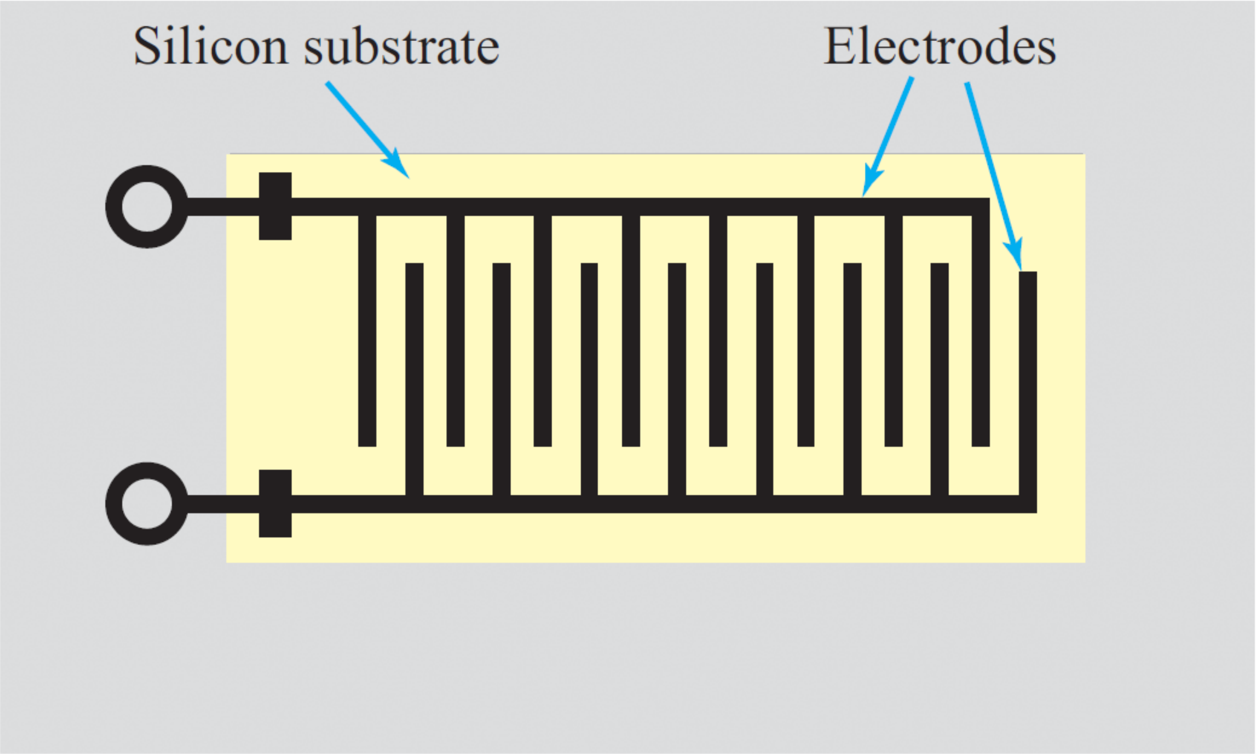


Figure TF9-2: Interdigital capacitor used as a humidity sensor.

Pressure Sensor

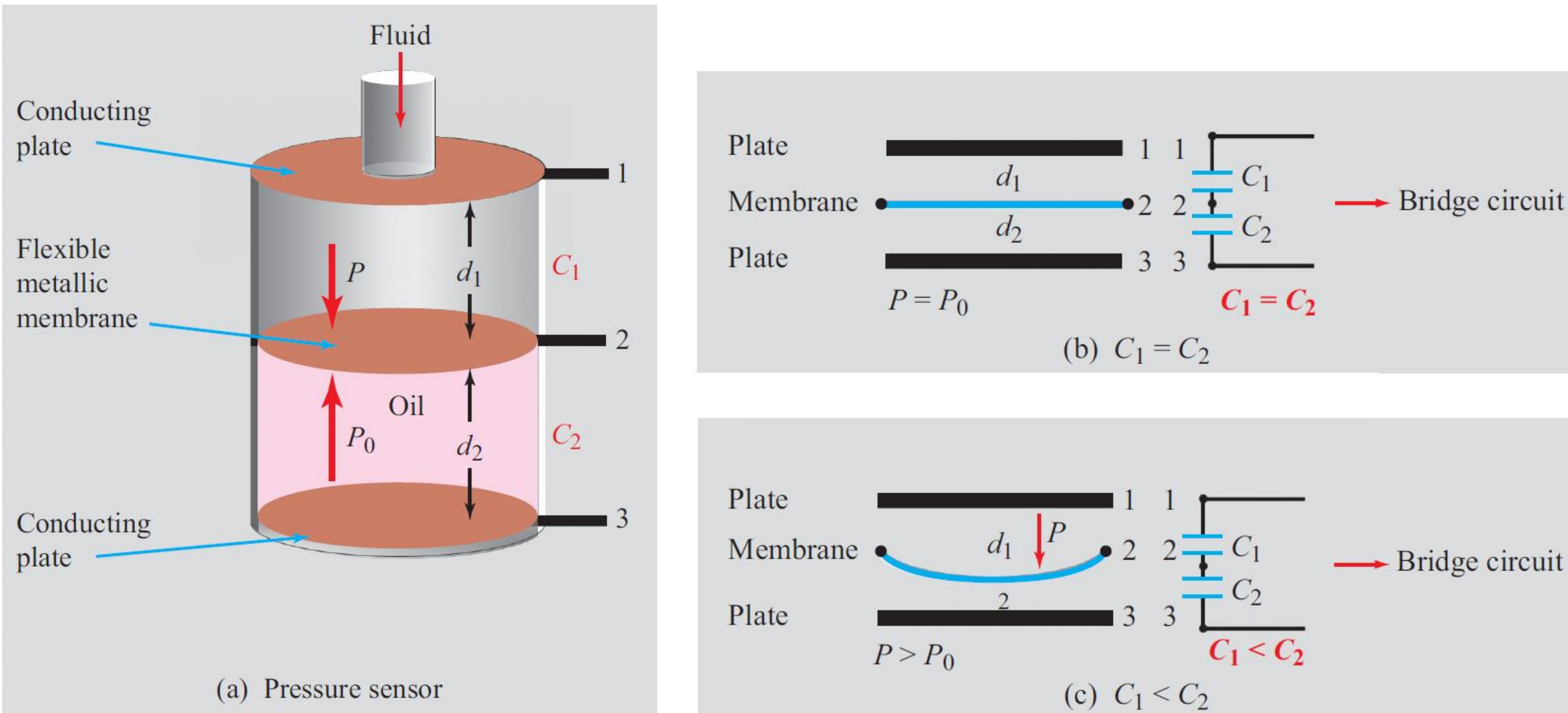


Figure TF9-3: Pressure sensor responds to deflection of metallic membrane.

Planar capacitors

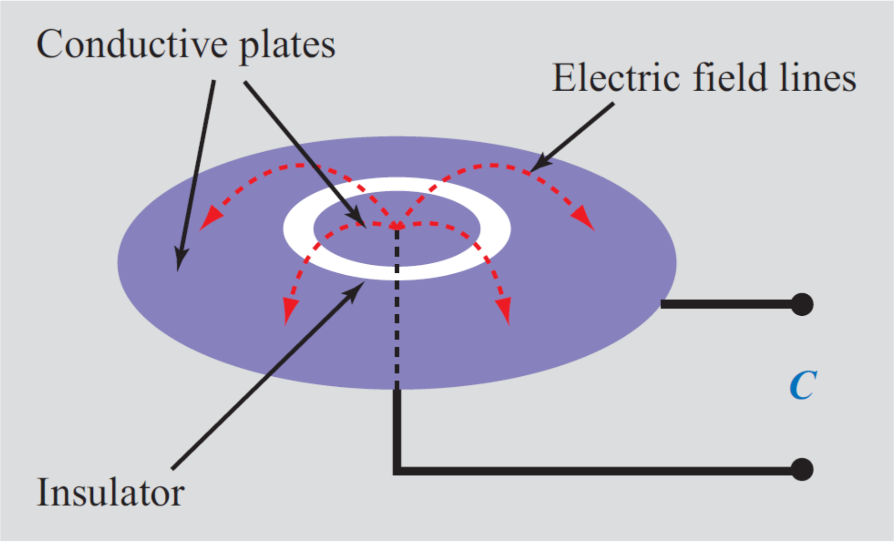


Figure TF9-4: Concentric-plate capacitor.

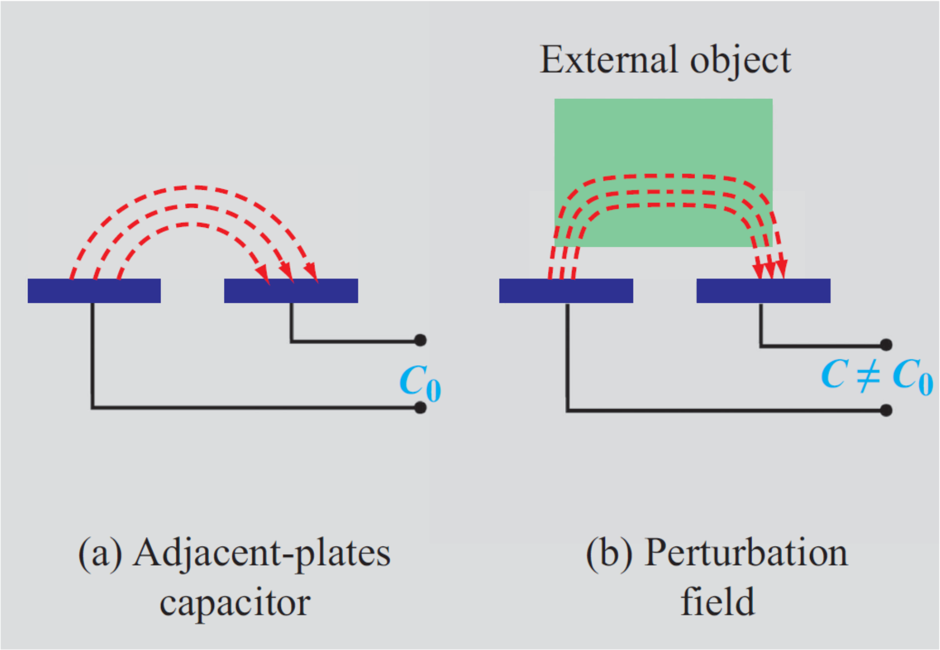


Figure TF9-5: (a) Adjacent-plates capacitor; (b) perturbation field.

Fingerprint Imager

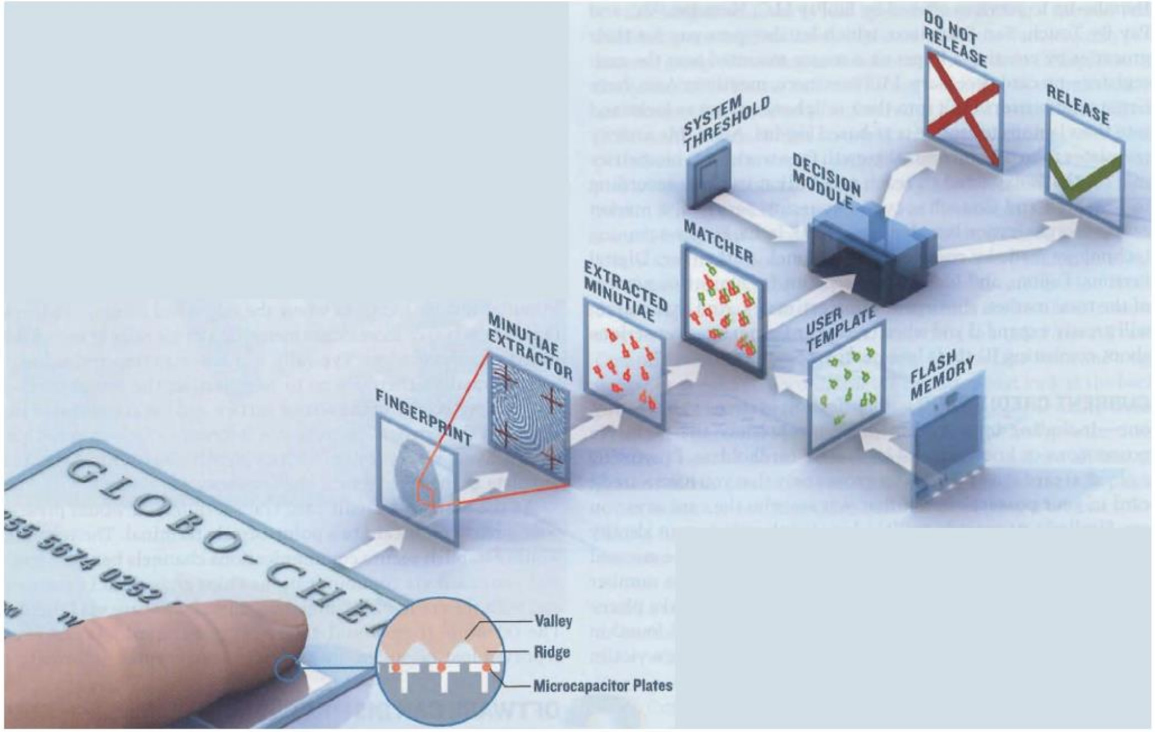


Figure TF9-6: Elements of a fingerprint matching system. (Courtesy of IEEE Spectrum.)

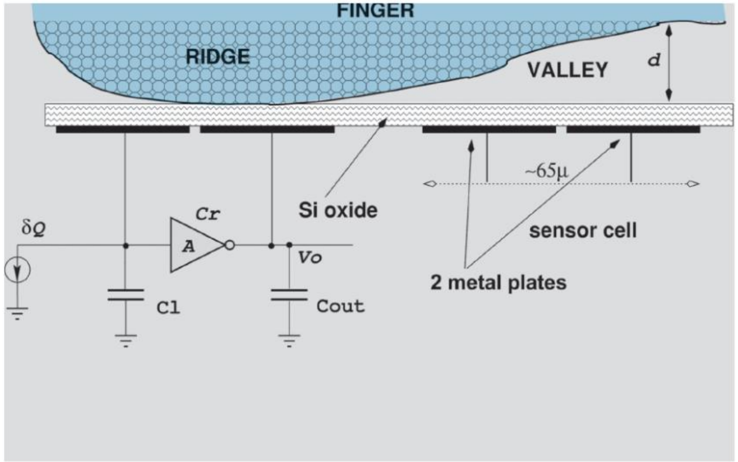


Figure TF9-7: Fingerprint representation. (Courtesy of Dr. M. Tartagni, University of Bologna, Italy.)

Chapter 4 Relationships

Maxwell's Equations for Electrostatics

Name	Differential Form	Integral Form
Gauss's law	$\nabla \cdot \mathbf{D} = \rho_v$	$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q$
Kirchhoff's law	$\nabla \times \mathbf{E} = 0$	$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$

Current density	$\mathbf{J} = \rho_v \mathbf{u}$
Poisson's equation	$\nabla^2 V = -\dfrac{\rho_v}{\varepsilon}$
Laplace's equation	$\nabla^2 V = 0$
Resistance	$R = \dfrac{-\int_l \mathbf{E} \cdot d\mathbf{l}}{\int_S \sigma \mathbf{E} \cdot d\mathbf{s}}$
Boundary conditions	Table 4-3
Capacitance	$C = \dfrac{\int_S \varepsilon \mathbf{E} \cdot d\mathbf{s}}{-\int_l \mathbf{E} \cdot d\mathbf{l}}$
RC relation	$RC = \dfrac{\varepsilon}{\sigma}$
Energy density	$w_e = \frac{1}{2} \varepsilon E^2$

Electric Field

Point charge	$\mathbf{E} = \hat{\mathbf{R}} \dfrac{q}{4\pi\varepsilon R^2}$
Many point charges	$\mathbf{E} = \dfrac{1}{4\pi\varepsilon} \sum_{i=1}^{N} \dfrac{q_i (\mathbf{R} - \mathbf{R}_i)}{\lvert \mathbf{R} - \mathbf{R}_i \rvert^3}$
Volume distribution	$\mathbf{E} = \dfrac{1}{4\pi\varepsilon} \int_{\mathcal{V}'} \hat{\mathbf{R}}' \dfrac{\rho_v \, d\mathcal{V}'}{R'^2}$
Surface distribution	$\mathbf{E} = \dfrac{1}{4\pi\varepsilon} \int_{S'} \hat{\mathbf{R}}' \dfrac{\rho_s \, ds'}{R'^2}$
Line distribution	$\mathbf{E} = \dfrac{1}{4\pi\varepsilon} \int_{l'} \hat{\mathbf{R}}' \dfrac{\rho_\ell \, dl'}{R'^2}$
Infinite sheet of charge	$\mathbf{E} = \hat{\mathbf{z}} \dfrac{\rho_s}{2\varepsilon_0}$
Infinite line of charge	$\mathbf{E} = \dfrac{\mathbf{D}}{\varepsilon_0} = \hat{\mathbf{r}} \dfrac{D_r}{\varepsilon_0} = \hat{\mathbf{r}} \dfrac{\rho_\ell}{2\pi\varepsilon_0 r}$
Dipole	$\mathbf{E} = \dfrac{qd}{4\pi\varepsilon_0 R^3} (\hat{\mathbf{R}}\, 2\cos\theta + \hat{\boldsymbol{\theta}} \sin\theta)$
Relation to *V*	$\mathbf{E} = -\nabla V$