ELECTROMAGNETICS II COURSE

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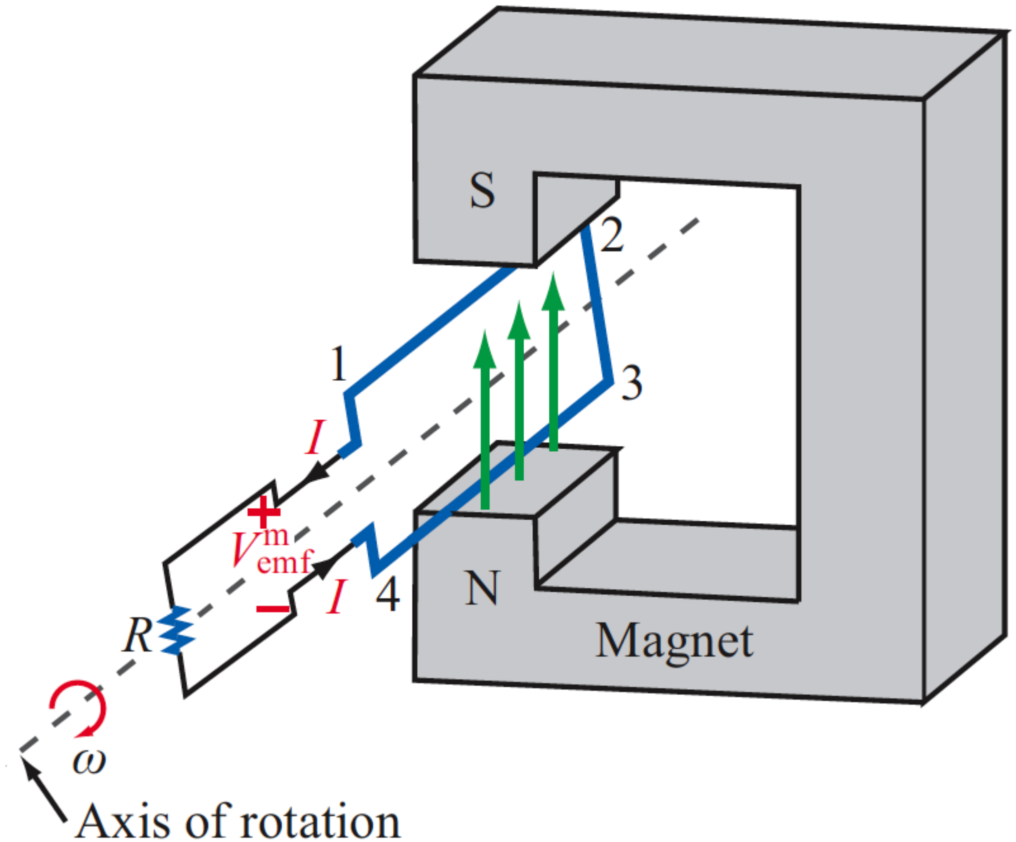


(b) ac generator

6. MAXWELL'S EQUATIONS IN TIME-VARYING FIELDS

7e Applied EM by Ulaby and Ravaioli

Chapter 6 Overview

Chapter Contents

Objectives

Upon learning the material presented in this chapter, you should be able to:

1. Apply Faraday's law to compute the voltage induced by a stationary coil placed in a time-varying magnetic field or moving in a medium containing a magnetic field.
2. Describe the operation of the electromagnetic generator.
3. Calculate the displacement current associated with a time-varying electric field.
4. Calculate the rate at which charge dissipates in a material with known ϵ and σ.

Maxwell's Equations

Table 6-1: Maxwell's equations.

Reference	Differential Form	Integral Form	
Gauss's law	$\nabla \cdot \mathbf{D} = \rho_v$	$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q$	(6.1)
Faraday's law	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}$	(6.2)*
Gauss's law for magnetism	$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{s} = 0$	(6.3)
Ampère's law	$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$	$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right) \cdot d\mathbf{s}$	(6.4)

*For a stationary surface S.

In this chapter, we will examine Faraday's and Ampère's laws

Faraday's Law

Electromotive force (voltage) induced by time-varying magnetic flux:

$$V_{\text{emf}} = -N \frac{d\Phi}{dt} = -N \frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{s} \qquad \text{(V)}$$

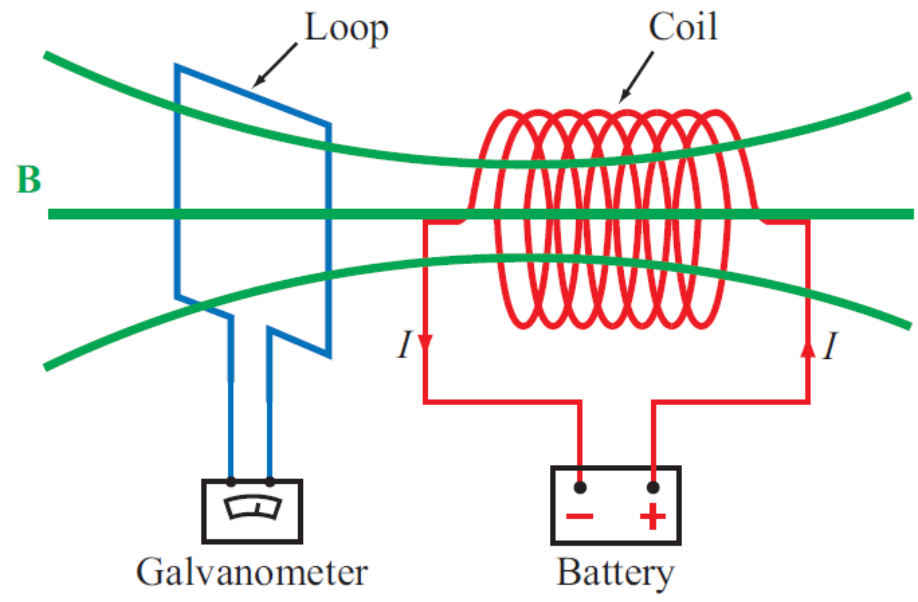


Figure 6-1: The galvanometer (predecessor of the ammeter) shows a deflection whenever the magnetic flux passing through the square loop changes with time.

Magnetic fields can produce an electric current in a closed loop, but only if the magnetic flux linking the surface area of the loop changes with time. The key to the induction process is change.

Three types of EMF

1. A time-varying magnetic field linking a stationary loop; the induced emf is then called the ***transformer emf***, $V_{\mathrm{emf}}^{\mathrm{tr}}$.

2. A moving loop with a time-varying surface area (relative to the normal component of **B**) in a static field **B**; the induced emf is then called the ***motional emf***, $V_{\mathrm{emf}}^{\mathrm{m}}$.

3. A moving loop in a time-varying field **B**.

The total emf is given by

$$V_{\mathrm{emf}} = V_{\mathrm{emf}}^{\mathrm{tr}} + V_{\mathrm{emf}}^{\mathrm{m}}, \tag{6.7}$$

Stationary Loop in Time-Varying B

It is important to remember that $\mathbf{B}_{\text{ind}}$ *serves to oppose the change in* $\mathbf{B}(t)$*, and not necessarily* $\mathbf{B}(t)$ *itself.*

$$V_{\text{emf}}^{\text{tr}} = -N \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s} \quad \text{(transformer emf)},$$

The connection between the direction of $d\mathbf{s}$ *and the polarity of* $V_{\text{emf}}^{\text{tr}}$ *is governed by the following right-hand rule: if* $d\mathbf{s}$ *points along the thumb of the right hand, then the direction of the contour C indicated by the four fingers is such that it always passes across the opening from the positive terminal of* $V_{\text{emf}}^{\text{tr}}$ *to the negative terminal.*

$$I = \frac{V_{\text{emf}}^{\text{tr}}}{R + R_{\text{i}}} . \qquad (6.9)$$

For good conductors, R_{i} usually is very small, and it may be ignored in comparison with practical values of R.

The polarity of $V_{\text{emf}}^{\text{tr}}$ *and hence the direction of I is governed by* ***Lenz's law****, which states that the current in the loop is always in a direction that opposes the change of magnetic flux* $\Phi(t)$ *that produced I.*

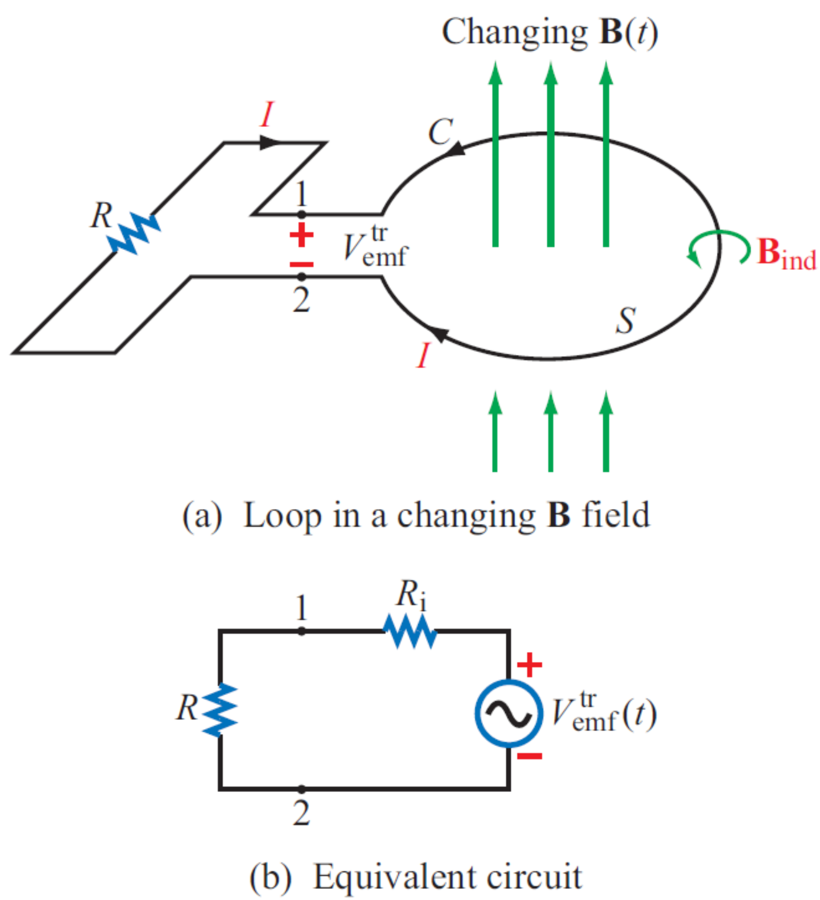


Figure 6-2: (a) Stationary circular loop in a changing magnetic field $\mathbf{B}(t)$, and (b) its equivalent circuit.

Example 6-1: Inductor in a Changing Magnetic Field

An inductor is formed by winding N turns of a thin conducting wire into a circular loop of radius a. The inductor loop is in the x–y plane with its center at the origin, and connected to a resistor R, as shown in Fig. 6-3. In the presence of a magnetic field $\mathbf{B} = B_0(\hat{\mathbf{y}}2 + \hat{\mathbf{z}}3)\sin\omega t$, where ω is the angular frequency, find

(a) the magnetic flux linking a single turn of the inductor,

(b) the transformer emf, given that $N = 10$, $B_0 = 0.2$ T, $a = 10$ cm, and $\omega = 10^3$ rad/s,

(c) the polarity of $V_{\text{emf}}^{\text{tr}}$ at $t = 0$, and

(d) the induced current in the circuit for $R = 1$ kΩ (assume the wire resistance to be much smaller than R).

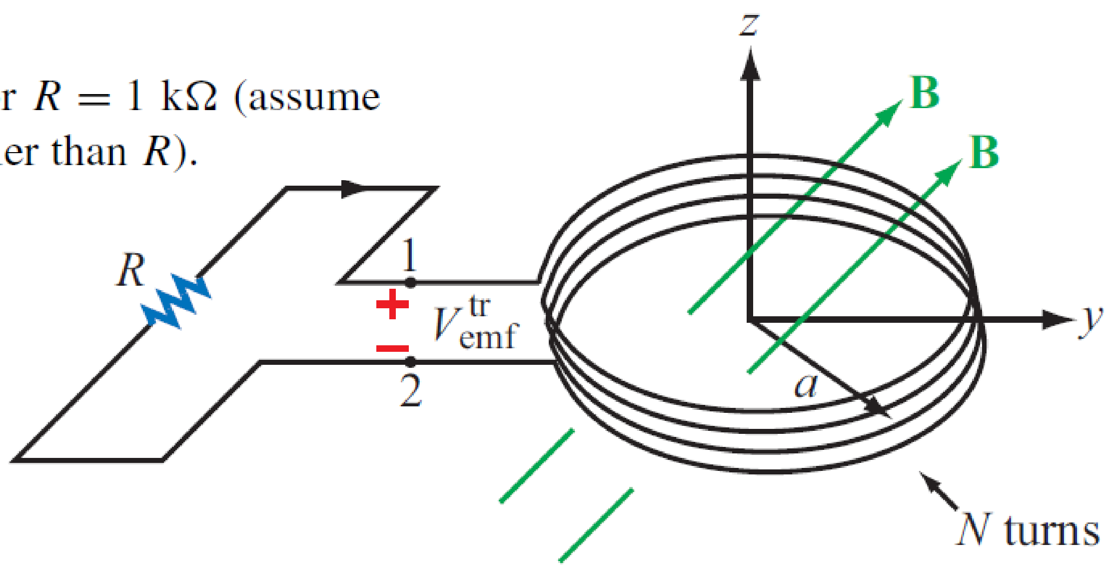


Figure 6-3: Circular loop with N turns in the x–y plane. The magnetic field is $\mathbf{B} = B_0(\hat{\mathbf{y}}2 + \hat{\mathbf{z}}3)\sin\omega t$ (Example 6-1).

cont.

Example 6-1 Solution

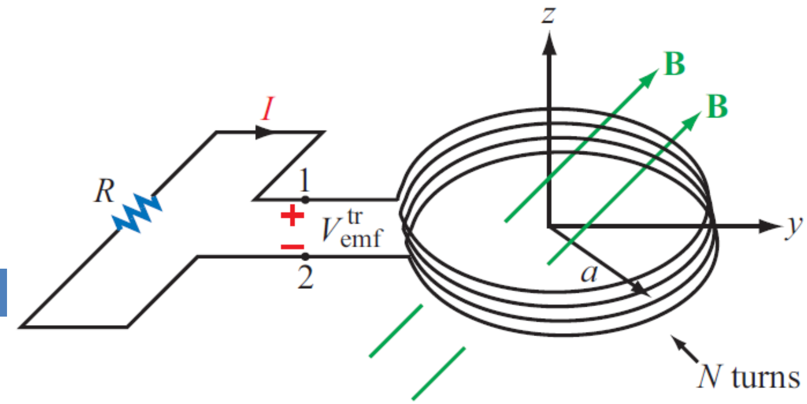


Figure 6-3: Circular loop with N turns in the x–y plane. The magnetic field is $\mathbf{B} = B_0(\hat{\mathbf{y}}2 + \hat{\mathbf{z}}3)\sin\omega t$ (Example 6-1).

Solution: **(a)** The magnetic flux linking each turn of the inductor is

$$\Phi = \int_S \mathbf{B}\cdot d\mathbf{s}$$
$$= \int_S [B_0(\hat{\mathbf{y}}\,2 + \hat{\mathbf{z}}\,3)\sin\omega t]\cdot\hat{\mathbf{z}}\,ds$$
$$= 3\pi a^2 B_0 \sin\omega t.$$

(b) To find V_{emf}^{tr}, we can apply Eq. (6.8) or we can apply the general expression given by Eq. (6.6) directly. The latter approach gives

$$V_{emf}^{tr} = -N\,\frac{d\Phi}{dt}$$
$$= -\frac{d}{dt}(3\pi N a^2 B_0 \sin\omega t)$$
$$= -3\pi N\omega a^2 B_0 \cos\omega t.$$

For $N = 10$, $a = 0.1$ m, $\omega = 10^3$ rad/s, and $B_0 = 0.2$ T,

$$V_{emf}^{tr} = -188.5\cos 10^3 t \qquad \text{(V)}.$$

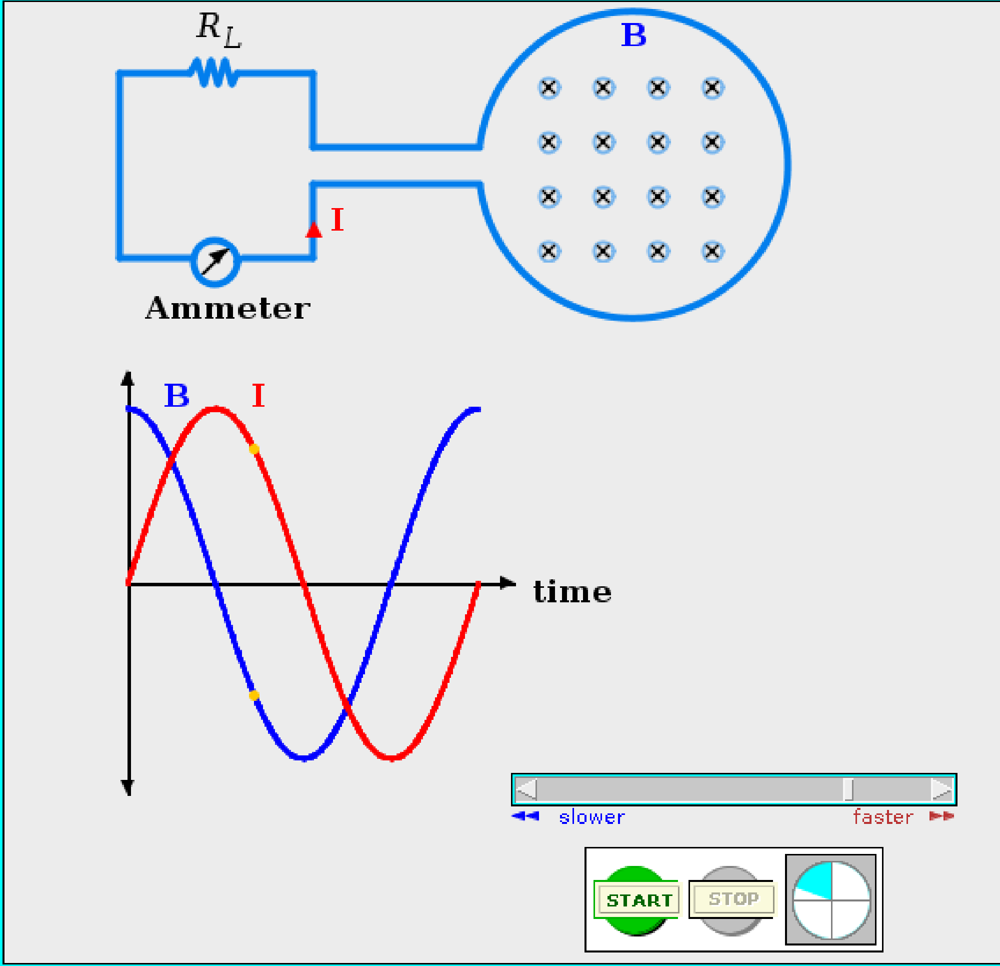
(c) At $t = 0$, $d\Phi/dt > 0$ and $V_{emf}^{tr} = -188.5$ V. Since the flux is increasing, the current I must be in the direction shown in Fig. 6-3 in order to satisfy Lenz's law. Consequently, terminal 2 is at a higher potential than terminal 1 and

$$V_{emf}^{tr} = V_1 - V_2$$
$$= -188.5 \qquad \text{(V)}.$$

(d) The current I is given by

$$I = \frac{V_2 - V_1}{R}$$
$$= \frac{188.5}{10^3}\cos 10^3 t$$
$$= 0.19\cos 10^3 t \qquad \text{(A)}.$$

Module 6.1 Circular Loop in Time-varying Magnetic Field



Demonstration of Faraday's Law

The circular wire loop shown in the figure is connected to a simple circuit composed of a resistor R_L in series with a current meter. The time-varying magnetic flux linking the surface of the loop induces a V_{emf}, and hence a current through R. The purpose of this demo is to illustrate, in the form of a slow-motion video, how the current I varies with time, in both magnitude and direction, when $B(t) = B_0 \cos \omega t$.

Note that $I(t)$ is a maximum when the slope of $B(t)$ is a maximum, which occurs when B itself is zero. The direction of $I(t)$ is dictated by Lenz's Law.

Applet Design: Janice Richards

Example 6-2: Lenz's Law

Determine voltages V_1 and V_2 across the 2-Ω and 4-Ω resistors shown in Fig. 6-4. The loop is located in the x–y plane, its area is 4 m^2, the magnetic flux density is $\mathbf{B} = -\hat{\mathbf{z}}0.3t$ (T), and the internal resistance of the wire may be ignored.

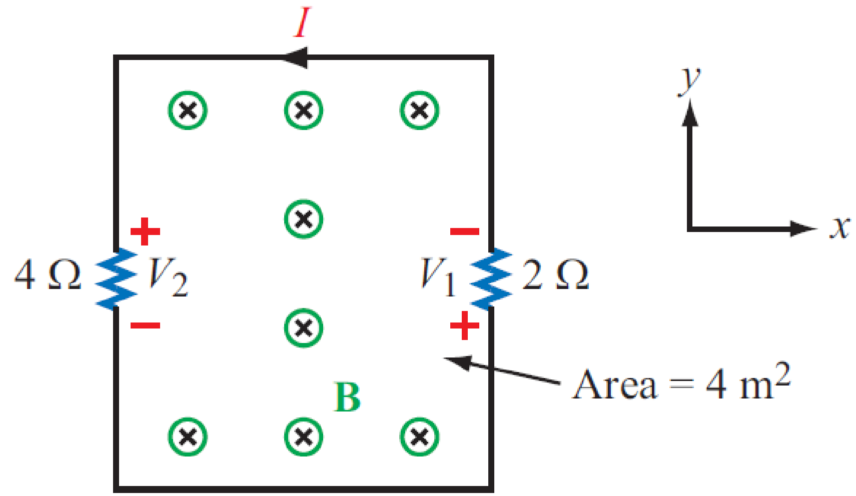


Figure 6-4: Circuit for Example 6-2.

Solution: The flux flowing through the loop is

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{s} = \int_S (-\hat{\mathbf{z}}0.3t) \cdot \hat{\mathbf{z}}\, ds$$

$$= -0.3t \times 4 = -1.2t \qquad \text{(Wb)},$$

and the corresponding transformer emf is

$$V_{\text{emf}}^{\text{tr}} = -\frac{d\Phi}{dt} = 1.2 \qquad \text{(V)}.$$

The total voltage of 1.2 V is distributed across two resistors in series. Consequently,

$$I = \frac{V_{\text{emf}}^{\text{tr}}}{R_1 + R_2}$$

$$= \frac{1.2}{2+4} = 0.2 \text{ A},$$

and

$$V_1 = IR_1 = 0.2 \times 2 = 0.4 \text{ V},$$

$$V_2 = IR_2 = 0.2 \times 4 = 0.8 \text{ V}.$$

Ideal Transformer

$$V_1 = -N_1 \frac{d\Phi}{dt} .$$

A similar relation holds true on the secondary side:

$$V_2 = -N_2 \frac{d\Phi}{dt} .$$

$$\frac{V_1}{V_2} = \frac{N_1}{N_2}$$

$$\frac{I_1}{I_2} = \frac{N_2}{N_1}$$

$$R_{\text{in}} = \frac{V_1}{I_1}$$

$$R_{\text{in}} = \frac{V_2}{I_2}\left(\frac{N_1}{N_2}\right)^2 = \left(\frac{N_1}{N_2}\right)^2 R_{\text{L}} . \qquad (6.20)$$

When the load is an impedance Z_L and V_1 is a sinusoidal source, the phasor-domain equivalent of Eq. (6.20) is

$$Z_{\text{in}} = \left(\frac{N_1}{N_2}\right)^2 Z_{\text{L}} . \qquad (6.21)$$

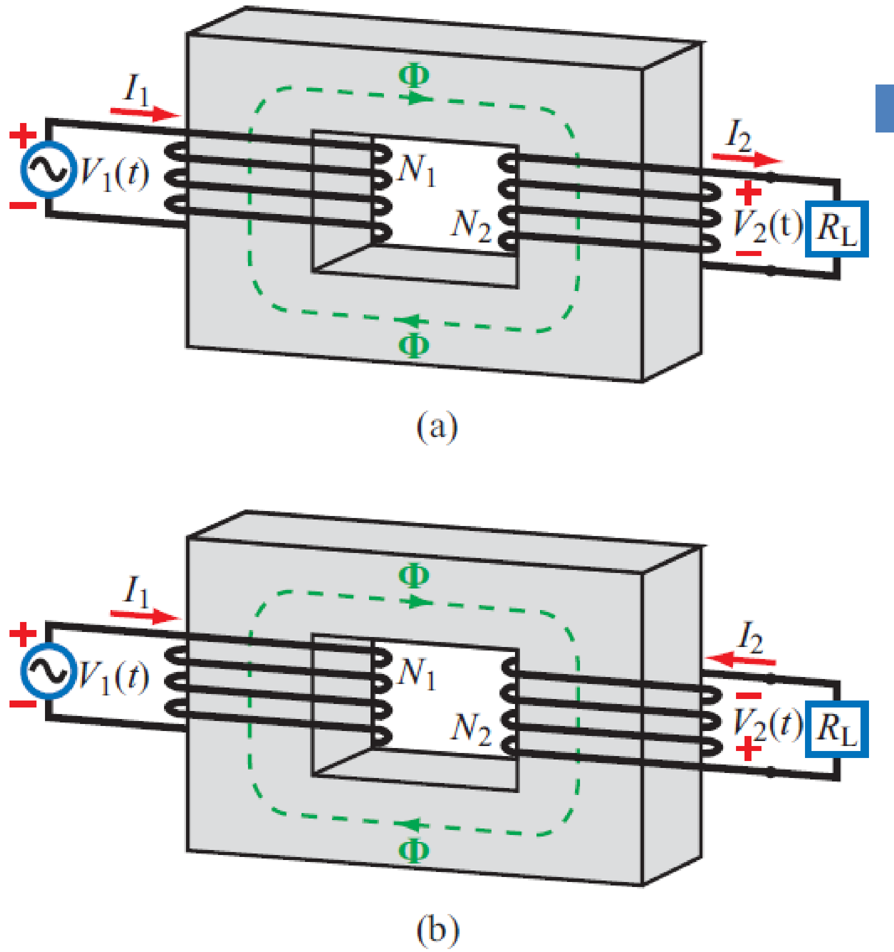


Figure 6-5: In a transformer, the directions of I_1 and I_2 are such that the flux Φ generated by one of them is opposite to that generated by the other. The direction of the secondary winding in (b) is opposite to that in (a), and so are the direction of I_2 and the polarity of V_2.

Motional EMF

Magnetic force on charge *q* moving with velocity **u** in a magnetic field **B**:

$$\mathbf{F}_m = q(\mathbf{u} \times \mathbf{B}).$$

This magnetic force is equivalent to the electrical force that would be exerted on the particle by the electric field Em given by

$$\mathbf{E}_m = \frac{\mathbf{F}_m}{q} = \mathbf{u} \times \mathbf{B}.$$

This, in turn, induces a voltage difference between ends 1 and 2, with end 2 being at the higher potential. The induced voltage is

$$V_{emf}^{m} = V_{12} = \int_2^1 \mathbf{E}_m \cdot d\mathbf{l} = \int_2^1 (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l}.$$

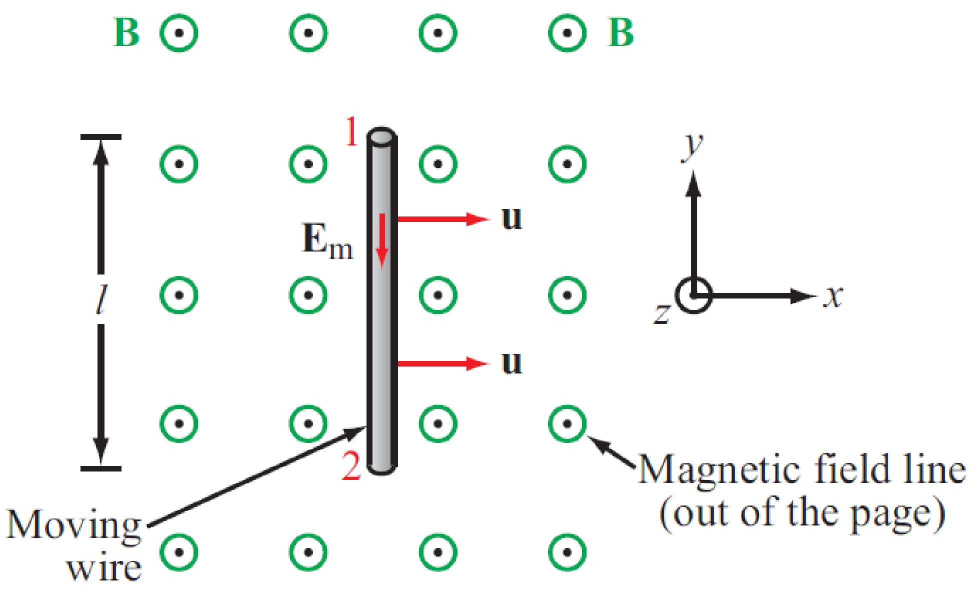


Figure 6-7: Conducting wire moving with velocity **u** in a static magnetic field.

For the conducting wire, $\mathbf{u} \times \mathbf{B} = \hat{\mathbf{x}}u \times \hat{\mathbf{z}}B_0 = -\hat{\mathbf{y}}uB_0$ and $d\mathbf{l} = \hat{\mathbf{y}}\, dl$. Hence,

$$V_{emf}^{m} = V_{12} = -uB_0 l. \tag{6.25}$$

Motional EMF

In general, if any segment of a closed circuit with contour C moves with a velocity $\mathbf{u}$ across a static magnetic field $\mathbf{B}$, then the induced motional emf is given by

$$V_{\text{emf}}^{\text{m}} = \oint_C (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l} \qquad \text{(motional emf).} \qquad (6.26)$$

Only those segments of the circuit that cross magnetic field lines contribute to $V_{\text{emf}}^{\text{m}}$.

Example 6-3: Sliding Bar

$$V_{\text{emf}}^{\text{m}} = V_{12} = V_{43} = \int_3^4 (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l}$$

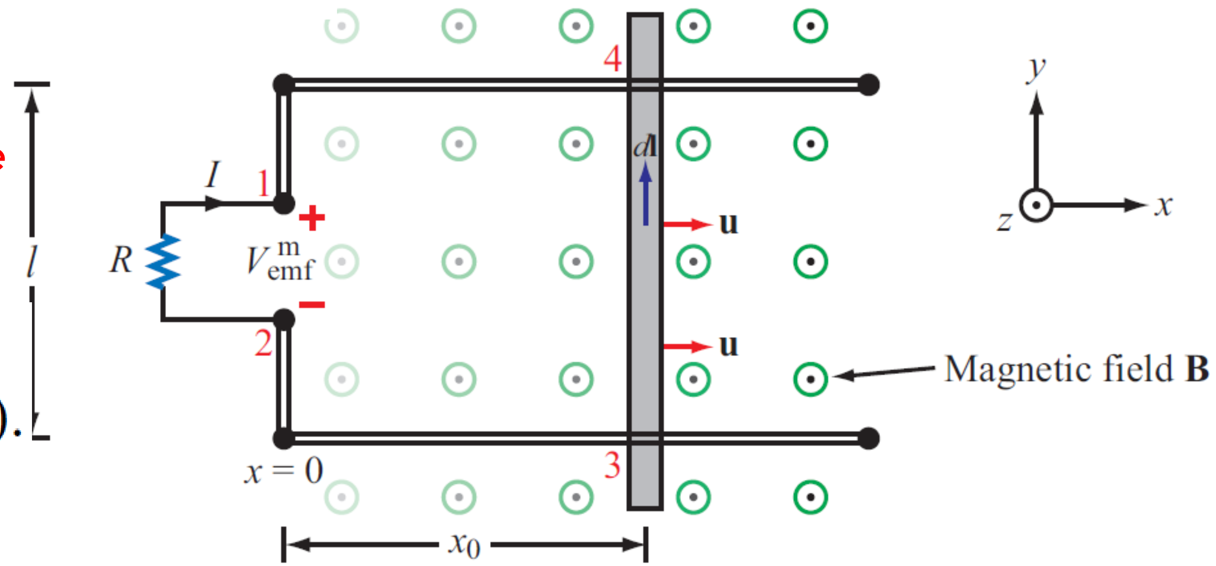
$$= \int_3^4 (\hat{\mathbf{x}}u \times \hat{\mathbf{z}}B_0x_0) \cdot \hat{\mathbf{y}}\, dl = -uB_0x_0l.$$

Note that B increases with x

$$\mathbf{B} = \hat{\mathbf{z}}B_0x$$

The length of the loop is related to *u* by *x0 = ut. Hence*

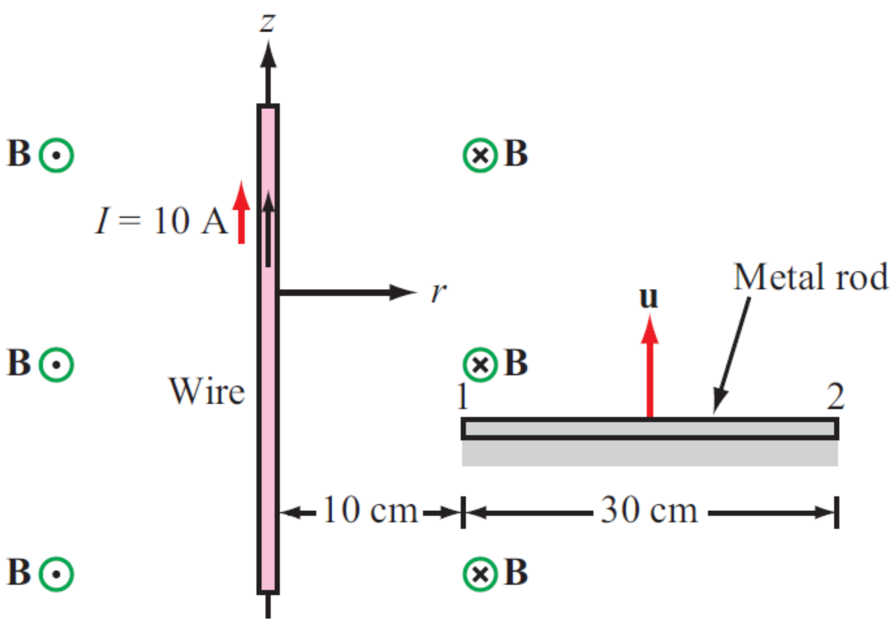
$$V_{\text{emf}}^{\text{m}} = -B_0u^2lt \qquad \text{(V)}.$$



Example 6-5: Moving Rod Next to a Wire

The wire shown in Fig. 6-10 carries a current $I = 10$ A. A 30-cm-long metal rod moves with a constant velocity $\mathbf{u} = \hat{\mathbf{z}}5$ m/s. Find V_{12}.

$$\mathbf{B} = \hat{\boldsymbol{\phi}} \frac{\mu_0 I}{2\pi r}$$



$$V_{12} = \int_{40\text{ cm}}^{10\text{ cm}} (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l}$$

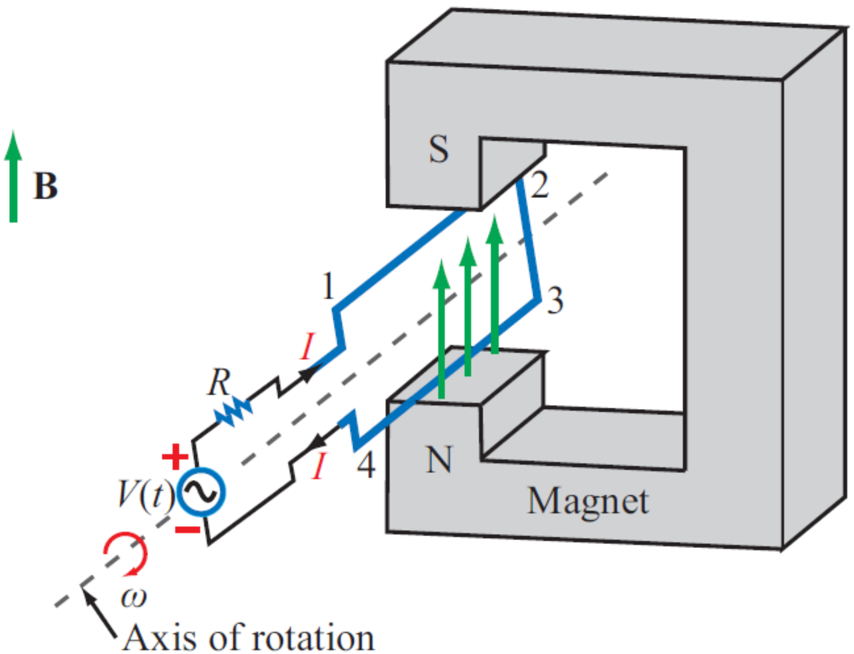
$$= \int_{40\text{ cm}}^{10\text{ cm}} \left(\hat{\mathbf{z}}5 \times \hat{\boldsymbol{\phi}} \frac{\mu_0 I}{2\pi r}\right) \cdot \hat{\mathbf{r}}\, dr$$

$$= -\frac{5\mu_0 I}{2\pi} \int_{40\text{ cm}}^{10\text{ cm}} \frac{dr}{r}$$

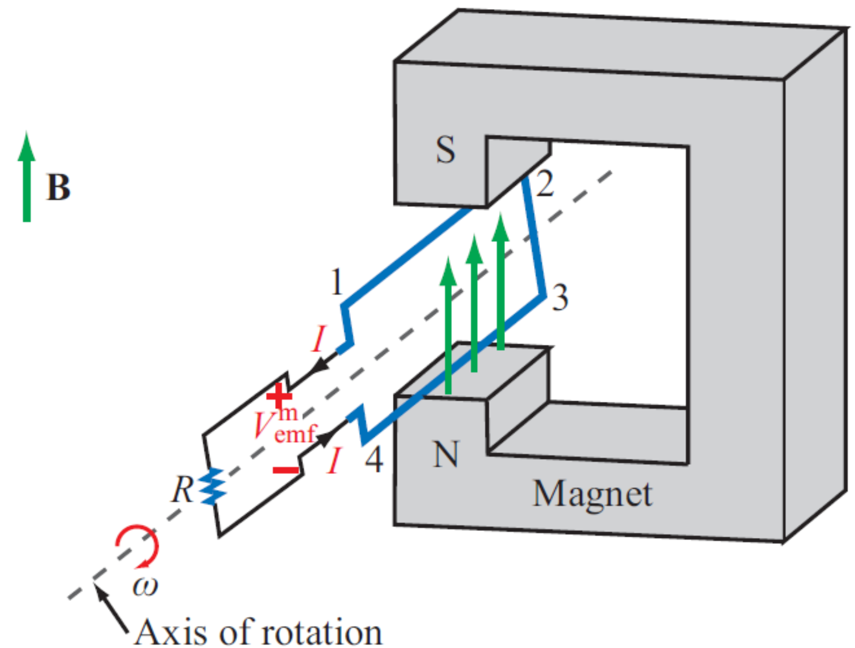
$$= -\frac{5 \times 4\pi \times 10^{-7} \times 10}{2\pi} \times \ln\left(\frac{10}{40}\right)$$

$$= 13.9 \qquad (\mu\text{V}).$$

EM Motor/ Generator Reciprocity



(a) ac motor



(b) ac generator

Motor: Electrical to mechanical energy conversion

Generator: Mechanical to electrical energy conversion

EM Generator EMF

As the loop rotates with an angular velocity *ω about its own* axis, segment 1–2 moves with velocity **u** given by

$$\mathbf{u} = \hat{\mathbf{n}}\omega\,\frac{w}{2}$$

Also: $$\hat{\mathbf{n}} \times \hat{\mathbf{z}} = \hat{\mathbf{x}}\sin\alpha.$$

Segment 3-4 moves with velocity **–u**. Hence:

$$V_{\mathrm{emf}}^{\mathrm{m}} = V_{14} = \int_{2}^{1} (\mathbf{u}\times\mathbf{B})\cdot d\mathbf{l} + \int_{4}^{3} (\mathbf{u}\times\mathbf{B})\cdot d\mathbf{l}$$

$$= \int_{-l/2}^{l/2} \left[\left(\hat{\mathbf{n}}\omega\frac{w}{2}\right)\times\hat{\mathbf{z}}B_0\right]\cdot\hat{\mathbf{x}}\,dx$$

$$+ \int_{l/2}^{-l/2} \left[\left(-\hat{\mathbf{n}}\omega\frac{w}{2}\right)\times\hat{\mathbf{z}}B_0\right]\cdot\hat{\mathbf{x}}\,dx.$$

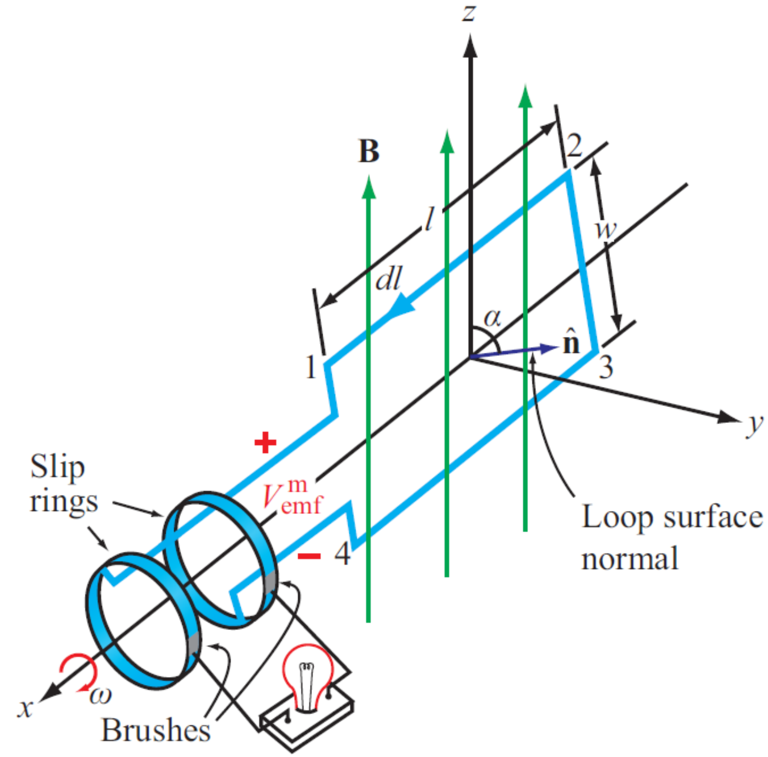


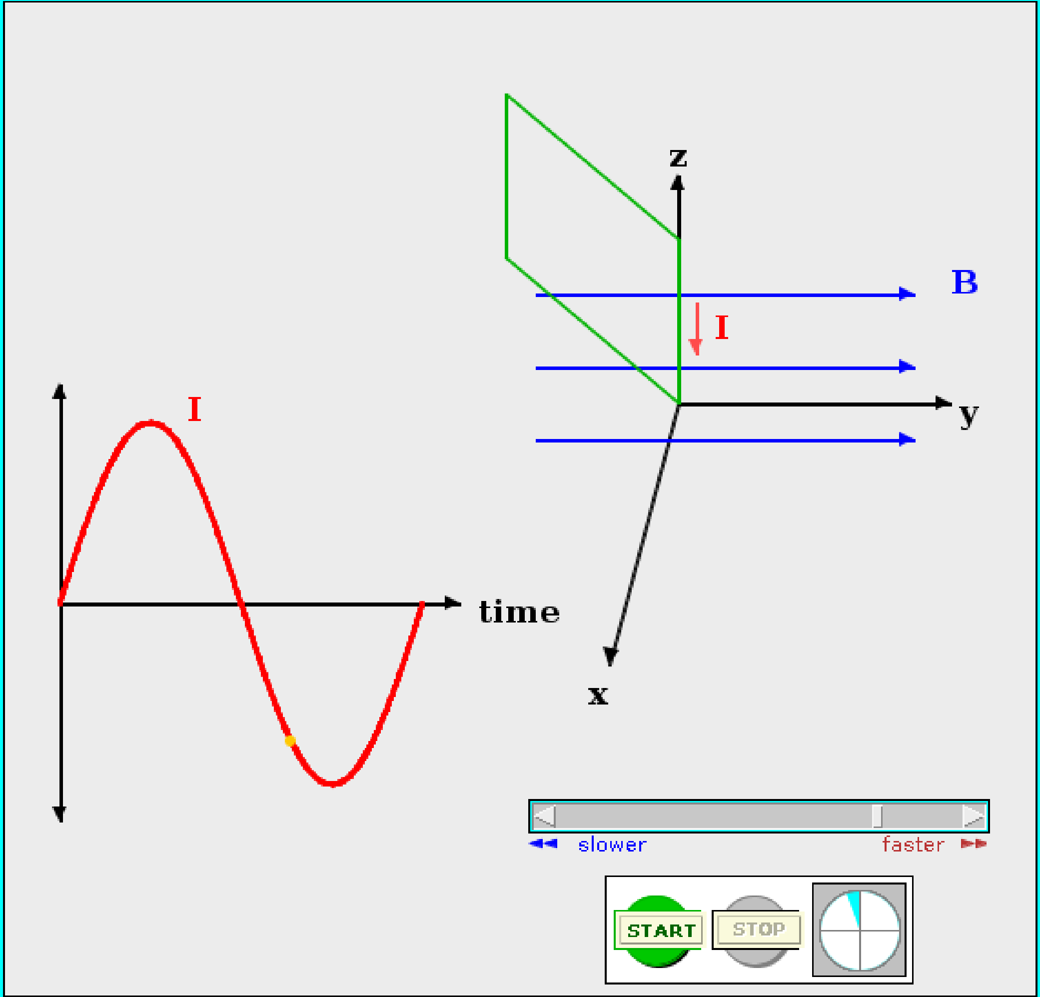
Figure 6-12: A loop rotating in a magnetic field induces an emf.

$$V_{\mathrm{emf}}^{\mathrm{m}} = wl\omega B_0\sin\alpha = A\omega B_0\sin\alpha,$$

$$\alpha = \omega t + C_0,$$

$$\boxed{V_{\mathrm{emf}}^{\mathrm{m}} = A\omega B_0\sin(\omega t + C_0) \qquad \text{(V).}}$$

Module 6.2 **Rotating Wire Loop in Constant Magnetic Field**



Demonstration of Motional EMF

A rectangular wire loop of area A rotates at an angular frequency ω in a constant magnetic flux density B_0. The purpose of the demo is to illustrate how the current varies in time relative to the loop's position.

Note the direction of the current and its magnitude, as indicated by its brightness.

$$I_{max} = \omega B_0 A$$

Applet Design: Janice Richards

Tech Brief 12: EMF Sensors

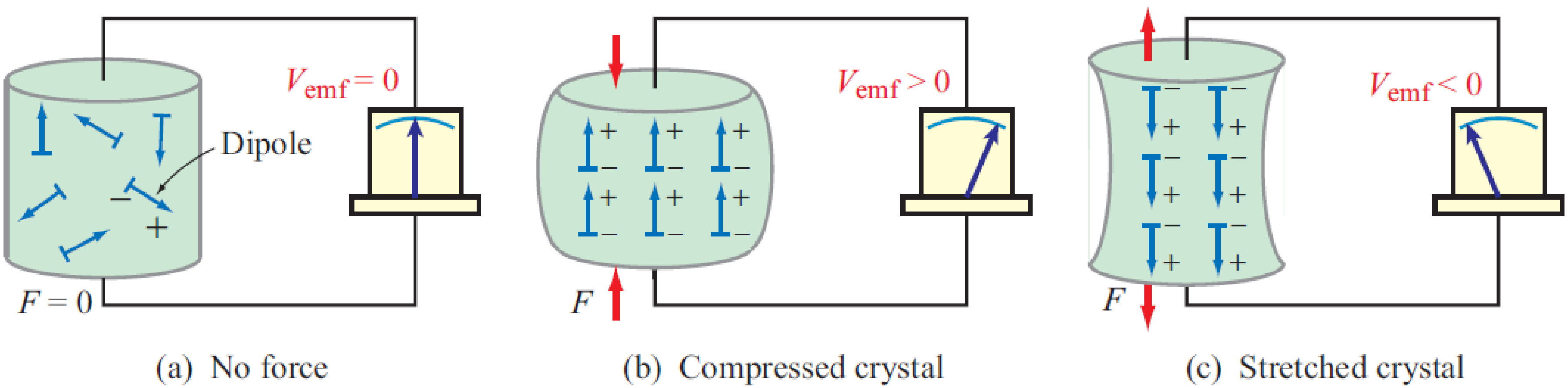


Figure TF12-1: Response of a piezoelectric crystal to an applied force.

- Piezoelectric crystals generate a voltage across them proportional to the compression or tensile (stretching) force applied across them.

- Piezoelectric transducers are used in medical ultrasound, microphones, loudspeakers, accelerometers, etc.

- Piezoelectric crystals are bidirectional: pressure generates emf, and conversely, emf generates pressure (through shape distortion).

Faraday Accelerometer

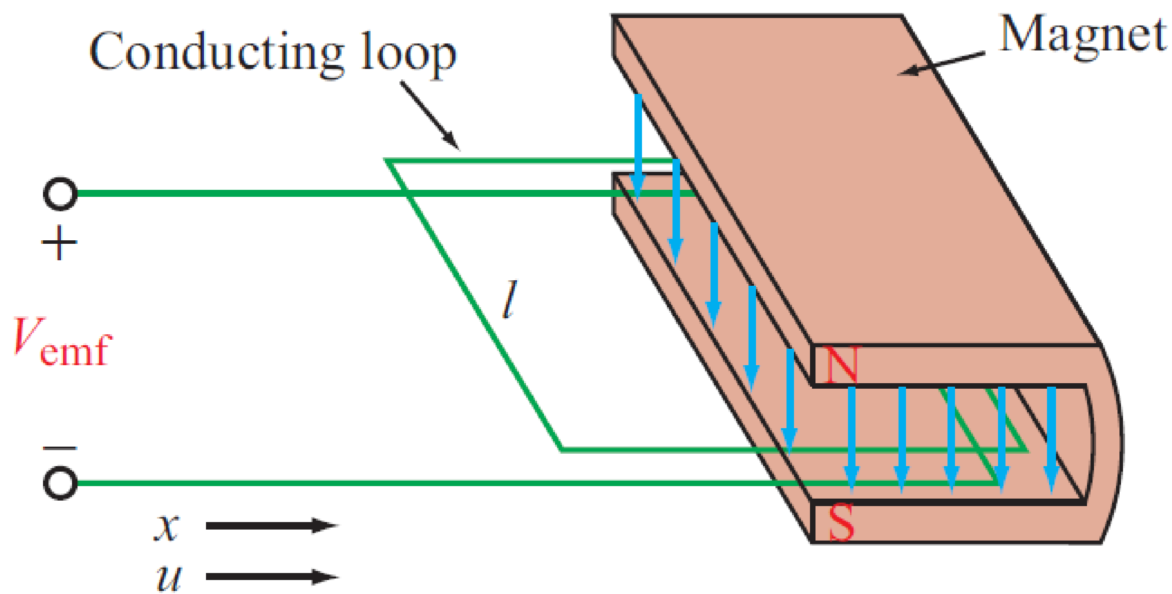


Figure TF12-3: In a Faraday accelerometer, the induced emf is directly proportional to the velocity of the loop (into and out of the magnet's cavity).

The acceleration **a** is determined by differentiating the velocity u with respect to time

The Thermocouple

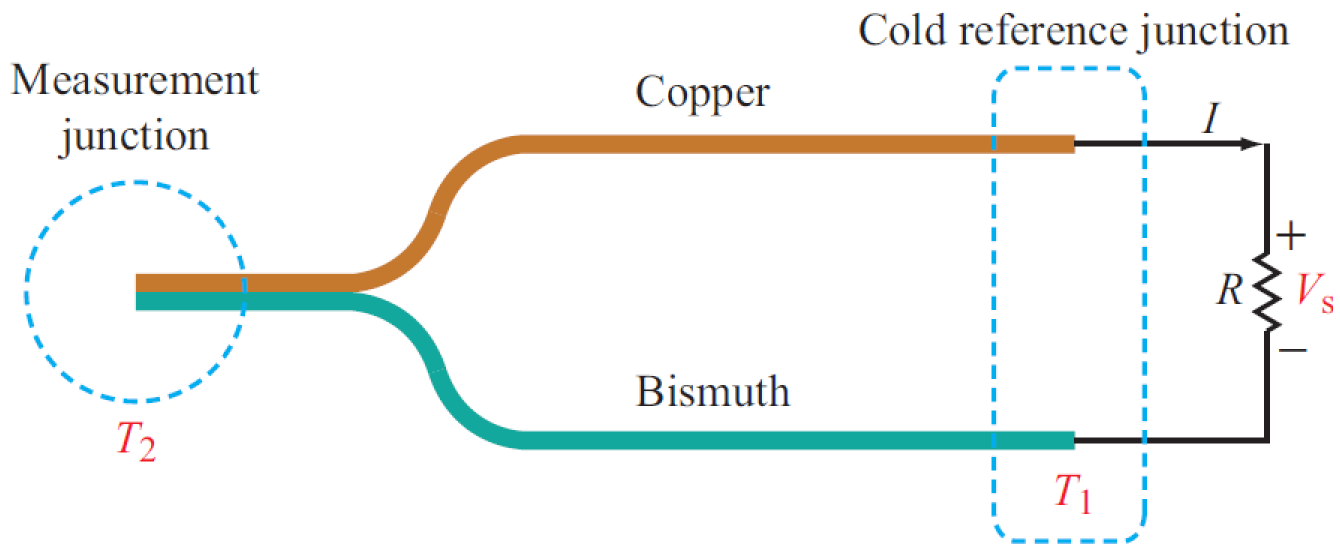


Figure TF12-4: Principle of the thermocouple.

- The thermocouple measures the unknown temperature T_2 at a junction connecting two metals with different thermal conductivities, relative to a reference temperature T_1.
- In today's temperature sensor designs, an artificial cold junction is used instead. The artificial junction is an electric circuit that generates a voltage equal to that expected from a reference junction at temperature T_1.

Displacement Current

Ampère's law in differential form is given by

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \qquad \text{(Ampère's law).} \tag{6.41}$$

Integrating both sides of Eq. (6.41) over an arbitrary open surface S with contour C, we have

$$\int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{s} = \int_S \mathbf{J} \cdot d\mathbf{s} + \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{s}. \tag{6.42}$$

This term is conduction current I_C

This term must represent a current

Application of Stokes's theorem gives:

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = I_c + \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{s} \qquad \text{(Ampère's law)}$$

Cont.

Displacement Current

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = I_c + \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{s} \quad \text{(Ampère's law)}$$

Define the displacement current as:

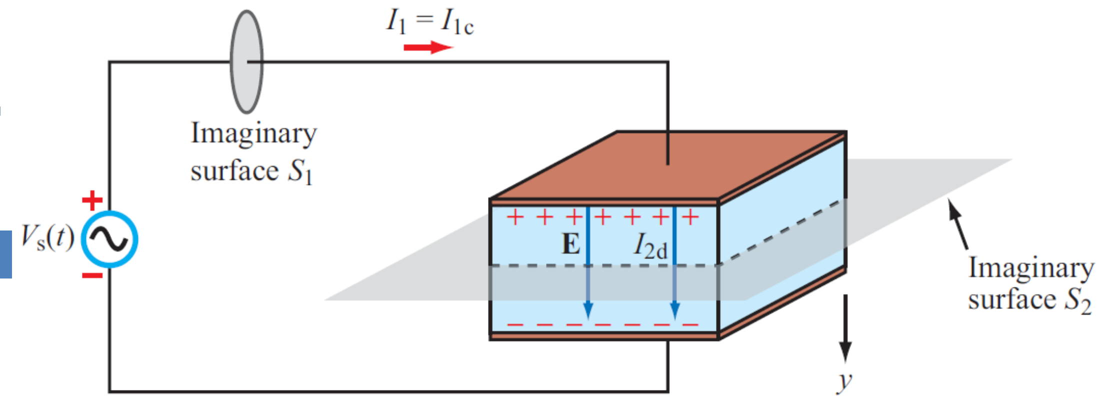
$$I_d = \int_S \mathbf{J}_d \cdot d\mathbf{s} = \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{s}, \qquad (6.44)$$

The displacement current does not involve real charges;
it is an equivalent current that depends on $\partial \mathbf{D}/\partial t$

where $\mathbf{J}_d = \partial \mathbf{D}/\partial t$ represents a ***displacement current density***. In view of Eq. (6.44),

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = I_c + I_d = I, \qquad (6.45)$$

Capacitor Circuit



Given: Wires are perfect conductors and capacitor insulator material is perfect dielectric.

For Surface S_1:

$$I_1 = I_{1c} + I_{1d}$$

$$I_{1c} = C\,\frac{dV_C}{dt} = C\,\frac{d}{dt}\,(V_0\cos\omega t) = -CV_0\omega\sin\omega t$$

$$I_{1d} = 0$$ (**D** = 0 in perfect conductor)

For Surface S_2:

$$I_2 = I_{2c} + I_{2d}$$

$I_{2c} = 0$ (perfect dielectric)

$$\mathbf{E} = \hat{\mathbf{y}}\,\frac{V_c}{d} = \hat{\mathbf{y}}\,\frac{V_0}{d}\cos\omega t$$

$$I_{2d} = \int_S \frac{\partial \mathbf{D}}{\partial t}\cdot d\mathbf{s}$$

$$= \int_A \left[\frac{\partial}{\partial t}\left(\hat{\mathbf{y}}\,\frac{\varepsilon V_0}{d}\cos\omega t\right)\right]\cdot(\hat{\mathbf{y}}\,ds)$$

$$= -\frac{\varepsilon A}{d}V_0\omega\sin\omega t = -CV_0\omega\sin\omega t$$

Conclusion: $I_1 = I_2$

Example 6-7: Displacement Current Density

The conduction current flowing through a wire with conductivity $\sigma = 2 \times 10^7$ S/m and relative permittivity $\varepsilon_r = 1$ is given by $I_c = 2 \sin \omega t$ (mA). If $\omega = 10^9$ rad/s, find the displacement current.

Solution: The conduction current $I_c = JA = \sigma EA$, where A is the cross section of the wire. Hence,

$$E = \frac{I_c}{\sigma A} = \frac{2 \times 10^{-3} \sin \omega t}{2 \times 10^7 A}$$
$$= \frac{1 \times 10^{-10}}{A} \sin \omega t \qquad \text{(V/m)}.$$

Application of Eq. (6.44), with $D = \varepsilon E$, leads to

$$\begin{aligned} I_d &= J_d A \\ &= \varepsilon A \frac{\partial E}{\partial t} \\ &= \varepsilon A \frac{\partial}{\partial t} \left(\frac{1 \times 10^{-10}}{A} \sin \omega t \right) \\ &= \varepsilon \omega \times 10^{-10} \cos \omega t = 0.885 \times 10^{-12} \cos \omega t \quad \text{(A)}, \end{aligned}$$

where we used $\omega = 10^9$ rad/s and $\varepsilon = \varepsilon_0 = 8.85 \times 10^{-12}$ F/m. Note that I_c and I_d are in phase quadrature (90° phase shift between them). Also, I_d is about nine orders of magnitude smaller than I_c, which is why the displacement current usually is ignored in good conductors.

Boundary Conditions

Table 6-2: Boundary conditions for the electric and magnetic fields.

Field Components	General Form	Medium 1 Dielectric	Medium 2 Dielectric	Medium 1 Dielectric	Medium 2 Conductor
Tangential E	$\hat{\mathbf{n}}_2 \times (\mathbf{E}_1 - \mathbf{E}_2) = 0$	$E_{1t} = E_{2t}$		$E_{1t} = E_{2t} = 0$	
Normal D	$\hat{\mathbf{n}}_2 \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s$	$D_{1n} - D_{2n} = \rho_s$		$D_{1n} = \rho_s$	$D_{2n} = 0$
Tangential H	$\hat{\mathbf{n}}_2 \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s$	$H_{1t} = H_{2t}$		$H_{1t} = J_s$	$H_{2t} = 0$
Normal B	$\hat{\mathbf{n}}_2 \cdot (\mathbf{B}_1 - \mathbf{B}_2) = 0$	$B_{1n} = B_{2n}$		$B_{1n} = B_{2n} = 0$	

Notes: (1) ρ_s is the surface charge density at the boundary; (2) $\mathbf{J}_s$ is the surface current density at the boundary; (3) normal components of all fields are along $\hat{\mathbf{n}}_2$, the outward unit vector of medium 2; (4) $E_{1t} = E_{2t}$ implies that the tangential components are equal in magnitude and parallel in direction; (5) direction of $\mathbf{J}_s$ is orthogonal to $(\mathbf{H}_1 - \mathbf{H}_2)$.

Charge Current Continuity Equation

Current I out of a volume is equal to rate of decrease of charge Q contained in that volume:

$$I = -\frac{dQ}{dt} = -\frac{d}{dt}\int_{\mathcal{V}} \rho_{\mathrm{v}}\, d\mathcal{V}$$

$$\oint_S \mathbf{J}\cdot d\mathbf{s} = -\frac{d}{dt}\int_{\mathcal{V}} \rho_{\mathrm{v}}\, d\mathcal{V}$$

$$\oint_S \mathbf{J}\cdot d\mathbf{s} = \int_{\mathcal{V}} \nabla\cdot\mathbf{J}\, d\mathcal{V} = -\frac{d}{dt}\int_{\mathcal{V}} \rho_{\mathrm{v}}\, d\mathcal{V}$$

Used Divergence Theorem

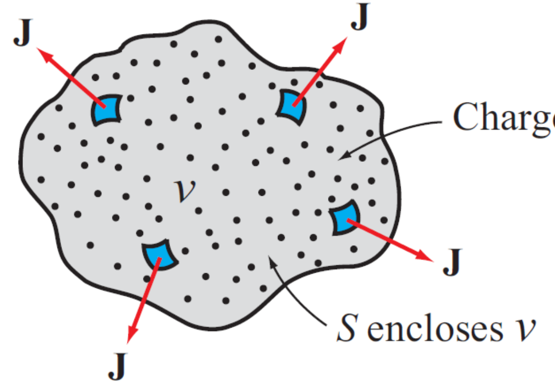


Figure 6-14: The total current flowing out of a volume $\mathcal{V}$ is equal to the flux of the current density **J** through the surface S, which in turn is equal to the rate of decrease of the charge enclosed in $\mathcal{V}$.

$$\boxed{\nabla\cdot\mathbf{J} = -\frac{\partial\rho_{\mathrm{v}}}{\partial t}\,, \qquad (6.54)}$$

which is known as the ***charge-current continuity relation***, or simply the ***charge continuity equation***.

Charge Dissipation

Question 1: What happens if you place a certain amount of free charge inside of a material?
Answer: The charge will move to the surface of the material, thereby returning its interior to a neutral state.

Question 2: How fast will this happen?
Answer: It depends on the material; in a good conductor, the charge dissipates in less than a femtosecond, whereas in a good dielectric, the process may take several hours.

Derivation of charge density equation:

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t} . \qquad (6.58)$$

In a conductor, the point form of Ohm's law, given by Eq. (4.63), states that $\mathbf{J} = \sigma \mathbf{E}$. Hence,

$$\sigma \nabla \cdot \mathbf{E} = -\frac{\partial \rho_v}{\partial t} . \qquad (6.59)$$

Next, we use Eq. (6.1), $\nabla \cdot \mathbf{E} = \rho_v / \varepsilon$, to obtain the partial differential equation

$$\frac{\partial \rho_v}{\partial t} + \frac{\sigma}{\varepsilon} \rho_v = 0. \qquad (6.60)$$

Cont.

Solution of Charge Dissipation Equation

$$\frac{\partial \rho_v}{\partial t} + \frac{\sigma}{\varepsilon}\rho_v = 0$$

Given that $\rho_v = \rho_{vo}$ at $t = 0$, the solution of Eq. (6.60) is

$$\rho_v(t) = \rho_{vo} e^{-(\sigma/\varepsilon)t} = \rho_{vo} e^{-t/\tau_r} \qquad (\text{C/m}^3),$$

where $\tau_r = \varepsilon/\sigma$ is called the ***relaxation time constant***.

For copper: $\tau_r = 1.53 \times 10^{-19}$ s

For mica: $\tau_r = 5.31 \times 10^{4}$ s = 15 hours

EM Potentials

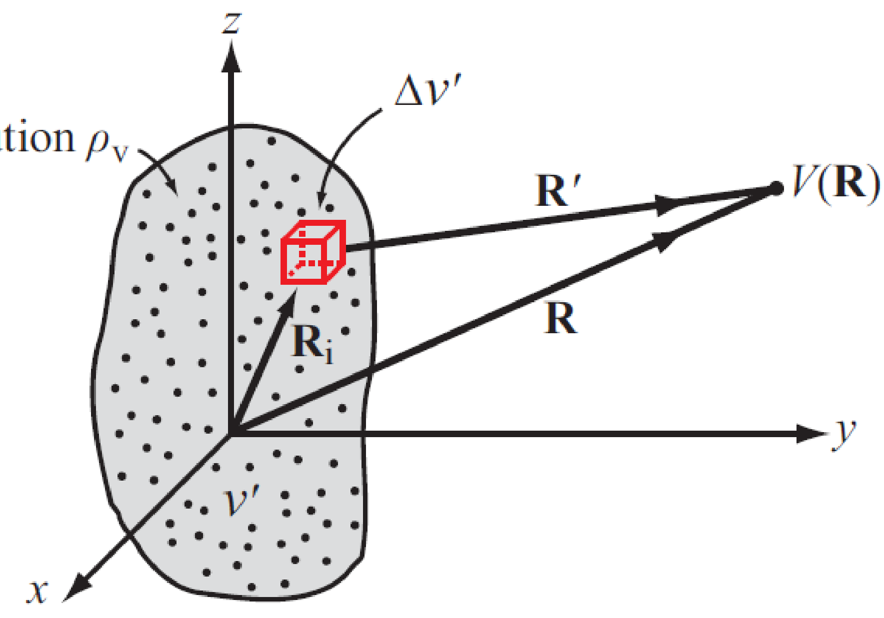


Figure 6-16: Electric potential $V(\mathbf{R})$ due to a charge distribution ρ_v over a volume $\mathcal{V}'$.

Static condition

$$V(\mathbf{R}) = \frac{1}{4\pi\varepsilon}\int_{\mathcal{V}'} \frac{\rho_v(\mathbf{R}_i)}{R'}\, d\mathcal{V}'$$

Dynamic condition

$$V(\mathbf{R}, t) = \frac{1}{4\pi\varepsilon}\int_{\mathcal{V}'} \frac{\rho_v(\mathbf{R}_i, t)}{R'}\, d\mathcal{V}'$$

Dynamic condition with propagation delay:

$$V(\mathbf{R}, t) = \frac{1}{4\pi\varepsilon}\int_{\mathcal{V}'} \frac{\rho_v(\mathbf{R}_i,\ t - R'/u_p)}{R'}\, d\mathcal{V}' \quad \text{(V)},$$

Similarly, for the magnetic vector potential:

$$\mathbf{A}(\mathbf{R}, t) = \frac{\mu}{4\pi}\int_{\mathcal{V}'} \frac{\mathbf{J}(\mathbf{R}_i,\ t - R'/u_p)}{R'}\, d\mathcal{V}' \quad \text{(Wb/m)}.$$

Time Harmonic Potentials

If charges and currents vary sinusoidally with time:

$$\rho_v(\mathbf{R}_i, t) = \rho_v(\mathbf{R}_i)\cos(\omega t + \phi)$$

we can use phasor notation:

$$\rho_v(\mathbf{R}_i, t) = \mathfrak{Re}\left[\tilde{\rho}_v(\mathbf{R}_i)\, e^{j\omega t}\right],$$

with

$$\tilde{\rho}_v(\mathbf{R}_i) = \rho_v(\mathbf{R}_i)\, e^{j\phi}.$$

Expressions for potentials become:

$$\widetilde{V}(\mathbf{R}) = \frac{1}{4\pi\varepsilon}\int_{\mathcal{V}'} \frac{\tilde{\rho}_v(\mathbf{R}_i)\, e^{-jkR'}}{R'}\, d\mathcal{V}' \quad \text{(V)}.$$

$$\widetilde{\mathbf{A}}(\mathbf{R}) = \frac{\mu}{4\pi}\int_{\mathcal{V}'} \frac{\widetilde{\mathbf{J}}(\mathbf{R}_i)\, e^{-jkR'}}{R'}\, d\mathcal{V}',$$

Also:

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \qquad \text{(dynamic case).}$$

$$\widetilde{\mathbf{H}} = \frac{1}{\mu}\nabla \times \widetilde{\mathbf{A}}.$$

Maxwell's equations become:

$$\nabla \times \widetilde{\mathbf{E}} = -j\omega\mu\widetilde{\mathbf{H}}$$

$$\text{or} \qquad \widetilde{\mathbf{H}} = -\frac{1}{j\omega\mu}\nabla \times \widetilde{\mathbf{E}}.$$

$$\nabla \times \widetilde{\mathbf{H}} = j\omega\varepsilon\widetilde{\mathbf{E}} \quad \text{or} \quad \widetilde{\mathbf{E}} = \frac{1}{j\omega\varepsilon}\nabla \times \widetilde{\mathbf{H}}.$$

$$k = \frac{\omega}{u_p}$$

Example 6-8: Relating $\mathbf{E}$ to $\mathbf{H}$

In a nonconducting medium with $\varepsilon = 16\varepsilon_0$ and $\mu = \mu_0$, the electric field intensity of an electromagnetic wave is

$$\mathbf{E}(z,t) = \hat{\mathbf{x}}\, 10 \sin(10^{10} t - kz) \qquad \text{(V/m)}. \tag{6.88}$$

Determine the associated magnetic field intensity **H** and find the value of k.

Solution: We begin by finding the phasor $\widetilde{\mathbf{E}}(z)$ of $\mathbf{E}(z,t)$. Since $\mathbf{E}(z,t)$ is given as a sine function and phasors are defined in this book with reference to the cosine function, we rewrite Eq. (6.88) as

$$\begin{aligned}\mathbf{E}(z,t) &= \hat{\mathbf{x}}\, 10 \cos(10^{10} t - kz - \pi/2) \qquad \text{(V/m)} \\ &= \mathfrak{Re}\left[\widetilde{\mathbf{E}}(z)\, e^{j\omega t}\right],\end{aligned} \tag{6.89}$$

with $\omega = 10^{10}$ (rad/s) and

Cont.

$$\widetilde{\mathbf{E}}(z) = \hat{\mathbf{x}}\, 10 e^{-jkz} e^{-j\pi/2} = -\hat{\mathbf{x}}\, j 10 e^{-jkz}. \tag{6.90}$$

Example 6-8: Relating E to H

To find both $\widetilde{\mathbf{H}}(z)$ and k, we will perform a "circle": we will use the given expression for $\widetilde{\mathbf{E}}(z)$ in Faraday's law to find $\widetilde{\mathbf{H}}(z)$; then we will use $\widetilde{\mathbf{H}}(z)$ in Ampère's law to find $\widetilde{\mathbf{E}}(z)$, which we will then compare with the original expression for $\widetilde{\mathbf{E}}(z)$; and the comparison will yield the value of k. Application of Eq. (6.87) gives

$$\begin{aligned}
\widetilde{\mathbf{H}}(z) &= -\frac{1}{j\omega\mu}\nabla\times\widetilde{\mathbf{E}} \\
&= -\frac{1}{j\omega\mu}\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -j10e^{-jkz} & 0 & 0 \end{vmatrix} \\
&= -\frac{1}{j\omega\mu}\left[\hat{\mathbf{y}}\,\frac{\partial}{\partial z}(-j10e^{-jkz})\right] \\
&= -\hat{\mathbf{y}}\,j\,\frac{10k}{\omega\mu}e^{-jkz}. \qquad (6.91)
\end{aligned}$$

Cont.

Example 6-8 cont.

So far, we have used Eq. (6.90) for $\tilde{\mathbf{E}}(z)$ to find $\tilde{\mathbf{H}}(z)$, but k remains unknown. To find k, we use $\tilde{\mathbf{H}}(z)$ in Eq. (6.86) to find $\tilde{\mathbf{E}}(z)$:

$$\begin{aligned}\tilde{\mathbf{E}}(z) &= \frac{1}{j\omega\varepsilon} \nabla \times \tilde{\mathbf{H}} \\ &= \frac{1}{j\omega\varepsilon}\left[-\hat{\mathbf{x}}\,\frac{\partial}{\partial z}\left(-j\,\frac{10k}{\omega\mu} e^{-jkz}\right)\right] \\ &= -\hat{\mathbf{x}} j\,\frac{10k^2}{\omega^2\mu\varepsilon} e^{-jkz}. \qquad (6.92)\end{aligned}$$

Equating Eqs. (6.90) and (6.92) leads to

$$k^2 = \omega^2\mu\varepsilon,$$

or

$$\begin{aligned}k &= \omega\sqrt{\mu\varepsilon} \\ &= 4\omega\sqrt{\mu_0\varepsilon_0} \\ &= \frac{4\omega}{c} = \frac{4 \times 10^{10}}{3 \times 10^8} = 133 \qquad \text{(rad/m)}. \qquad (6.93)\end{aligned}$$

Cont.

Example 6-8 cont.

With k known, the instantaneous magnetic field intensity is then given by

$$
\begin{aligned}
\mathbf{H}(z,t) &= \mathfrak{Re}\left[\widetilde{\mathbf{H}}(z)\; e^{j\omega t}\right] \\
&= \mathfrak{Re}\left[-\hat{\mathbf{y}} j\;\frac{10k}{\omega\mu} e^{-jkz} e^{j\omega t}\right] \\
&= \hat{\mathbf{y}}\, 0.11 \sin(10^{10} t - 133z) \qquad \text{(A/m)}. \qquad (6.94)
\end{aligned}
$$

We note that k has the same expression as the phase constant of a lossless transmission line [Eq. (2.49)].

Summary

Chapter 6 Relationships

Faraday's Law

$$V_{\text{emf}} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{s} = V_{\text{emf}}^{\text{tr}} + V_{\text{emf}}^{\text{m}}$$

Transformer

$$V_{\text{emf}}^{\text{tr}} = -N\int_S \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{s} \qquad (N \text{ loops})$$

Motional

$$V_{\text{emf}}^{\text{m}} = \oint_C (\mathbf{u}\times\mathbf{B})\cdot d\mathbf{l}$$

Charge-Current Continuity

$$\nabla\cdot\mathbf{J} = -\frac{\partial \rho_{\text{v}}}{\partial t}$$

EM Potentials

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$$

$$\mathbf{B} = \nabla\times\mathbf{A}$$

Current Density

Conduction $\quad \mathbf{J}_{\text{c}} = \sigma\,\mathbf{E}$

Displacement $\quad \mathbf{J}_{\text{d}} = \dfrac{\partial \mathbf{D}}{\partial t}$

Conductor Charge Dissipation

$$\rho_{\text{v}}(t) = \rho_{\text{vo}}e^{-(\sigma/\varepsilon)t} = \rho_{\text{vo}}e^{-t/\tau_{\text{r}}}$$

https://www.youtube.com/watch?v=bxHs9I3lbZc