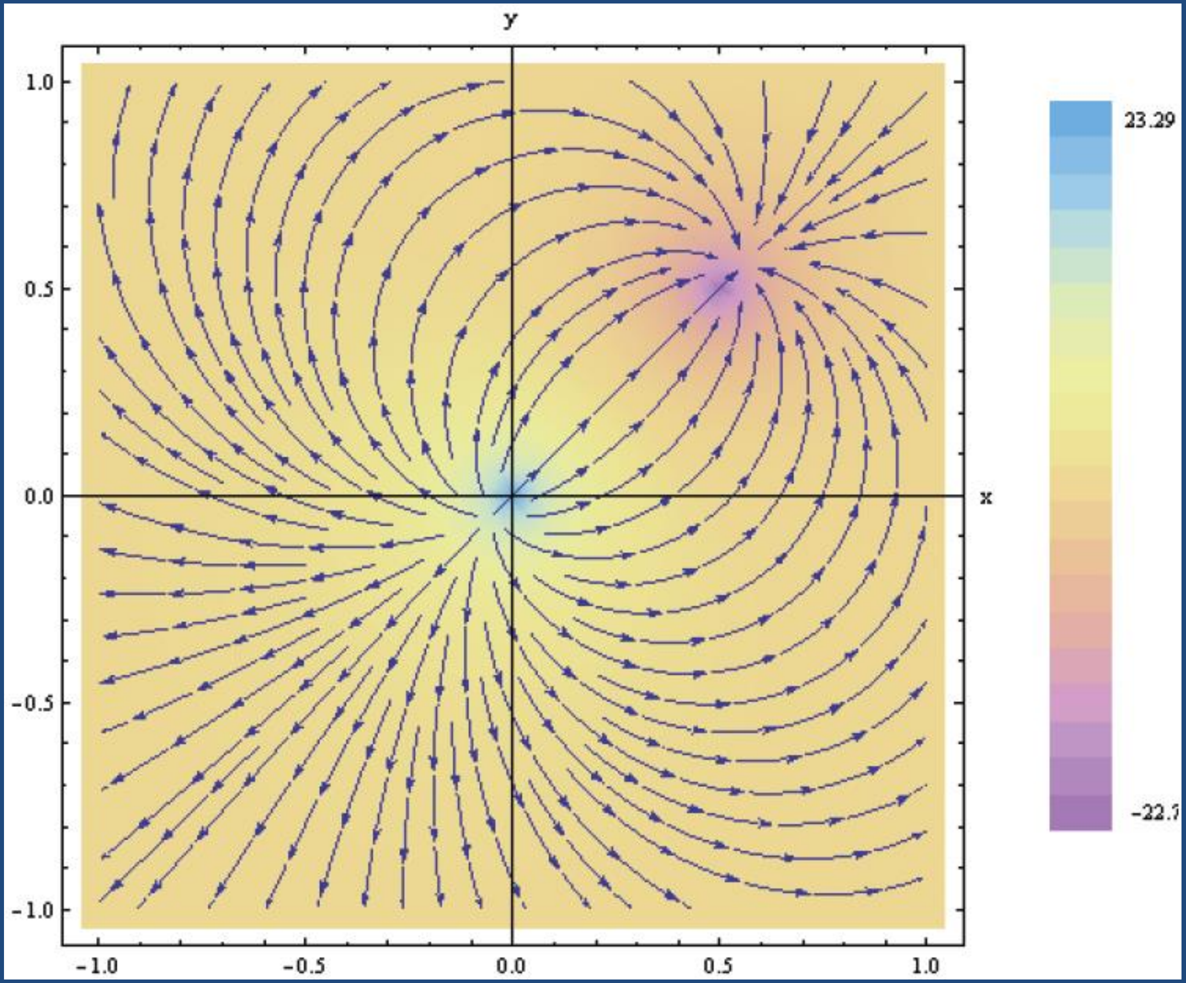


3. VECTOR ANALYSIS

7e Applied EM by Ulaby and Ravaioli

Chapter 3 Overview

Chapter Contents

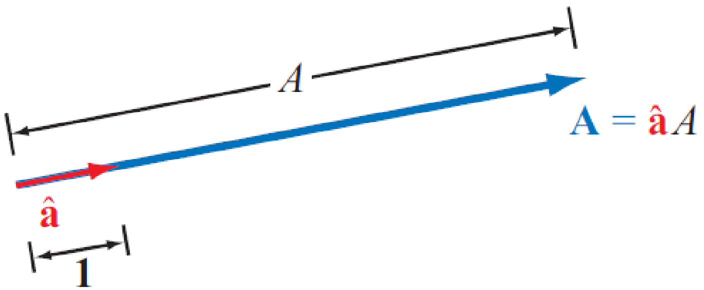
Objectives

Upon learning the material presented in this chapter, you should be able to:

1. Use vector algebra in Cartesian, cylindrical, and spherical coordinate systems.
2. Transform vectors between the three primary coordinate systems.
3. Calculate the gradient of a scalar function and the divergence and curl of a vector function in any of the three primary coordinate systems.
4. Apply the divergence theorem and Stokes's theorem.

https://www.youtube.com/watch?v=rB83DpBJQsE

Laws of Vector Algebra

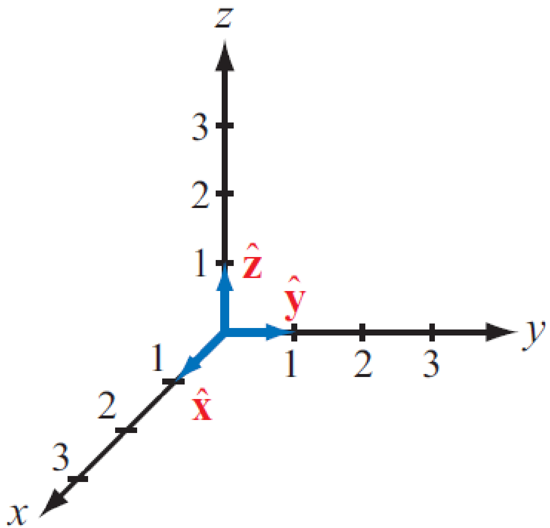


$$\mathbf{A} = \hat{\mathbf{a}}|\mathbf{A}| = \hat{\mathbf{a}}A$$

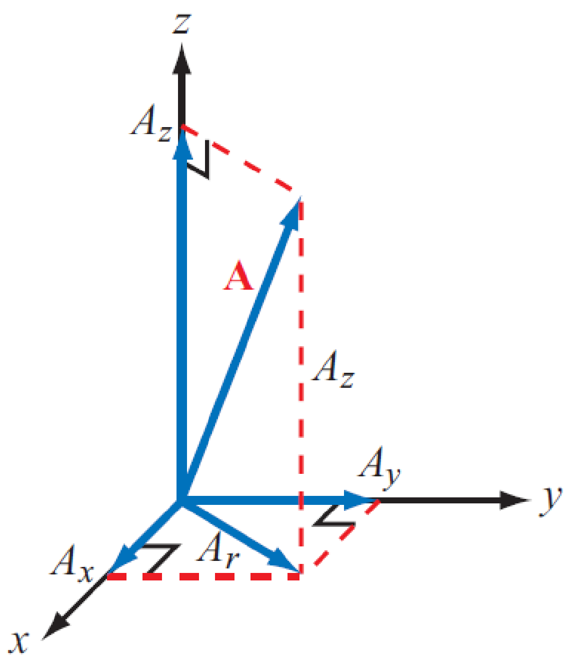
$$\mathbf{A} = \hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$$

$$A = |\mathbf{A}| = \sqrt[+]{A_x^2 + A_y^2 + A_z^2}$$

$$\hat{\mathbf{a}} = \frac{\mathbf{A}}{A} = \frac{\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z}{\sqrt[+]{A_x^2 + A_y^2 + A_z^2}}$$



(a) Base vectors



(b) Components of **A**

Properties of Vector Operations

Equality of Two Vectors

$$\mathbf{A} = \hat{\mathbf{a}}A = \hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z, \quad (3.6a)$$

$$\mathbf{B} = \hat{\mathbf{b}}B = \hat{\mathbf{x}}B_x + \hat{\mathbf{y}}B_y + \hat{\mathbf{z}}B_z, \quad (3.6b)$$

then $\mathbf{A} = \mathbf{B}$ if and only if $A = B$ and $\hat{\mathbf{a}} = \hat{\mathbf{b}}$, which requires that $A_x = B_x$, $A_y = B_y$, and $A_z = B_z$.

Equality of two vectors does not necessarily imply that they are identical; in Cartesian coordinates, two displaced parallel vectors of equal magnitude and pointing in the same direction are equal, but they are identical only if they lie on top of one another.

Commutative property

$$\mathbf{C} = \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$$

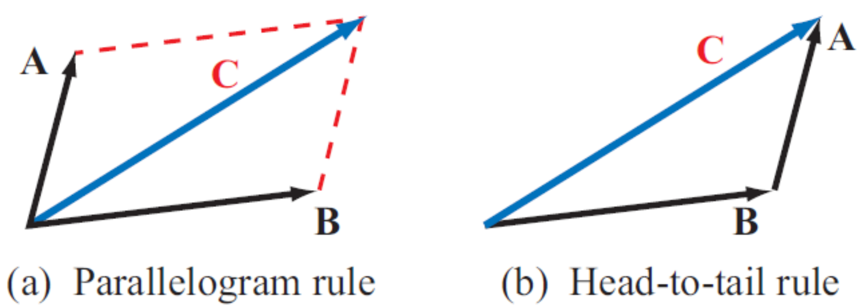
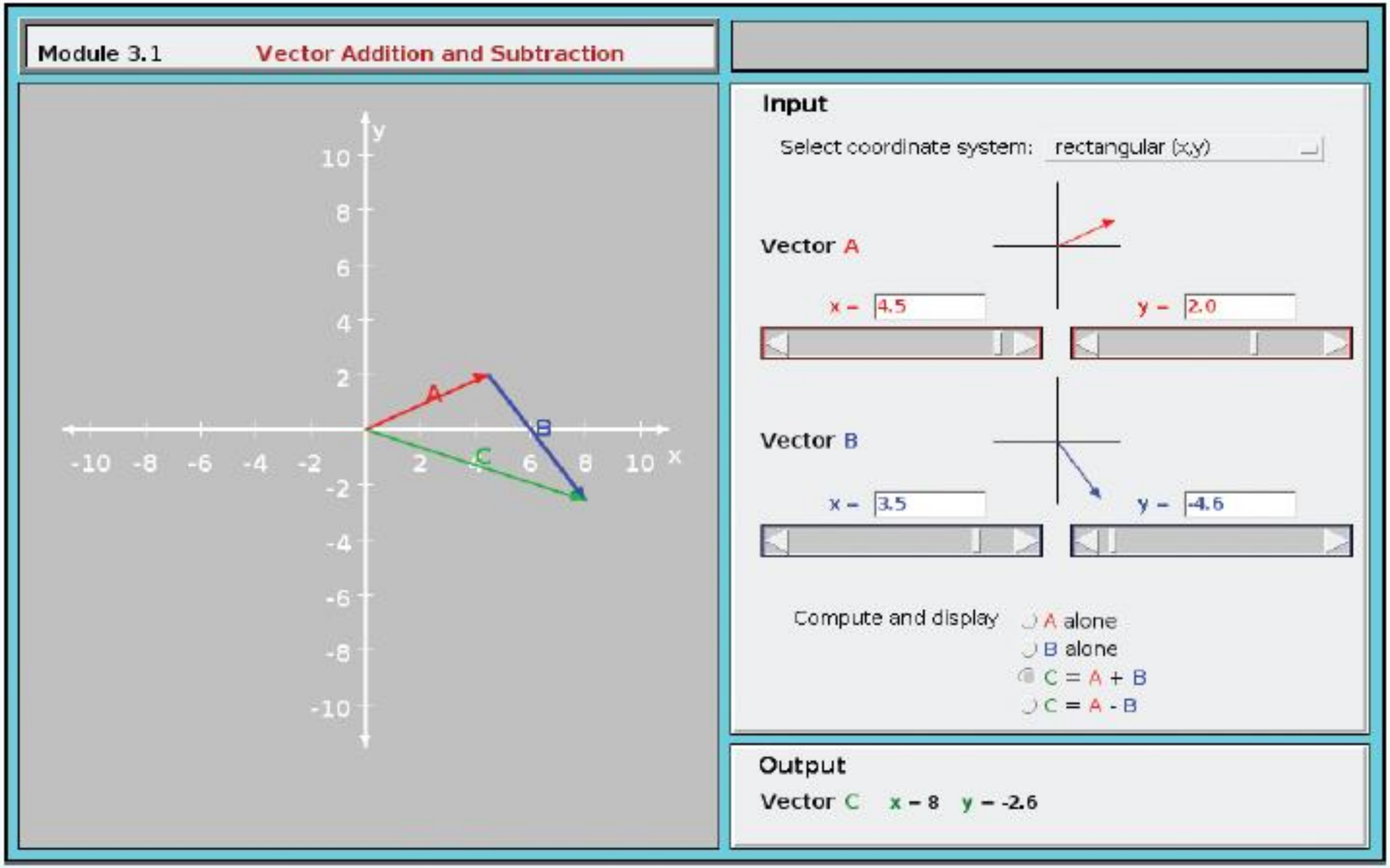


Figure 3-3: Vector addition by (a) the parallelogram rule and (b) the head-to-tail rule.

Module 3.1 Vector Addition and Subtraction Display two vectors in rectangular or cylindrical coordinates, and compute their sum and difference.



Position & Distance Vectors

Position Vector: From origin to point P

$$\mathbf{R}_1 = \overrightarrow{OP_1} = \hat{\mathbf{x}}x_1 + \hat{\mathbf{y}}y_1 + \hat{\mathbf{z}}z_1$$

$$\mathbf{R}_2 = \overrightarrow{OP_2} = \hat{\mathbf{x}}x_2 + \hat{\mathbf{y}}y_2 + \hat{\mathbf{z}}z_2$$

Distance Vector: Between two points

$$\mathbf{R}_{12} = \overrightarrow{P_1P_2} = \mathbf{R}_2 - \mathbf{R}_1 = \hat{\mathbf{x}}(x_2 - x_1) + \hat{\mathbf{y}}(y_2 - y_1) + \hat{\mathbf{z}}(z_2 - z_1)$$

the distance d between P_1 and P_2 equals the magnitude of $\mathbf{R}_{12}$:

$$d = |\mathbf{R}_{12}| = [(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]^{1/2}. \qquad (3.12)$$

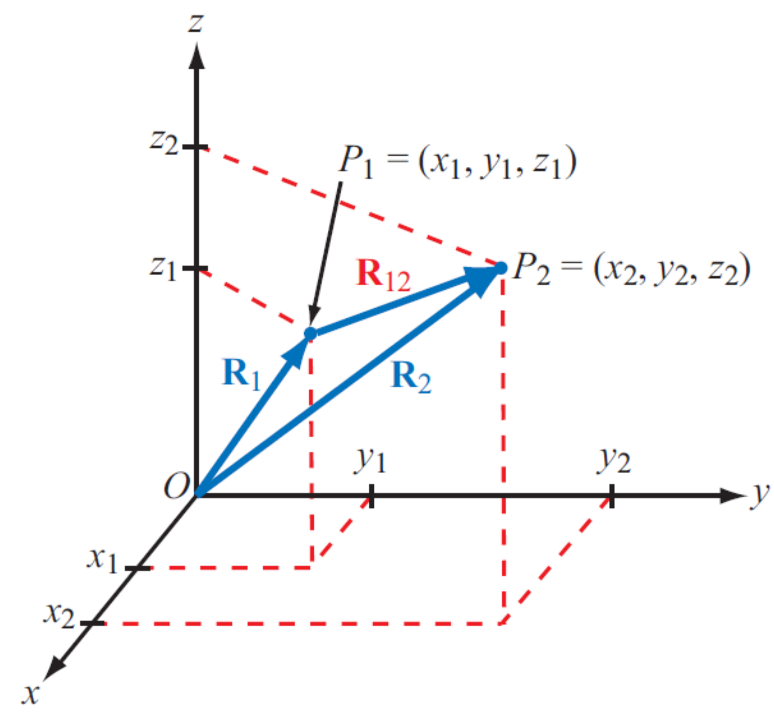


Figure 3-4: Distance vector $\mathbf{R}_{12} = \overrightarrow{P_1P_2} = \mathbf{R}_2 - \mathbf{R}_1$, where $\mathbf{R}_1$ and $\mathbf{R}_2$ are the position vectors of points P_1 and P_2, respectively.

Vector Multiplication: Scalar Product or "Dot Product"

$$\mathbf{A} \cdot \mathbf{B} = AB\cos\theta_{AB}$$

$$A = |\mathbf{A}| = \sqrt[+]{\mathbf{A} \cdot \mathbf{A}}$$

$$\theta_{AB} = \cos^{-1}\left[\frac{\mathbf{A} \cdot \mathbf{B}}{\sqrt[+]{\mathbf{A} \cdot \mathbf{A}}\ \sqrt[+]{\mathbf{B} \cdot \mathbf{B}}}\right]$$

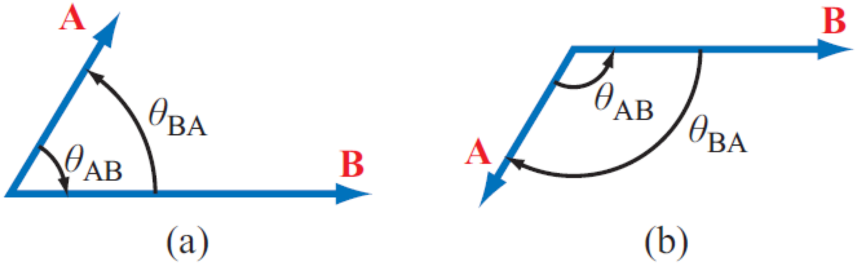


Figure 3-5: The angle θ_{AB} is the angle between **A** and **B**, measured from **A** to **B** between vector tails. The dot product is positive if $0 \leq \theta_{AB} < 90°$, as in (a), and it is negative if $90° < \theta_{AB} \leq 180°$, as in (b).

$$\hat{\mathbf{x}} \cdot \hat{\mathbf{x}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{z}} = 1,$$

$$\hat{\mathbf{x}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{z}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{x}} = 0.$$

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} \quad \text{(commutative property)},$$

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C} \quad \text{(distributive property)}$$

If $\mathbf{A} = (A_x, A_y, A_z)$ and $\mathbf{B} = (B_x, B_y, B_z)$, then

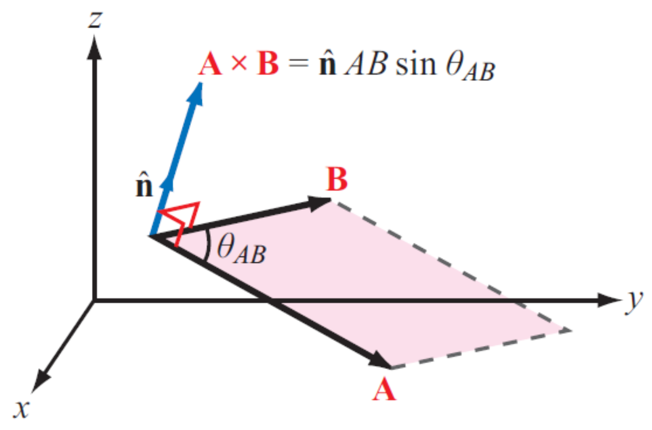
$$\mathbf{A} \cdot \mathbf{B} = (\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z) \cdot (\hat{\mathbf{x}}B_x + \hat{\mathbf{y}}B_y + \hat{\mathbf{z}}B_z).$$

Hence:

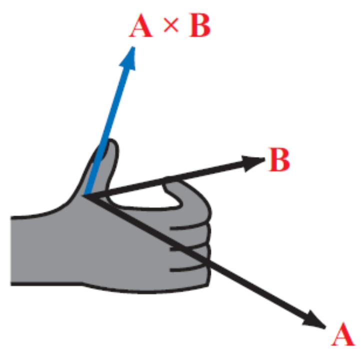
$$\mathbf{A} \cdot \mathbf{B} = A_xB_x + A_yB_y + A_zB_z.$$

Vector Multiplication: Vector Product or "Cross Product"

$$\mathbf{A} \times \mathbf{B} = \hat{\mathbf{n}}\, AB \sin\theta_{AB}$$



(a) Cross product



(b) Right-hand rule

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A} \qquad \text{(anticommutative)}$$

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C} \qquad \text{(distributive)}$$

$$\mathbf{A} \times \mathbf{A} = 0$$

$$\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}}, \qquad \hat{\mathbf{y}} \times \hat{\mathbf{z}} = \hat{\mathbf{x}}, \qquad \hat{\mathbf{z}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}}. \qquad (3.25)$$

Note the cyclic order $(xyzxyz\ldots)$. Also,

$$\hat{\mathbf{x}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}} \times \hat{\mathbf{z}} = 0. \qquad (3.26)$$

If $\mathbf{A} = (A_x, A_y, A_z)$ and $\mathbf{B} = (B_x, B_y, B_z)$,

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}.$$

Example 3-1: Vectors and Angles

In Cartesian coordinates, vector **A** points from the origin to point $P_1 = (2, 3, 3)$, and vector **B** is directed from P_1 to point $P_2 = (1, -2, 2)$. Find

(a) vector **A**, its magnitude A, and unit vector $\hat{\mathbf{a}}$,
(b) the angle between **A** and the y-axis,
(c) vector **B**,
(d) the angle θ_{AB} between **A** and **B**, and
(e) the perpendicular distance from the origin to vector **B**.

Solution: (a) Vector **A** is given by the position vector of $P_1 = (2, 3, 3)$ as shown in Fig. 3-7. Thus,

$$\mathbf{A} = \hat{\mathbf{x}}2 + \hat{\mathbf{y}}3 + \hat{\mathbf{z}}3,$$

$$A = |\mathbf{A}| = \sqrt{2^2 + 3^2 + 3^2} = \sqrt{22}\,,$$

$$\hat{\mathbf{a}} = \frac{\mathbf{A}}{A} = (\hat{\mathbf{x}}2 + \hat{\mathbf{y}}3 + \hat{\mathbf{z}}3)/\sqrt{22}\,.$$

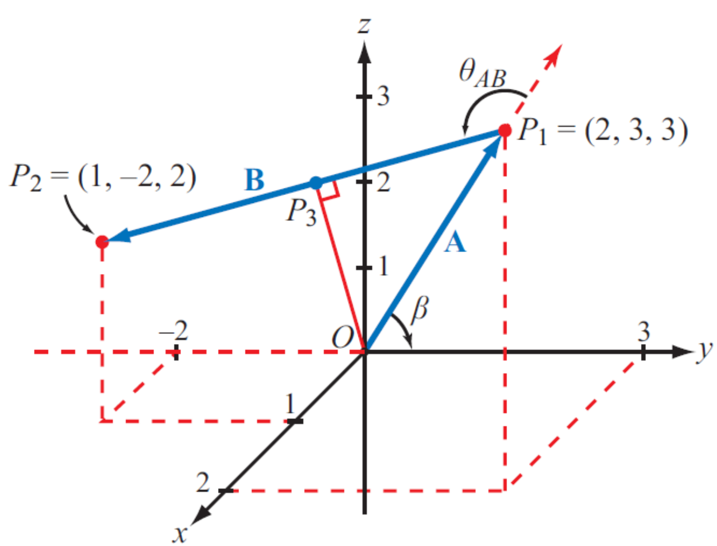


Figure 3-7: Geometry of Example 3-1.

(b) The angle β between **A** and the y-axis is obtained from

$$\mathbf{A} \cdot \hat{\mathbf{y}} = |\mathbf{A}||\hat{\mathbf{y}}| \cos\beta = A\cos\beta,$$

or

$$\beta = \cos^{-1}\left(\frac{\mathbf{A}\cdot\hat{\mathbf{y}}}{A}\right) = \cos^{-1}\left(\frac{3}{\sqrt{22}}\right) = 50.2^\circ.$$

(c)

$$\mathbf{B} = \hat{\mathbf{x}}(1-2) + \hat{\mathbf{y}}(-2-3) + \hat{\mathbf{z}}(2-3) = -\hat{\mathbf{x}} - \hat{\mathbf{y}}5 - \hat{\mathbf{z}}.$$

(d)

$$\theta_{AB} = \cos^{-1}\left[\frac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{A}||\mathbf{B}|}\right] = \cos^{-1}\left[\frac{(-2-15-3)}{\sqrt{22}\,\sqrt{27}}\right] = 145.1^\circ.$$

(e) The perpendicular distance between the origin and vector **B** is the distance $|\overrightarrow{OP_3}|$ shown in Fig. 3-7. From right triangle OP_1P_3,

$$|\overrightarrow{OP_3}| = |\mathbf{A}|\sin(180^\circ - \theta_{AB})$$
$$= \sqrt{22}\,\sin(180^\circ - 145.1^\circ) = 2.68.$$

Triple Products

Scalar Triple Product

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B}).$$

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

Vector Triple Product

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B}),$$

which is known as the "bac-cab" rule.

Example 3-2: Vector Triple Product

Given $\mathbf{A} = \hat{\mathbf{x}} - \hat{\mathbf{y}} + \hat{\mathbf{z}}2$, $\mathbf{B} = \hat{\mathbf{y}} + \hat{\mathbf{z}}$, and $\mathbf{C} = -\hat{\mathbf{x}}2 + \hat{\mathbf{z}}3$, find $(\mathbf{A}\times\mathbf{B})\times\mathbf{C}$ and compare it with $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$.

Solution:

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 1 & -1 & 2 \\ 0 & 1 & 1 \end{vmatrix} = -\hat{\mathbf{x}}3 - \hat{\mathbf{y}} + \hat{\mathbf{z}}$$

and

$$(\mathbf{A}\times\mathbf{B})\times\mathbf{C} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ -3 & -1 & 1 \\ -2 & 0 & 3 \end{vmatrix} = -\hat{\mathbf{x}}3 + \hat{\mathbf{y}}7 - \hat{\mathbf{z}}2.$$

A similar procedure gives $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \hat{\mathbf{x}}2 + \hat{\mathbf{y}}4 + \hat{\mathbf{z}}$.

Hence:

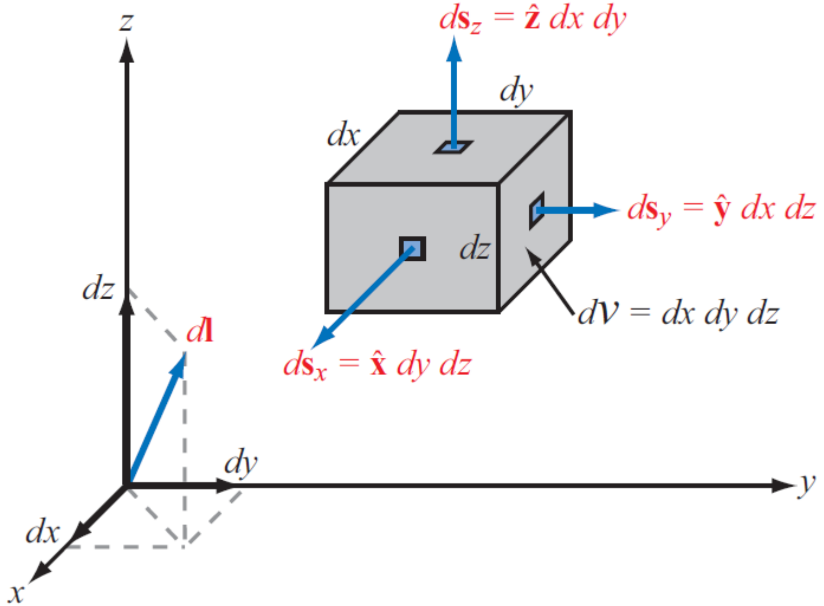
$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) \neq (\mathbf{A}\times\mathbf{B})\times\mathbf{C}$$

Cartesian Coordinate System

Differential length vector

$$d\mathbf{l} = \hat{\mathbf{x}}\, dl_x + \hat{\mathbf{y}}\, dl_y + \hat{\mathbf{z}}\, dl_z = \hat{\mathbf{x}}\, dx + \hat{\mathbf{y}}\, dy + \hat{\mathbf{z}}\, dz, \quad (3.34)$$

where $dl_x = dx$ is a differential length along $\hat{\mathbf{x}}$, and similar interpretations apply to $dl_y = dy$ and $dl_z = dz$.



Differential area vectors

$$d\mathbf{s}_x = \hat{\mathbf{x}}\, dl_y\, dl_z = \hat{\mathbf{x}}\, dy\, dz \qquad (y\text{–}z \text{ plane}), \qquad (3.35a)$$

with the subscript on $d\mathbf{s}$ denoting its direction. Similarly,

$$d\mathbf{s}_y = \hat{\mathbf{y}}\, dx\, dz \qquad (x\text{–}z \text{ plane}), \qquad (3.35b)$$

$$d\mathbf{s}_z = \hat{\mathbf{z}}\, dx\, dy \qquad (x\text{–}y \text{ plane}). \qquad (3.35c)$$

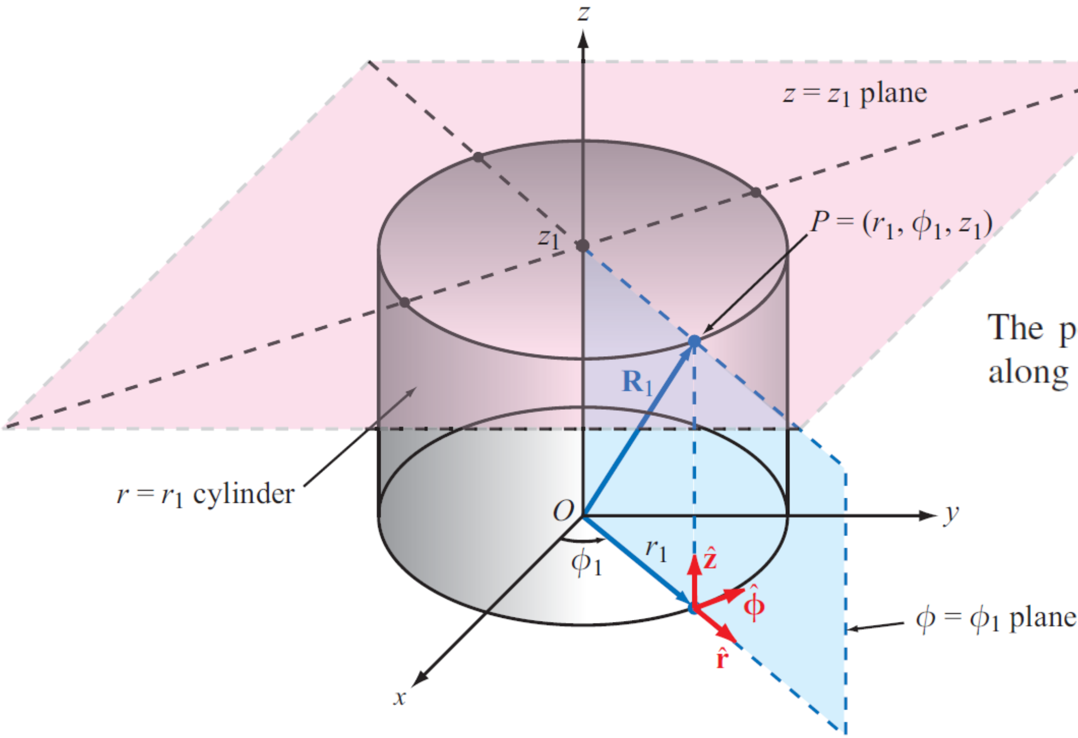
A ***differential volume*** equals the product of all three differential lengths:

$$dV = dx\, dy\, dz. \qquad (3.36)$$

Table 3-1: Summary of vector relations.

	Cartesian Coordinates	Cylindrical Coordinates	Spherical Coordinates
Coordinate variables	x, y, z	r, ϕ, z	R, θ, ϕ
Vector representation $\mathbf{A} =$	$\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$	$\hat{\mathbf{r}}A_r + \hat{\boldsymbol{\phi}}A_\phi + \hat{\mathbf{z}}A_z$	$\hat{\mathbf{R}}A_R + \hat{\boldsymbol{\theta}}A_\theta + \hat{\boldsymbol{\phi}}A_\phi$
Magnitude of A $\lvert\mathbf{A}\rvert =$	$\sqrt[+]{A_x^2 + A_y^2 + A_z^2}$	$\sqrt[+]{A_r^2 + A_\phi^2 + A_z^2}$	$\sqrt[+]{A_R^2 + A_\theta^2 + A_\phi^2}$
Position vector $\overrightarrow{OP_1} =$	$\hat{\mathbf{x}}x_1 + \hat{\mathbf{y}}y_1 + \hat{\mathbf{z}}z_1$, for $P = (x_1, y_1, z_1)$	$\hat{\mathbf{r}}r_1 + \hat{\mathbf{z}}z_1$, for $P = (r_1, \phi_1, z_1)$	$\hat{\mathbf{R}}R_1$, for $P = (R_1, \theta_1, \phi_1)$
Base vectors properties	$\hat{\mathbf{x}}\cdot\hat{\mathbf{x}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{z}} = 1$ $\hat{\mathbf{x}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{z}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{x}} = 0$ $\hat{\mathbf{x}}\times\hat{\mathbf{y}} = \hat{\mathbf{z}}$ $\hat{\mathbf{y}}\times\hat{\mathbf{z}} = \hat{\mathbf{x}}$ $\hat{\mathbf{z}}\times\hat{\mathbf{x}} = \hat{\mathbf{y}}$	$\hat{\mathbf{r}}\cdot\hat{\mathbf{r}} = \hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{z}} = 1$ $\hat{\mathbf{r}}\cdot\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}\cdot\hat{\mathbf{z}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{r}} = 0$ $\hat{\mathbf{r}}\times\hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}$ $\hat{\boldsymbol{\phi}}\times\hat{\mathbf{z}} = \hat{\mathbf{r}}$ $\hat{\mathbf{z}}\times\hat{\mathbf{r}} = \hat{\boldsymbol{\phi}}$	$\hat{\mathbf{R}}\cdot\hat{\mathbf{R}} = \hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\phi}} = 1$ $\hat{\mathbf{R}}\cdot\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}\cdot\hat{\mathbf{R}} = 0$ $\hat{\mathbf{R}}\times\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}$ $\hat{\boldsymbol{\theta}}\times\hat{\boldsymbol{\phi}} = \hat{\mathbf{R}}$ $\hat{\boldsymbol{\phi}}\times\hat{\mathbf{R}} = \hat{\boldsymbol{\theta}}$
Dot product $\mathbf{A}\cdot\mathbf{B} =$	$A_xB_x + A_yB_y + A_zB_z$	$A_rB_r + A_\phi B_\phi + A_zB_z$	$A_RB_R + A_\theta B_\theta + A_\phi B_\phi$
Cross product $\mathbf{A}\times\mathbf{B} =$	$\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\phi}} & \hat{\mathbf{z}} \\ A_r & A_\phi & A_z \\ B_r & B_\phi & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{R}} & \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\phi}} \\ A_R & A_\theta & A_\phi \\ B_R & B_\theta & B_\phi \end{vmatrix}$
Differential length $d\mathbf{l} =$	$\hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz$	$\hat{\mathbf{r}}\,dr + \hat{\boldsymbol{\phi}}r\,d\phi + \hat{\mathbf{z}}\,dz$	$\hat{\mathbf{R}}\,dR + \hat{\boldsymbol{\theta}}R\,d\theta + \hat{\boldsymbol{\phi}}R\sin\theta\,d\phi$
Differential surface areas	$d\mathbf{s}_x = \hat{\mathbf{x}}\,dy\,dz$ $d\mathbf{s}_y = \hat{\mathbf{y}}\,dx\,dz$ $d\mathbf{s}_z = \hat{\mathbf{z}}\,dx\,dy$	$d\mathbf{s}_r = \hat{\mathbf{r}}r\,d\phi\,dz$ $d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}\,dr\,dz$ $d\mathbf{s}_z = \hat{\mathbf{z}}r\,dr\,d\phi$	$d\mathbf{s}_R = \hat{\mathbf{R}}R^2\sin\theta\,d\theta\,d\phi$ $d\mathbf{s}_\theta = \hat{\boldsymbol{\theta}}R\sin\theta\,dR\,d\phi$ $d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}R\,dR\,d\theta$
Differential volume $dV =$	$dx\,dy\,dz$	$r\,dr\,d\phi\,dz$	$R^2\sin\theta\,dR\,d\theta\,d\phi$

Cylindrical Coordinate System



The position vector $\overrightarrow{OP}$ shown in Fig. 3-9 has components along r and z only. Thus,

$$\mathbf{R}_1 = \overrightarrow{OP} = \hat{\mathbf{r}} r_1 + \hat{\mathbf{z}} z_1. \quad (3.40)$$

The mutually perpendicular base vectors are $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\phi}}$, and $\hat{\mathbf{z}}$, with $\hat{\mathbf{r}}$ pointing away from the origin along r, $\hat{\boldsymbol{\phi}}$ pointing in a direction tangential to the cylindrical surface, and $\hat{\mathbf{z}}$ pointing along the vertical. Unlike the Cartesian system, in which the base vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ are independent of the location of P, in the cylindrical system both $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\phi}}$ are functions of ϕ.

Cylindrical Coordinate System

The base unit vectors obey the following right-hand cyclic relations:

$$\hat{\mathbf{r}} \times \hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}, \qquad \hat{\boldsymbol{\phi}} \times \hat{\mathbf{z}} = \hat{\mathbf{r}}, \qquad \hat{\mathbf{z}} \times \hat{\mathbf{r}} = \hat{\boldsymbol{\phi}}, \qquad (3.37)$$

and like all unit vectors, $\hat{\mathbf{r}} \cdot \hat{\mathbf{r}} = \hat{\boldsymbol{\phi}} \cdot \hat{\boldsymbol{\phi}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{z}} = 1$, and $\hat{\mathbf{r}} \times \hat{\mathbf{r}} = \hat{\boldsymbol{\phi}} \times \hat{\boldsymbol{\phi}} = \hat{\mathbf{z}} \times \hat{\mathbf{z}} = 0$.

In cylindrical coordinates, a vector is expressed as

$$\mathbf{A} = \hat{\mathbf{a}}|\mathbf{A}| = \hat{\mathbf{r}}A_r + \hat{\boldsymbol{\phi}}A_\phi + \hat{\mathbf{z}}A_z, \qquad (3.38)$$

$$dl_r = dr, \qquad dl_\phi = r\,d\phi, \qquad dl_z = dz. \qquad (3.41)$$

Note that the differential length along $\hat{\boldsymbol{\phi}}$ is $r\,d\phi$, not just $d\phi$. The differential length $d\mathbf{l}$ in cylindrical coordinates is given by

$$d\mathbf{l} = \hat{\mathbf{r}}\,dl_r + \hat{\boldsymbol{\phi}}\,dl_\phi + \hat{\mathbf{z}}\,dl_z = \hat{\mathbf{r}}\,dr + \hat{\boldsymbol{\phi}}r\,d\phi + \hat{\mathbf{z}}\,dz. \qquad (3.42)$$

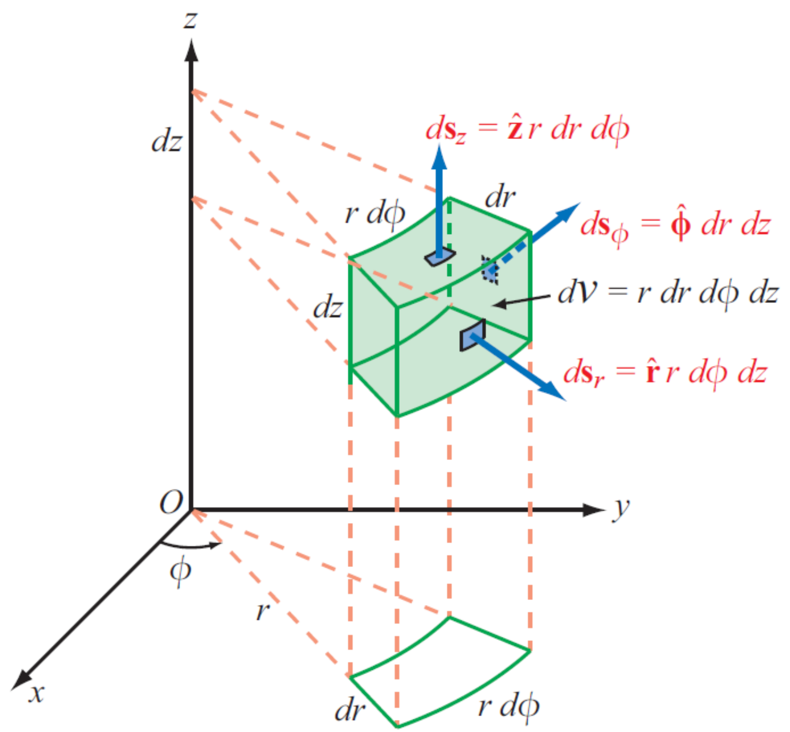


Figure 3-10: Differential areas and volume in cylindrical coordinates.

Example 3-3: Distance Vector in Cylindrical Coordinates

Find an expression for the unit vector of vector **A** shown in Fig. 3-11 in cylindrical coordinates.

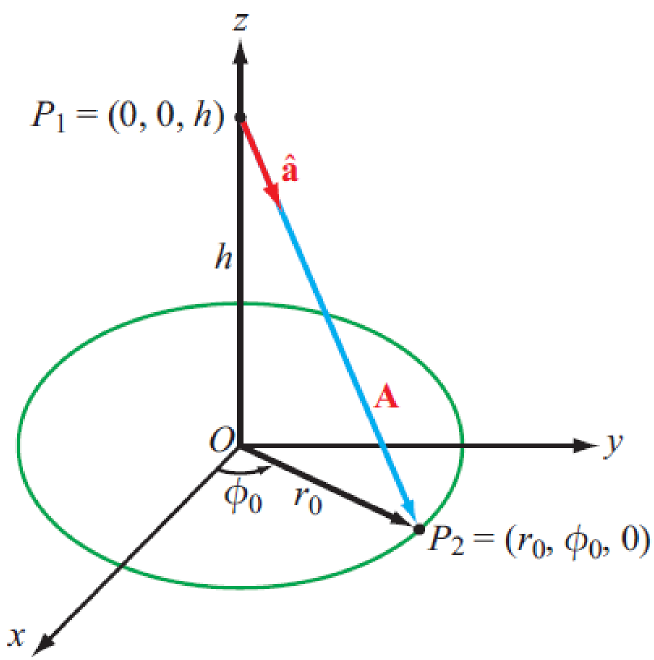


Figure 3-11: Geometry of Example 3-3.

Solution: In triangle OP_1P_2,

$$\overrightarrow{OP_2} = \overrightarrow{OP_1} + \mathbf{A}.$$

Hence,

$$\mathbf{A} = \overrightarrow{OP_2} - \overrightarrow{OP_1} = \hat{\mathbf{r}} r_0 - \hat{\mathbf{z}} h,$$

and

$$\hat{\mathbf{a}} = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{\hat{\mathbf{r}} r_0 - \hat{\mathbf{z}} h}{\sqrt{r_0^2 + h^2}}.$$

We note that the expression for **A** is independent of ϕ_0. That is, all vectors from point P_1 to any point on the circle defined by $r = r_0$ in the x–y plane are equal in the cylindrical coordinate system. The ambiguity can be eliminated by specifying that **A** passes through a point whose $\phi = \phi_0$.

Example 3-4: Cylindrical Area

Find the area of a cylindrical surface described by $r = 5$, $30° \leq \phi \leq 60°$, and $0 \leq z \leq 3$ (Fig. 3-12).

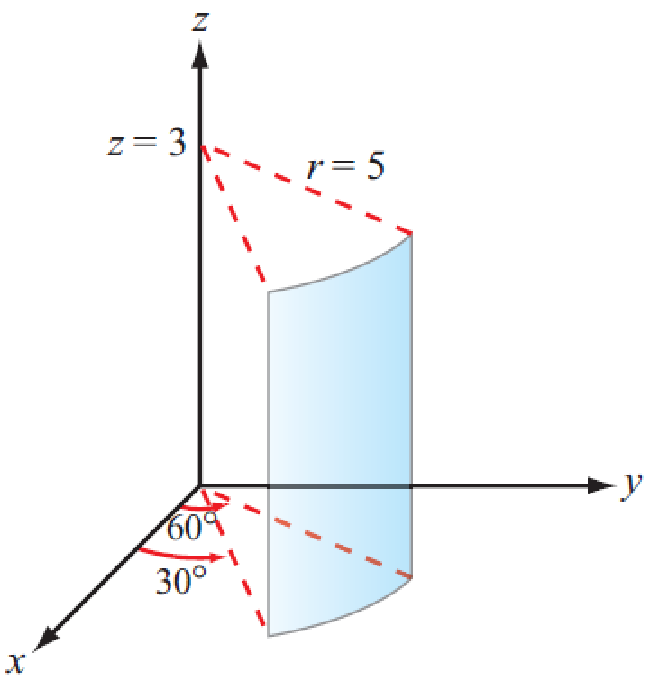


Figure 3-12: Cylindrical surface of Example 3-4.

Solution: The prescribed surface is shown in Fig. 3-12. Use of Eq. (3.43a) for a surface element with constant r gives

$$
\begin{aligned}
S &= r \int_{\phi=30°}^{60°} d\phi \int_{z=0}^{3} dz \\
&= 5\phi\Big|_{\pi/6}^{\pi/3} \; z\Big|_{0}^{3} \\
&= \frac{5\pi}{2} .
\end{aligned}
$$

Note that ϕ had to be converted to radians before evaluating the integration limits.

Spherical Coordinate System

$$\hat{\mathbf{R}} \times \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}, \quad \hat{\boldsymbol{\theta}} \times \hat{\boldsymbol{\phi}} = \hat{\mathbf{R}}, \quad \hat{\boldsymbol{\phi}} \times \hat{\mathbf{R}} = \hat{\boldsymbol{\theta}}. \qquad (3.45)$$

A vector with components A_R, A_θ, and A_ϕ is written as

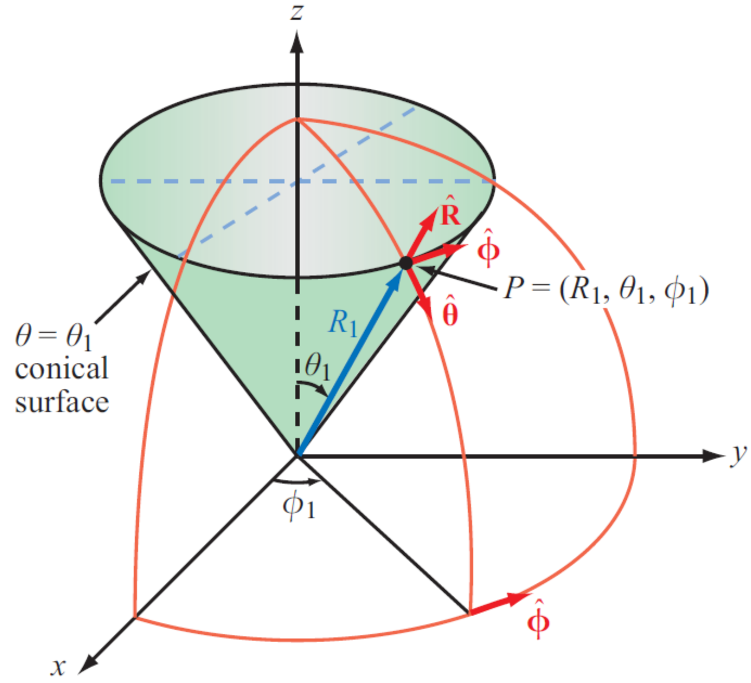
$$\mathbf{A} = \hat{\mathbf{a}}|\mathbf{A}| = \hat{\mathbf{R}}A_R + \hat{\boldsymbol{\theta}}A_\theta + \hat{\boldsymbol{\phi}}A_\phi, \qquad (3.46)$$

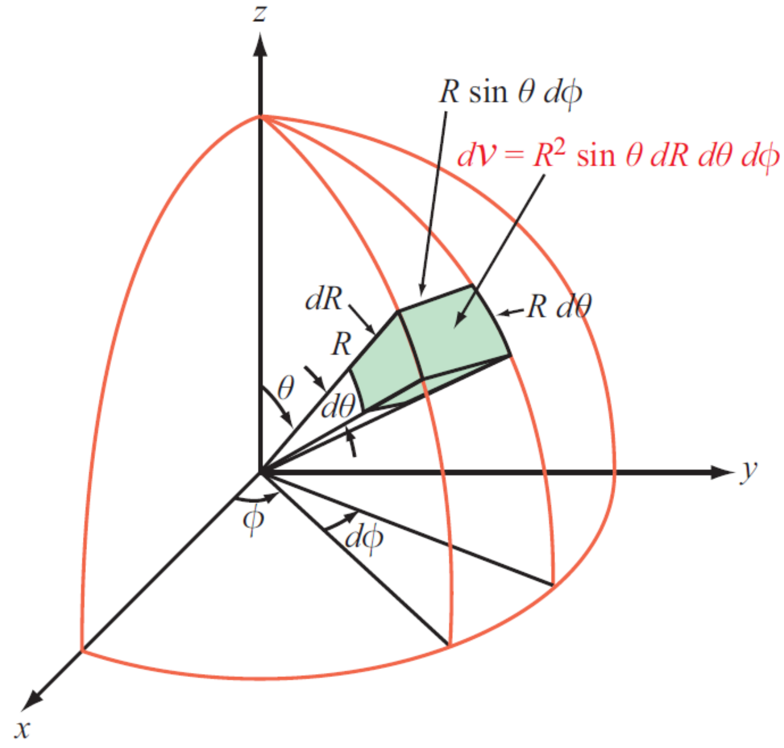
and its magnitude is

$$|\mathbf{A}| = \sqrt[+]{\mathbf{A} \cdot \mathbf{A}} = \sqrt[+]{A_R^2 + A_\theta^2 + A_\phi^2}. \qquad (3.47)$$

The position vector of point $P = (R_1, \theta_1, \phi_1)$ is simply

$$\mathbf{R}_1 = \overrightarrow{OP} = \hat{\mathbf{R}}R_1, \qquad (3.48)$$





Example 3-5: Surface Area in Spherical Coordinates

The spherical strip shown in Fig. 3-15 is a section of a sphere of radius 3 cm. Find the area of the strip.

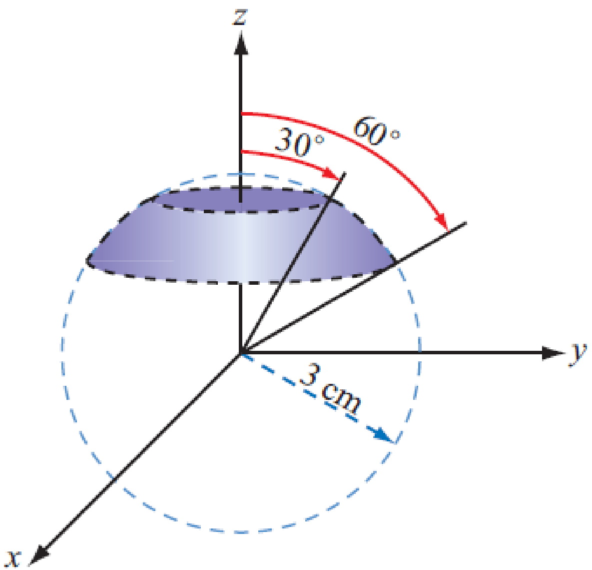


Figure 3-15: Spherical strip of Example 3-5.

Solution: Use of Eq. (3.50b) for the area of an elemental spherical area with constant radius R gives

$$S = R^2 \int_{\theta=30^\circ}^{60^\circ} \sin\theta \, d\theta \int_{\phi=0}^{2\pi} d\phi$$

$$= 9(-\cos\theta)\Big|_{30^\circ}^{60^\circ} \; \phi\Big|_0^{2\pi} \qquad (\text{cm}^2)$$

$$= 18\pi(\cos 30^\circ - \cos 60^\circ) = 20.7 \text{ cm}^2.$$

Example 3-6: Charge in a Sphere

A sphere of radius 2 cm contains a volume charge density ρ_v given by

$$\rho_v = 4\cos^2\theta \qquad (\text{C/m}^3).$$

Find the total charge Q contained in the sphere.

Solution:

$$Q = \int_{\mathcal{V}} \rho_v \, d\mathcal{V}$$

$$= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{R=0}^{2\times10^{-2}} (4\cos^2\theta) R^2 \sin\theta \, dR \, d\theta \, d\phi$$

$$= 4 \int_0^{2\pi} \int_0^{\pi} \left(\frac{R^3}{3}\right)\Big|_0^{2\times10^{-2}} \sin\theta\cos^2\theta \, d\theta \, d\phi$$

$$= \frac{32}{3} \times 10^{-6} \int_0^{2\pi} \left(-\frac{\cos^3\theta}{3}\right)\Big|_0^{\pi} d\phi$$

$$= \frac{64}{9} \times 10^{-6} \int_0^{2\pi} d\phi$$

$$= \frac{128\pi}{9} \times 10^{-6} = 44.68 \qquad (\mu\text{C}).$$

Technology Brief 5: GPS

Figure TF5-1: iPhone map feature.

Figure TF5-2: GPS nominal satellite constellation. Four satellites in each plane, 20,200 km altitudes, 55° inclination.

How does a GPS receiver determine its location?

GPS: Minimum of 4 Satellites Needed

Unknown: location of receiver (x_0, y_0, z_0)
Also unknown: time offset of receiver clock t_0

Quantities known with high precision: locations of satellites and their atomic clocks (satellites use expensive high precision clocks, whereas receivers do not)

Solving for 4 unknowns requires at least 4 equations (four satellites)

$$d_1^2 = (x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2 = c\,[(t_1 + t_0)]^2\,,$$
$$d_2^2 = (x_2 - x_0)^2 + (y_2 - y_0)^2 + (z_2 - z_0)^2 = c\,[(t_2 + t_0)]^2\,,$$
$$d_3^2 = (x_3 - x_0)^2 + (y_3 - y_0)^2 + (z_3 - z_0)^2 = c\,[(t_3 + t_0)]^2\,,$$
$$d_4^2 = (x_4 - x_0)^2 + (y_4 - y_0)^2 + (z_4 - z_0)^2 = c\,[(t_4 + t_0)]^2\,.$$

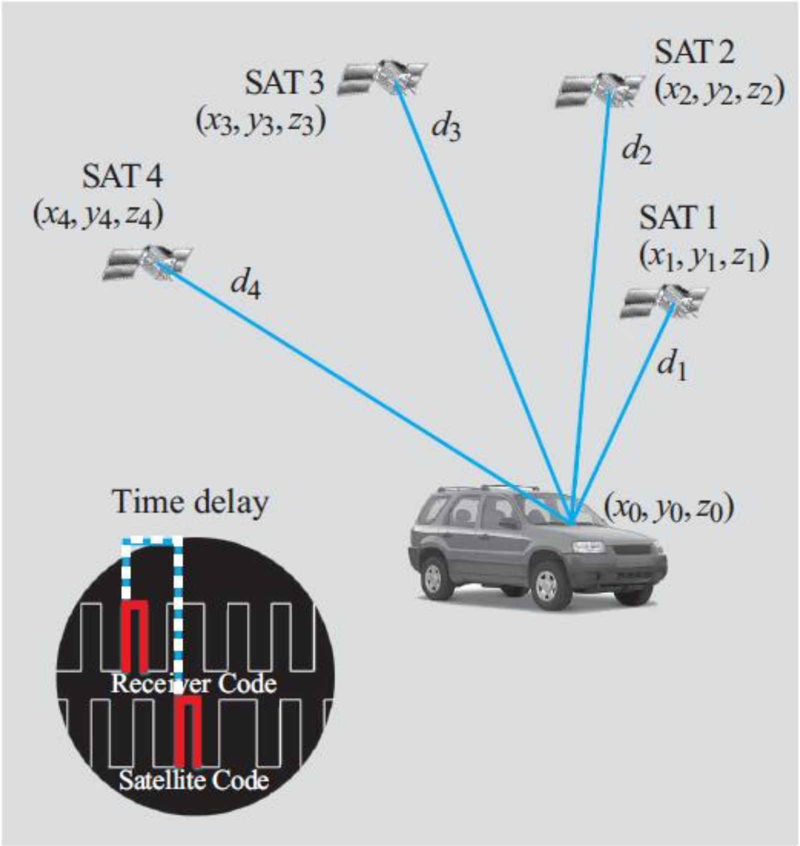
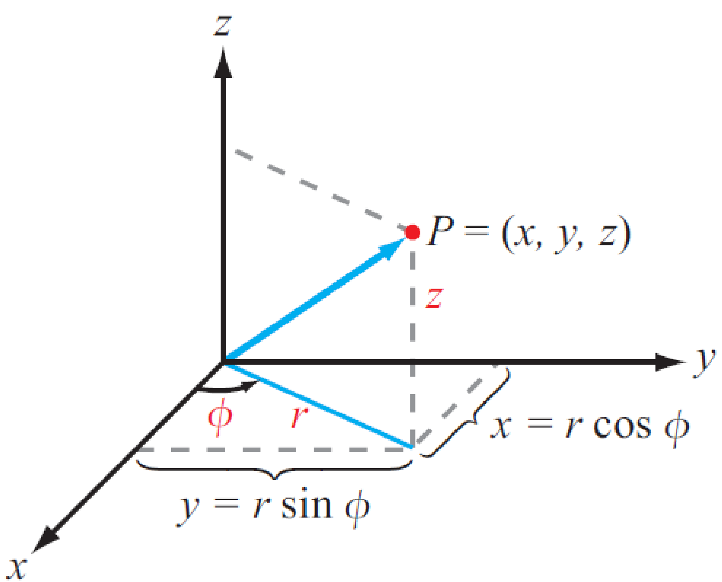


Figure TF5-3: Automobile GPS receiver at location (x_0, y_0, z_0).

Coordinate Transformations: Coordinates

- To solve a problem, we select the coordinate system that best fits its geometry
- Sometimes we need to transform between coordinate systems



$$r = \sqrt[+]{x^2 + y^2}, \qquad \phi = \tan^{-1}\left(\frac{y}{x}\right),$$

and the inverse relations are

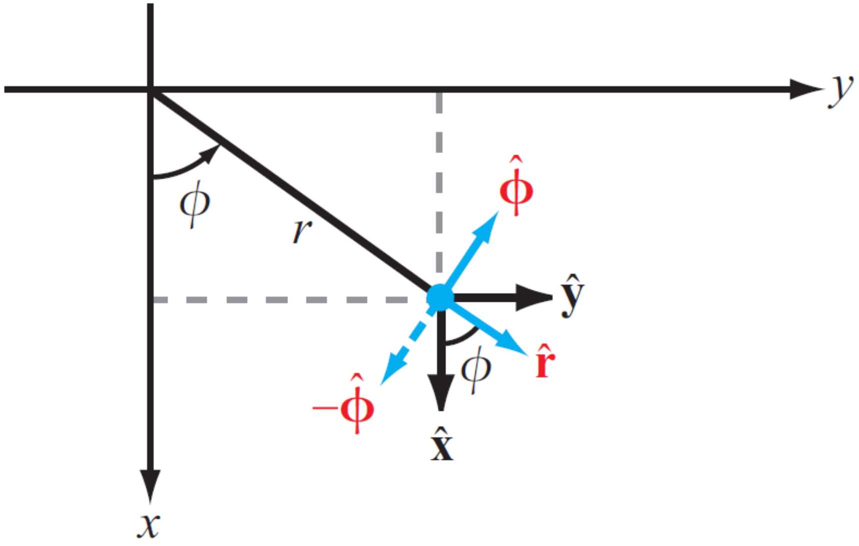
$$x = r\cos\phi, \qquad y = r\sin\phi.$$

Figure 3-16: Interrelationships between Cartesian coordinates (x, y, z) and cylindrical coordinates (r, ϕ, z).

Coordinate Transformations: Unit Vectors

$$\hat{\mathbf{r}} \cdot \hat{\mathbf{x}} = \cos\phi, \qquad \hat{\mathbf{r}} \cdot \hat{\mathbf{y}} = \sin\phi,$$
$$\hat{\boldsymbol{\phi}} \cdot \hat{\mathbf{x}} = -\sin\phi, \qquad \hat{\boldsymbol{\phi}} \cdot \hat{\mathbf{y}} = \cos\phi.$$

$$\hat{\mathbf{r}} = \hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi.$$
$$\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi.$$



$$\hat{\mathbf{x}} = \hat{\mathbf{r}}\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi,$$
$$\hat{\mathbf{y}} = \hat{\mathbf{r}}\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi.$$

Table 3-2: Coordinate transformation relations.

Transformation	Coordinate Variables	Unit Vectors	Vector Components
Cartesian to cylindrical	$r = \sqrt[+]{x^2 + y^2}$ $\phi = \tan^{-1}(y/x)$ $z = z$	$\hat{\mathbf{r}} = \hat{\mathbf{x}} \cos\phi + \hat{\mathbf{y}} \sin\phi$ $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}} \sin\phi + \hat{\mathbf{y}} \cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$A_r = A_x \cos\phi + A_y \sin\phi$ $A_\phi = -A_x \sin\phi + A_y \cos\phi$ $A_z = A_z$
Cylindrical to Cartesian	$x = r\cos\phi$ $y = r\sin\phi$ $z = z$	$\hat{\mathbf{x}} = \hat{\mathbf{r}} \cos\phi - \hat{\boldsymbol{\phi}} \sin\phi$ $\hat{\mathbf{y}} = \hat{\mathbf{r}} \sin\phi + \hat{\boldsymbol{\phi}} \cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$A_x = A_r \cos\phi - A_\phi \sin\phi$ $A_y = A_r \sin\phi + A_\phi \cos\phi$ $A_z = A_z$
Cartesian to spherical	$R = \sqrt[+]{x^2 + y^2 + z^2}$ $\theta = \tan^{-1}[\sqrt[+]{x^2 + y^2}/z]$ $\phi = \tan^{-1}(y/x)$	$\hat{\mathbf{R}} = \hat{\mathbf{x}} \sin\theta \cos\phi + \hat{\mathbf{y}} \sin\theta \sin\phi + \hat{\mathbf{z}} \cos\theta$ $\hat{\boldsymbol{\theta}} = \hat{\mathbf{x}} \cos\theta \cos\phi + \hat{\mathbf{y}} \cos\theta \sin\phi - \hat{\mathbf{z}} \sin\theta$ $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}} \sin\phi + \hat{\mathbf{y}} \cos\phi$	$A_R = A_x \sin\theta \cos\phi + A_y \sin\theta \sin\phi + A_z \cos\theta$ $A_\theta = A_x \cos\theta \cos\phi + A_y \cos\theta \sin\phi - A_z \sin\theta$ $A_\phi = -A_x \sin\phi + A_y \cos\phi$
Spherical to Cartesian	$x = R\sin\theta\cos\phi$ $y = R\sin\theta\sin\phi$ $z = R\cos\theta$	$\hat{\mathbf{x}} = \hat{\mathbf{R}} \sin\theta \cos\phi + \hat{\boldsymbol{\theta}} \cos\theta \cos\phi - \hat{\boldsymbol{\phi}} \sin\phi$ $\hat{\mathbf{y}} = \hat{\mathbf{R}} \sin\theta \sin\phi + \hat{\boldsymbol{\theta}} \cos\theta \sin\phi + \hat{\boldsymbol{\phi}} \cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{R}} \cos\theta - \hat{\boldsymbol{\theta}} \sin\theta$	$A_x = A_R \sin\theta \cos\phi + A_\theta \cos\theta \cos\phi - A_\phi \sin\phi$ $A_y = A_R \sin\theta \sin\phi + A_\theta \cos\theta \sin\phi + A_\phi \cos\phi$ $A_z = A_R \cos\theta - A_\theta \sin\theta$
Cylindrical to spherical	$R = \sqrt[+]{r^2 + z^2}$ $\theta = \tan^{-1}(r/z)$ $\phi = \phi$	$\hat{\mathbf{R}} = \hat{\mathbf{r}} \sin\theta + \hat{\mathbf{z}} \cos\theta$ $\hat{\boldsymbol{\theta}} = \hat{\mathbf{r}} \cos\theta - \hat{\mathbf{z}} \sin\theta$ $\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}$	$A_R = A_r \sin\theta + A_z \cos\theta$ $A_\theta = A_r \cos\theta - A_z \sin\theta$ $A_\phi = A_\phi$
Spherical to cylindrical	$r = R\sin\theta$ $\phi = \phi$ $z = R\cos\theta$	$\hat{\mathbf{r}} = \hat{\mathbf{R}} \sin\theta + \hat{\boldsymbol{\theta}} \cos\theta$ $\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}$ $\hat{\mathbf{z}} = \hat{\mathbf{R}} \cos\theta - \hat{\boldsymbol{\theta}} \sin\theta$	$A_r = A_R \sin\theta + A_\theta \cos\theta$ $A_\phi = A_\phi$ $A_z = A_R \cos\theta - A_\theta \sin\theta$

Example 3-7: Cartesian to Cylindrical Transformations

Given point $P_1 = (3, -4, 3)$ and vector $\mathbf{A} = \hat{\mathbf{x}}2 - \hat{\mathbf{y}}3 + \hat{\mathbf{z}}4$, defined in Cartesian coordinates, express P_1 and $\mathbf{A}$ in cylindrical coordinates and evaluate $\mathbf{A}$ at P_1.

Solution: For point P_1, $x = 3$, $y = -4$, and $z = 3$. Using Eq. (3.51), we have

$$r = \sqrt[+]{x^2 + y^2} = 5, \quad \phi = \tan^{-1}\frac{y}{x} = -53.1^\circ = 306.9^\circ,$$

and z remains unchanged. Hence, $P_1 = (5, 306.9^\circ, 3)$ in cylindrical coordinates.

The cylindrical components of vector $\mathbf{A} = \hat{\mathbf{r}}A_r + \hat{\boldsymbol{\phi}}A_\phi + \hat{\mathbf{z}}A_z$ can be determined by applying Eqs. (3.58a) and (3.58b):

$$A_r = A_x \cos\phi + A_y \sin\phi = 2\cos\phi - 3\sin\phi,$$

$$A_\phi = -A_x \sin\phi + A_y \cos\phi = -2\sin\phi - 3\cos\phi,$$

$$A_z = 4.$$

Hence,

$$\mathbf{A} = \hat{\mathbf{r}}(2\cos\phi - 3\sin\phi) - \hat{\boldsymbol{\phi}}(2\sin\phi + 3\cos\phi) + \hat{\mathbf{z}}4.$$

At point P, $\phi = 306.9^\circ$, which gives

$$\mathbf{A} = \hat{\mathbf{r}}3.60 - \hat{\boldsymbol{\phi}}0.20 + \hat{\mathbf{z}}4.$$

Example 3-8: Cartesian to Spherical Transformation

Express vector $\mathbf{A} = \hat{\mathbf{x}}(x+y) + \hat{\mathbf{y}}(y-x) + \hat{\mathbf{z}}z$ in spherical coordinates.

Solution: Using the transformation relation for A_R given in Table 3-2, we have

$$A_R = A_x \sin\theta\cos\phi + A_y \sin\theta\sin\phi + A_z\cos\theta$$
$$= (x+y)\sin\theta\cos\phi + (y-x)\sin\theta\sin\phi + z\cos\theta.$$

Using the expressions for x, y, and z given by Eq. (3.61c), we have

$$A_R = (R\sin\theta\cos\phi + R\sin\theta\sin\phi)\sin\theta\cos\phi$$
$$+ (R\sin\theta\sin\phi - R\sin\theta\cos\phi)\sin\theta\sin\phi + R\cos^2\theta$$
$$= R\sin^2\theta\,(\cos^2\phi + \sin^2\phi) + R\cos^2\theta$$
$$= R\sin^2\theta + R\cos^2\theta = R.$$

Similarly,

$$A_\theta = (x+y)\cos\theta\cos\phi + (y-x)\cos\theta\sin\phi - z\sin\theta,$$
$$A_\phi = -(x+y)\sin\phi + (y-x)\cos\phi,$$

Using the relations:

$$x = R\sin\theta\cos\phi,$$
$$y = R\sin\theta\sin\phi,$$
$$z = R\cos\theta.$$

leads to:

$$A_\theta = 0,$$
$$A_\phi = -R\sin\theta.$$

$$\mathbf{A} = \hat{\mathbf{R}}A_R + \hat{\boldsymbol{\theta}}A_\theta + \hat{\boldsymbol{\phi}}A_\phi = \hat{\mathbf{R}}R - \hat{\boldsymbol{\phi}}R\sin\theta.$$

Distance Between 2 Points

$$d = |\mathbf{R}_{12}| = [(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]^{1/2}. \quad (3.66)$$

$$d = \Big[(r_2 \cos\phi_2 - r_1 \cos\phi_1)^2 + (r_2 \sin\phi_2 - r_1 \sin\phi_1)^2 + (z_2 - z_1)^2\Big]^{1/2} = \Big[r_2^2 + r_1^2 - 2r_1 r_2 \cos(\phi_2 - \phi_1) + (z_2 - z_1)^2\Big]^{1/2} \quad \text{(cylindrical).} \quad (3.67)$$

$$d = \Big\{R_2^2 + R_1^2 - 2R_1 R_2[\cos\theta_2 \cos\theta_1 + \sin\theta_1 \sin\theta_2 \cos(\phi_2 - \phi_1)]\Big\}^{1/2} \quad \text{(spherical).} \quad (3.68)$$

Gradient of A Scalar Field

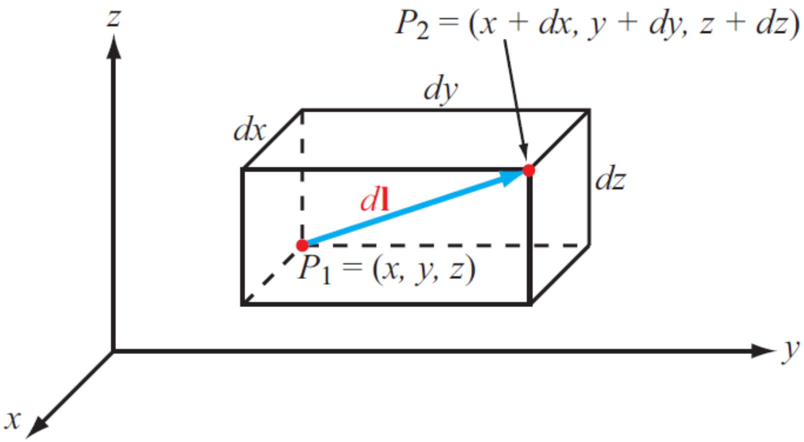


Figure 3-19: Differential distance vector $d\mathbf{l}$ between points P_1 and P_2.

From differential calculus, the temperature difference between points P_1 and P_2, $dT = T_2 - T_1$, is

$$dT = \frac{\partial T}{\partial x}\,dx + \frac{\partial T}{\partial y}\,dy + \frac{\partial T}{\partial z}\,dz. \qquad (3.70)$$

Because $dx = \hat{\mathbf{x}} \cdot d\mathbf{l}$, $dy = \hat{\mathbf{y}} \cdot d\mathbf{l}$, and $dz = \hat{\mathbf{z}} \cdot d\mathbf{l}$, Eq. (3.70) can be rewritten as

$$dT = \hat{\mathbf{x}}\frac{\partial T}{\partial x} \cdot d\mathbf{l} + \hat{\mathbf{y}}\frac{\partial T}{\partial y} \cdot d\mathbf{l} + \hat{\mathbf{z}}\frac{\partial T}{\partial z} \cdot d\mathbf{l}$$

$$= \left[\hat{\mathbf{x}}\frac{\partial T}{\partial x} + \hat{\mathbf{y}}\frac{\partial T}{\partial y} + \hat{\mathbf{z}}\frac{\partial T}{\partial z}\right] \cdot d\mathbf{l}. \qquad (3.71)$$

$$\nabla T = \text{grad } T = \hat{\mathbf{x}}\frac{\partial T}{\partial x} + \hat{\mathbf{y}}\frac{\partial T}{\partial y} + \hat{\mathbf{z}}\frac{\partial T}{\partial z}\,. \qquad (3.72)$$

Equation (3.71) can then be expressed as

$$dT = \nabla T \cdot d\mathbf{l}. \qquad (3.73)$$

The symbol ∇ is called the ***del*** or ***gradient operator*** and is defined as

$$\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z} \qquad \text{(Cartesian)}. \qquad (3.74)$$

Gradient (cont.)

With $d\mathbf{l} = \hat{\mathbf{a}}_l\, dl$, where $\hat{\mathbf{a}}_l$ is the unit vector of $d\mathbf{l}$, the ***directional derivative*** of T along $\hat{\mathbf{a}}_l$ is

$$\frac{dT}{dl} = \nabla T \cdot \hat{\mathbf{a}}_l. \qquad (3.75)$$

We can find the difference $(T_2 - T_1)$, where $T_1 = T(x_1, y_1, z_1)$ and $T_2 = T(x_2, y_2, z_2)$ are the values of T at points $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ not necessarily infinitesimally close to one another, by integrating both sides of Eq. (3.73). Thus,

$$T_2 - T_1 = \int_{P_1}^{P_2} \nabla T \cdot d\mathbf{l}. \qquad (3.76)$$

Example 3-9: Directional Derivative

Find the directional derivative of $T = x^2 + y^2 z$ along direction $\hat{\mathbf{x}}2 + \hat{\mathbf{y}}3 - \hat{\mathbf{z}}2$ and evaluate it at $(1, -1, 2)$.

Solution: First, we find the gradient of T:

$$\nabla T = \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right)(x^2 + y^2 z)$$
$$= \hat{\mathbf{x}}2x + \hat{\mathbf{y}}2yz + \hat{\mathbf{z}}y^2.$$

We denote $\mathbf{l}$ as the given direction,

$$\mathbf{l} = \hat{\mathbf{x}}2 + \hat{\mathbf{y}}3 - \hat{\mathbf{z}}2.$$

Its unit vector is

$$\hat{\mathbf{a}}_l = \frac{\mathbf{l}}{|\mathbf{l}|} = \frac{\hat{\mathbf{x}}2 + \hat{\mathbf{y}}3 - \hat{\mathbf{z}}2}{\sqrt{2^2 + 3^2 + 2^2}} = \frac{\hat{\mathbf{x}}2 + \hat{\mathbf{y}}3 - \hat{\mathbf{z}}2}{\sqrt{17}}.$$

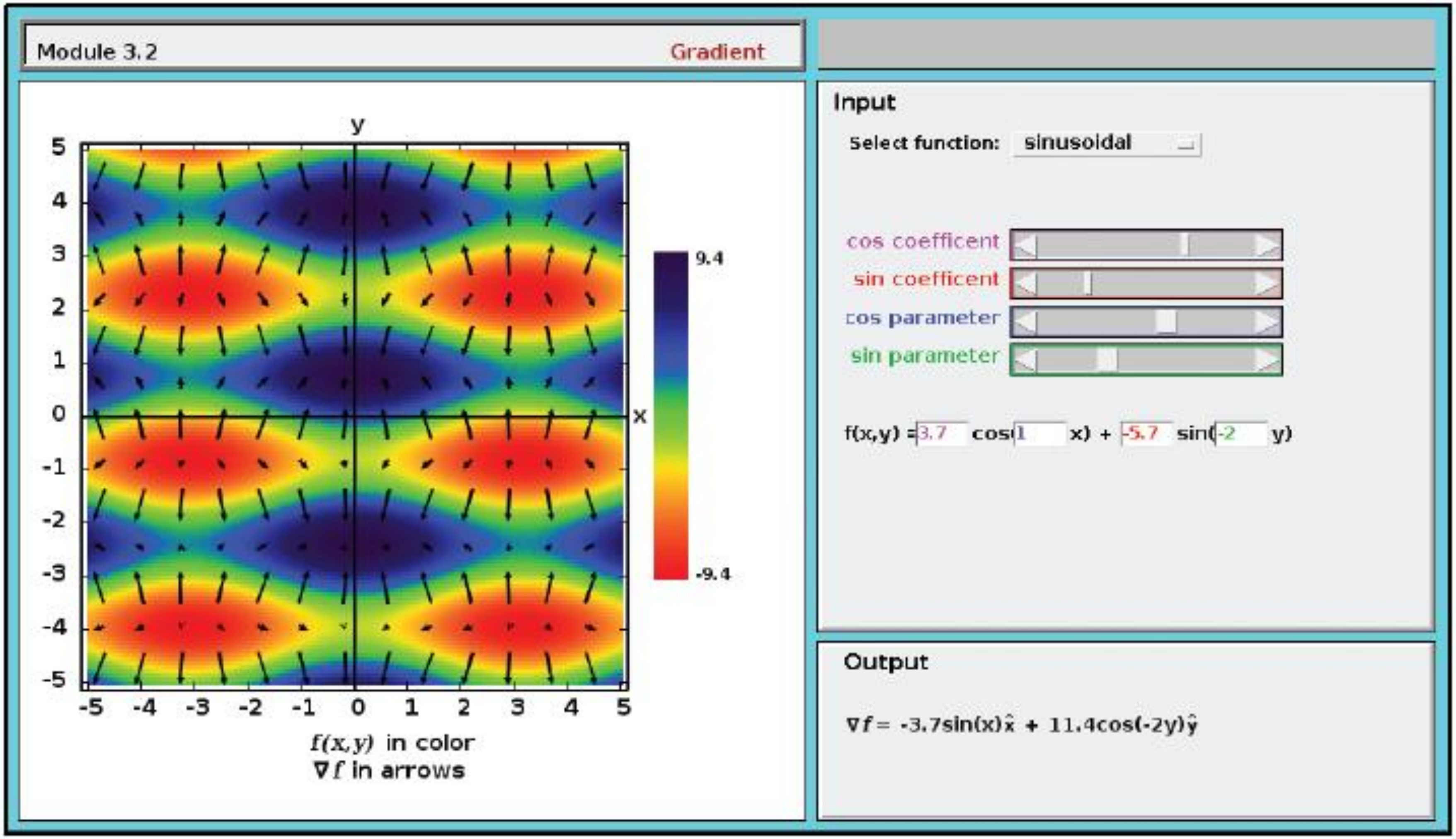
Application of Eq. (3.75) gives

$$\frac{dT}{dl} = \nabla T \cdot \hat{\mathbf{a}}_l = (\hat{\mathbf{x}}2x + \hat{\mathbf{y}}2yz + \hat{\mathbf{z}}y^2) \cdot \left(\frac{\hat{\mathbf{x}}2 + \hat{\mathbf{y}}3 - \hat{\mathbf{z}}2}{\sqrt{17}}\right)$$
$$= \frac{4x + 6yz - 2y^2}{\sqrt{17}}.$$

At $(1, -1, 2)$,

$$\left.\frac{dT}{dl}\right|_{(1,-1,2)} = \frac{4 - 12 - 2}{\sqrt{17}} = \frac{-10}{\sqrt{17}}.$$

Module 3.2 Gradient Select a scalar function $f(x, y, z)$, evaluate its gradient, and display both in an appropriate 2-D plane.



Divergence of a Vector Field

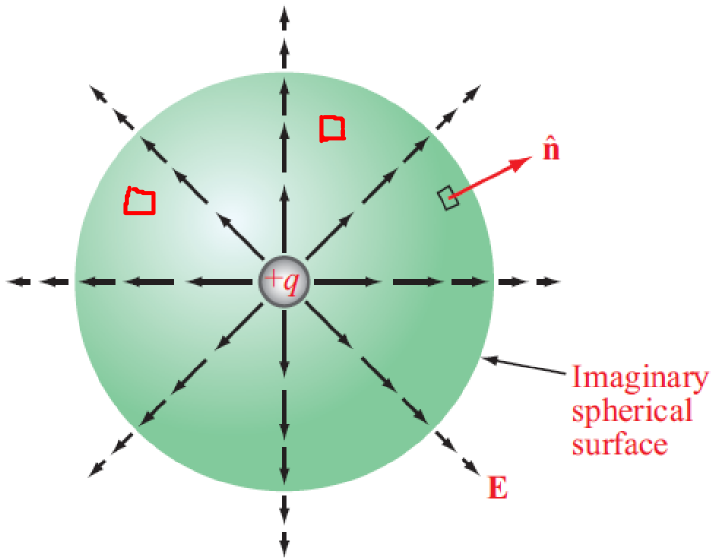


Figure 3-20: Flux lines of the electric field **E** due to a positive charge q.

At a surface boundary, ***flux density*** is defined as the amount of outward flux crossing a unit surface $d\mathbf{s}$:

$$\text{Flux density of } \mathbf{E} = \frac{\mathbf{E} \cdot d\mathbf{s}}{|d\mathbf{s}|} = \frac{\mathbf{E} \cdot \hat{\mathbf{n}}\, ds}{ds} = \mathbf{E} \cdot \hat{\mathbf{n}}, \qquad (3.85)$$

where $\hat{\mathbf{n}}$ is the normal to $d\mathbf{s}$. The ***total flux*** outwardly crossing a closed surface S, such as the enclosed surface of the imaginary sphere outlined in Fig. 3-20, is

$$\text{Total flux} = \oint_S \mathbf{E} \cdot d\mathbf{s}. \qquad (3.86)$$

$$\text{div}\, \mathbf{E} \triangleq \lim_{\Delta V \to 0} \frac{\oint_S \mathbf{E} \cdot d\mathbf{s}}{\Delta V}, \qquad (3.95)$$

where S encloses the elemental volume ΔV. Instead of denoting the divergence of **E** by div **E**, it is common practice to denote it as $\nabla \cdot \mathbf{E}$. That is,

$$\nabla \cdot \mathbf{E} = \text{div}\, \mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \qquad (3.96)$$

for a vector **E** in Cartesian coordinates.

From the definition of the divergence of **E** *given by Eq. (3.95), field* **E** *has positive divergence if the net flux out of surface S is positive, which may be "viewed" as if volume ΔV contains a* ***source*** *of field lines. If the divergence is negative, ΔV may be viewed as containing a* ***sink*** *of field lines because the net flux is into ΔV. For a uniform field* **E**, *the same amount of flux enters ΔV as leaves it; hence, its divergence is zero and the field is said to be* ***divergenceless***.

Divergence Theorem

$$\int_V \nabla \cdot \mathbf{E}\, dV = \oint_S \mathbf{E} \cdot d\mathbf{s} \qquad \text{(divergence theorem).} \tag{3.98}$$

Useful tool for converting integration over a volume to one over the surface enclosing that volume, and vice versa

Example 3-11: Calculating the Divergence

Determine the divergence of each of the following vector fields and then evaluate them at the indicated points:

(a) $\mathbf{E} = \hat{\mathbf{x}}3x^2 + \hat{\mathbf{y}}2z + \hat{\mathbf{z}}x^2 z$ at $(2, -2, 0)$;

(b) $\mathbf{E} = \hat{\mathbf{R}}(a^3 \cos\theta / R^2) - \hat{\boldsymbol{\theta}}(a^3 \sin\theta / R^2)$ at $(a/2, 0, \pi)$.

Solution:

(a)
$$\begin{aligned}\nabla \cdot \mathbf{E} &= \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \\ &= \frac{\partial}{\partial x}(3x^2) + \frac{\partial}{\partial y}(2z) + \frac{\partial}{\partial z}(x^2 z) \\ &= 6x + 0 + x^2 \\ &= x^2 + 6x.\end{aligned}$$

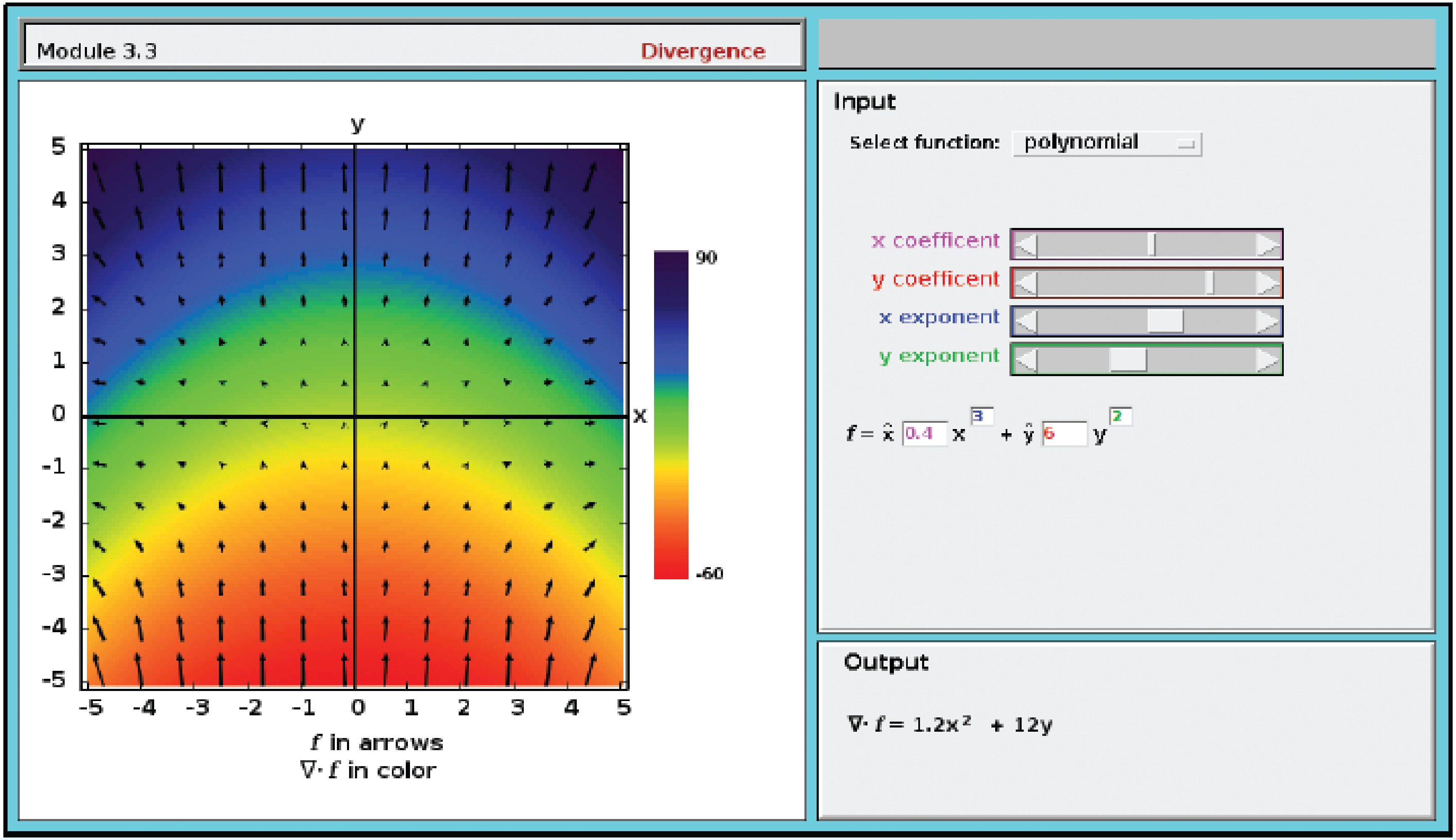
At $(2, -2, 0)$, $\nabla \cdot \mathbf{E}\Big|_{(2,-2,0)} = 16$.

(b) From the expression given on the inside of the back cover of the book for the divergence of a vector in spherical coordinates, it follows that

$$\begin{aligned}\nabla \cdot \mathbf{E} &= \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 E_R) + \frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}(E_\theta \sin\theta) \\ &\quad + \frac{1}{R\sin\theta}\frac{\partial E_\phi}{\partial\phi} \\ &= \frac{1}{R^2}\frac{\partial}{\partial R}(a^3\cos\theta) + \frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}\left(-\frac{a^3\sin^2\theta}{R^2}\right) \\ &= 0 - \frac{2a^3\cos\theta}{R^3} \\ &= -\frac{2a^3\cos\theta}{R^3}.\end{aligned}$$

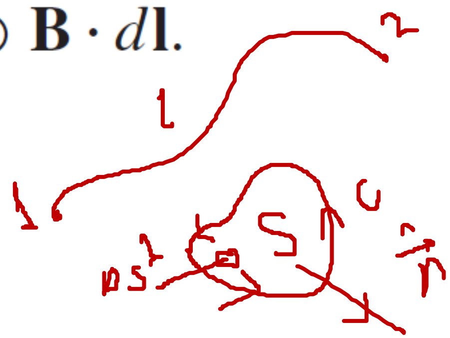
At $R = a/2$ and $\theta = 0$, $\nabla \cdot \mathbf{E}\Big|_{(a/2,0,\pi)} = -16$.

Module 3.3 Divergence Select a vector function $\mathbf{f}(x, y, z)$, evaluate its divergence, and display both in an appropriate 2-D plane.



Curl of a Vector Field

$$\text{Circulation} = \oint_C \mathbf{B} \cdot d\mathbf{l}.$$



$$\nabla \times \mathbf{B} = \text{curl}\, \mathbf{B} = \lim_{\Delta s \to 0} \frac{1}{\Delta s} \left[\hat{\mathbf{n}} \oint_C \mathbf{B} \cdot d\mathbf{l} \right]_{\max} . \quad (3.103)$$

Thus, curl **B** *is the circulation of* **B** *per unit area, with the area* Δs *of the contour C being oriented such that the circulation is maximum.*

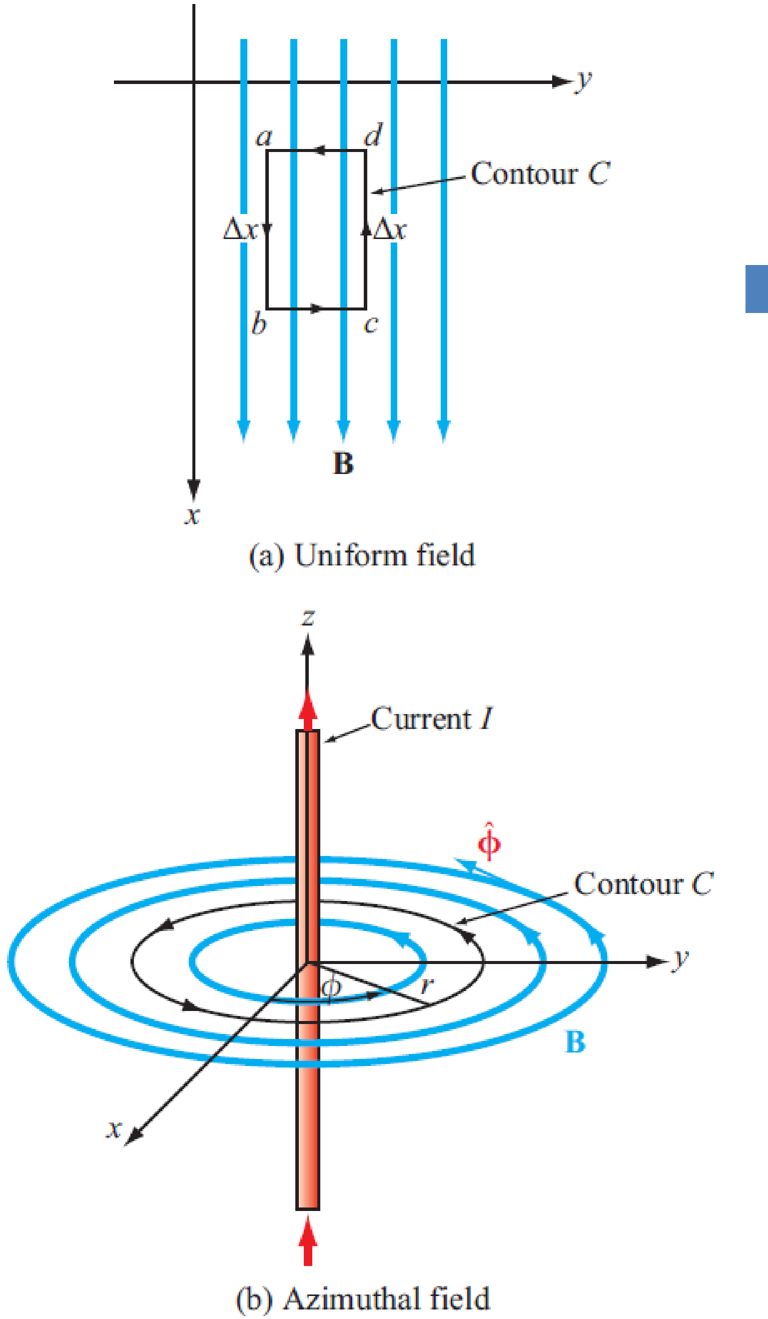


Figure 3-22: Circulation is zero for the uniform field in (a), but it is not zero for the azimuthal field in (b).

Stokes's Theorem

Stokes's theorem converts the surface integral of the curl of a vector over an open surface S into a line integral of the vector along the contour C bounding the surface S.

For the geometry shown in Fig. 3-23, ***Stokes's theorem*** states

$$\int_S (\nabla \times \mathbf{B}) \cdot d\mathbf{s} = \oint_C \mathbf{B} \cdot d\mathbf{l} \qquad \text{(Stokes's theorem)}, \tag{3.107}$$

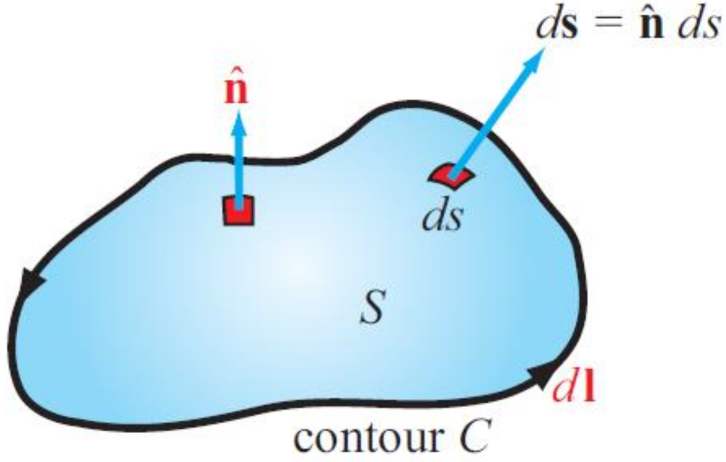
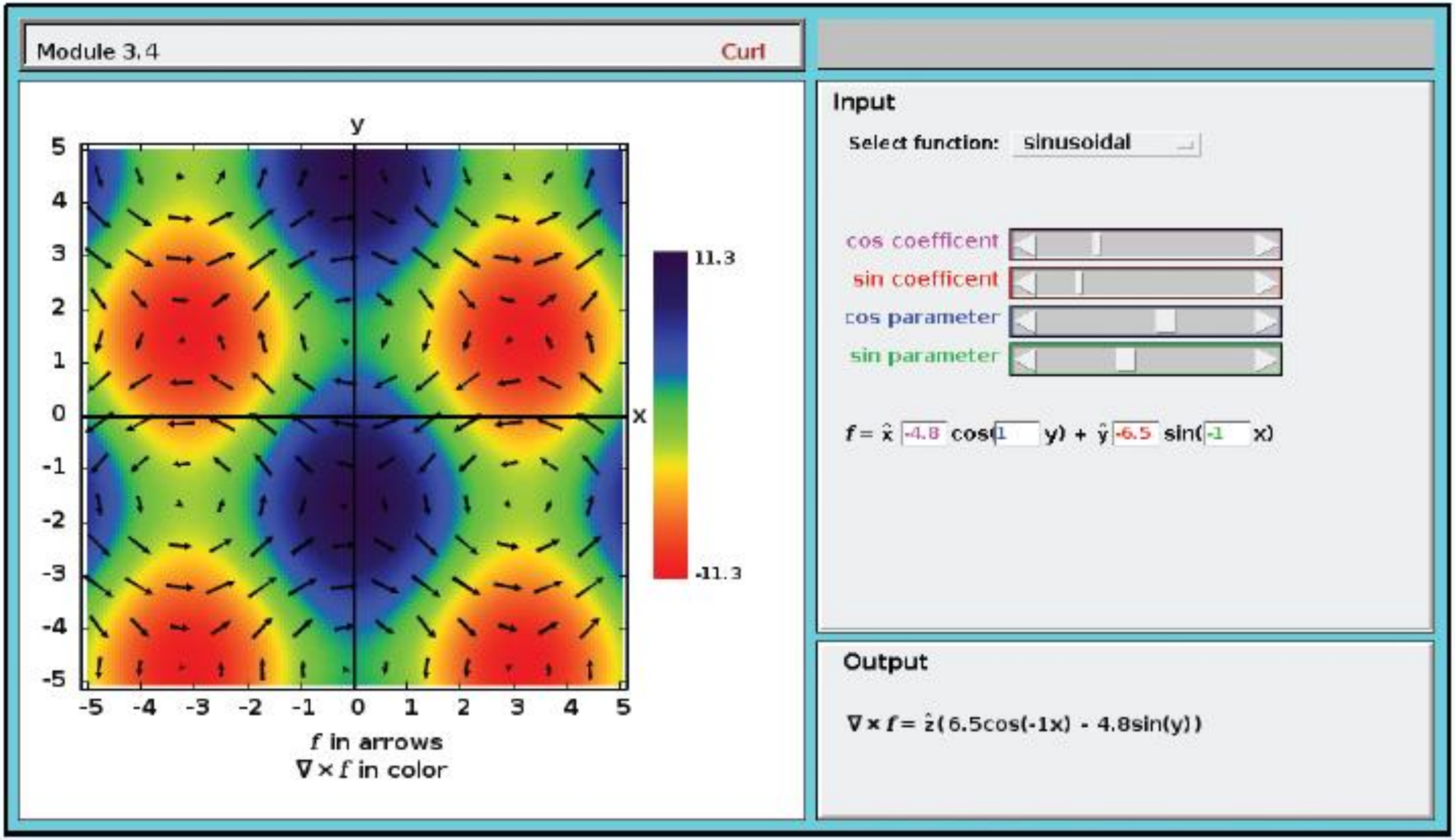


Figure 3-23: The direction of the unit vector $\hat{\mathbf{n}}$ is along the thumb when the other four fingers of the right hand follow $d\mathbf{l}$.

Module 3.4 Curl Select a vector $\mathbf{f}(x, y)$, evaluate its curl, and display both in the x-y plane.



Laplacian Operator

Laplacian of a Scalar Field

$$\nabla^2 V = \nabla \cdot (\nabla V) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} . \qquad (3.110)$$

Laplacian of a Vector Field

$$\nabla^2 \mathbf{E} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \mathbf{E}$$

$$= \hat{\mathbf{x}} \nabla^2 E_x + \hat{\mathbf{y}} \nabla^2 E_y + \hat{\mathbf{z}} \nabla^2 E_z$$

Useful Relation

$$\nabla^2 \mathbf{E} = \nabla (\nabla \cdot \mathbf{E}) - \nabla \times (\nabla \times \mathbf{E}). \qquad (3.113)$$

Tech Brief 6: X-Ray Computed Tomography

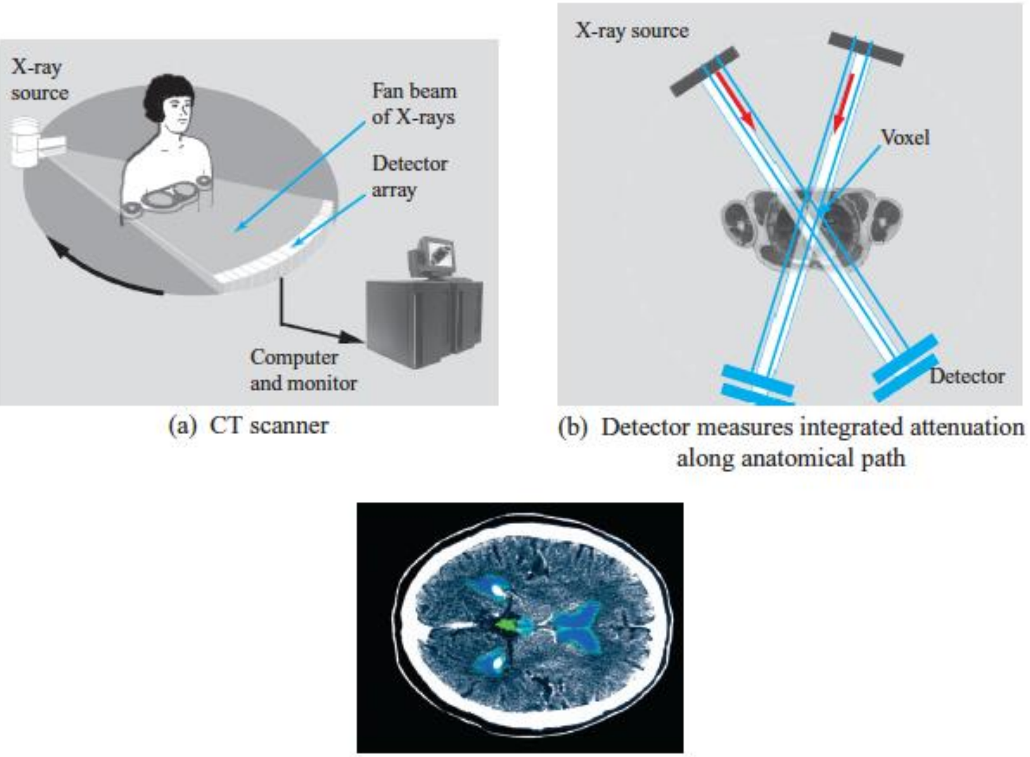
How does a CT scanner generate a 3-D image?

Figure TF6-1 2-D X-ray image.

Figure TF6-2 CT scanner.

Tech Brief 6: X-Ray Computed Tomography

- For each anatomical slice, the CT scanner generates on the order of 7 x 10^5 measurements (1,000 angular orientations x 700 detector channels)
- Use of vector calculus allows the extraction of the 2-D image of a slice
- Combining multiple slices generates a 3-D scan



(a) CT scanner (b) Detector measures integrated attenuation along anatomical path (c) CT image of a normal brain

Figure TF6-3 Basic elements of a CT scanner.

Chapter 3 Relationships

Distance Between Two Points

$$d = [(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]^{1/2}$$

$$d = \left[r_2^2 + r_1^2 - 2r_1r_2\cos(\phi_2 - \phi_1) + (z_2 - z_1)^2\right]^{1/2}$$

$$d = \left\{R_2^2 + R_1^2 - 2R_1R_2[\cos\theta_2\cos\theta_1 + \sin\theta_1\sin\theta_2\cos(\phi_2 - \phi_1)]\right\}^{1/2}$$

Coordinate Systems

Table 3-1

Coordinate Transformations

Table 3-2

Vector Products

$$\mathbf{A}\cdot\mathbf{B} = AB\cos\theta_{AB}$$

$$\mathbf{A}\times\mathbf{B} = \hat{\mathbf{n}}\, AB\sin\theta_{AB}$$

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$$

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$$

Divergence Theorem

$$\int_V \nabla\cdot\mathbf{E}\, dV = \oint_S \mathbf{E}\cdot d\mathbf{s}$$

Vector Operators

$$\nabla T = \hat{\mathbf{x}}\frac{\partial T}{\partial x} + \hat{\mathbf{y}}\frac{\partial T}{\partial y} + \hat{\mathbf{z}}\frac{\partial T}{\partial z}$$

$$\nabla\cdot\mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

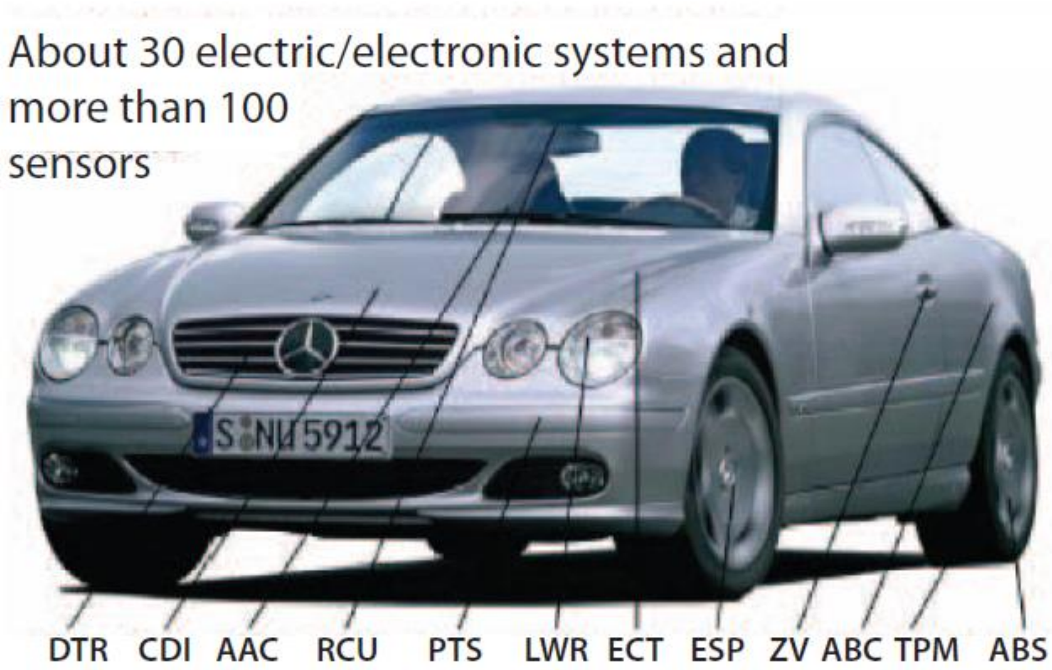
$$\nabla\times\mathbf{B} = \hat{\mathbf{x}}\left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

(see back cover for cylindrical and spherical coordinates)

Stokes's Theorem

$$\int_S (\nabla\times\mathbf{B})\cdot d\mathbf{s} = \oint_C \mathbf{B}\cdot d\mathbf{l}$$



System	Abbrev.	Sensors	System	Abbrev.	Sensors
Distronic	DTR	3	Common-rail diesel injection	CDI	11
Electronic controlled transmission	ECT	9	Automatic air condition	AAC	13
Roof control unit	RCU	7	Active body control	ABC	12
Antilock braking system	ABS	4	Tire pressure monitoring	TPM	11
Central locking system	ZV	3	Elektron. stability program	ESP	14
Dyn. beam levelling	LWR	6	Parktronic system	PTS	12

Figure TF7-1: Most cars use on the order of 100 sensors. (Courtesy Mercedes-Benz.)

4. ELECTROSTATICS

7e Applied EM by Ulaby and Ravaioli

Chapter 4 Overview

Chapter Contents

Objectives

Upon learning the material presented in this chapter, you should be able to:

1. Evaluate the electric field and electric potential due to any distribution of electric charges.
2. Apply Gauss's law.
3. Calculate the resistance R of any shaped object, given the electric field at every point in its volume.
4. Describe the operational principles of resistive and capacitive sensors.
5. Calculate the capacitance of two-conductor configurations.

Maxwell's Equations

God said:

$$\nabla \cdot \mathbf{D} = \rho_v,$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

$$\nabla \cdot \mathbf{B} = 0,$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}.$$

And there was light!

Under ***static*** conditions, none of the quantities appearing in Maxwell's equations are functions of time (i.e., $\partial/\partial t = 0$). ***This happens when all charges are permanently fixed in space, or, if they move, they do so at a steady rate so that ρ_v and J are constant in time.*** Under these circumstances, the time derivatives of **B** and **D** in Eqs. (4.1b) and (4.1d) vanish, and Maxwell's equations reduce to

Electrostatics

$$\nabla \cdot \mathbf{D} = \rho_v, \quad (4.2a)$$

$$\nabla \times \mathbf{E} = 0. \quad (4.2b)$$

Magnetostatics

$$\nabla \cdot \mathbf{B} = 0, \quad (4.3a)$$

$$\nabla \times \mathbf{H} = \mathbf{J}. \quad (4.3b)$$

Electric and magnetic fields become decoupled under static conditions.

Charge Distributions

Volume charge density:

$$\rho_v = \lim_{\Delta \mathcal{V} \to 0} \frac{\Delta q}{\Delta \mathcal{V}} = \frac{dq}{d\mathcal{V}} \qquad (\text{C/m}^3)$$

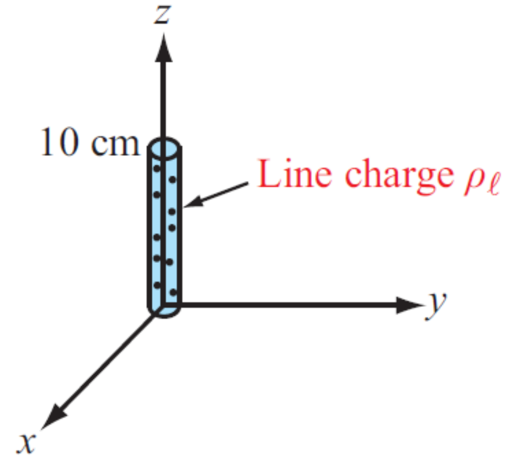
Total Charge in a Volume

$$Q = \int_{\mathcal{V}} \rho_v \, d\mathcal{V} \qquad (\text{C})$$

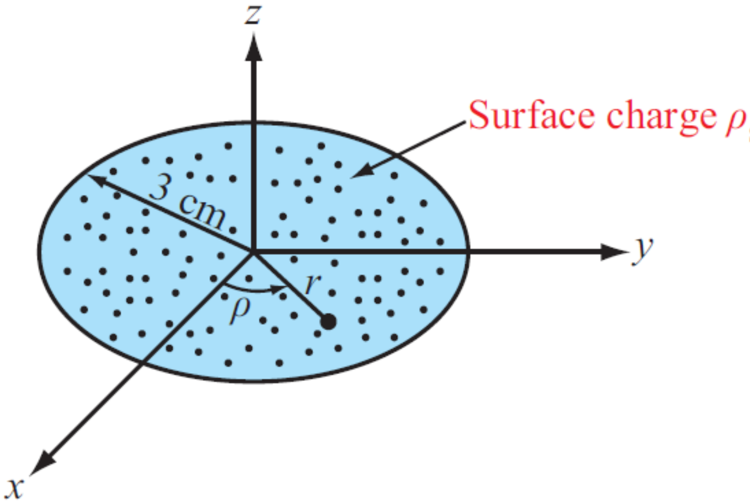
Surface and Line Charge Densities

$$\rho_s = \lim_{\Delta s \to 0} \frac{\Delta q}{\Delta s} = \frac{dq}{ds} \qquad (\text{C/m}^2)$$

$$\rho_\ell = \lim_{\Delta l \to 0} \frac{\Delta q}{\Delta l} = \frac{dq}{dl} \qquad (\text{C/m})$$



(a) Line charge distribution



Current Density

The amount of charge that crosses the tube's cross-sectional surface $\Delta s'$ in time Δt is therefore

$$\Delta q' = \rho_v\, \Delta \mathcal{V} = \rho_v\, \Delta l\, \Delta s' = \rho_v u\, \Delta s'\, \Delta t. \qquad (4.8)$$

For a surface with any orientation:

$$\Delta q = \rho_v \mathbf{u} \cdot \Delta \mathbf{s}\, \Delta t, \qquad (4.9)$$

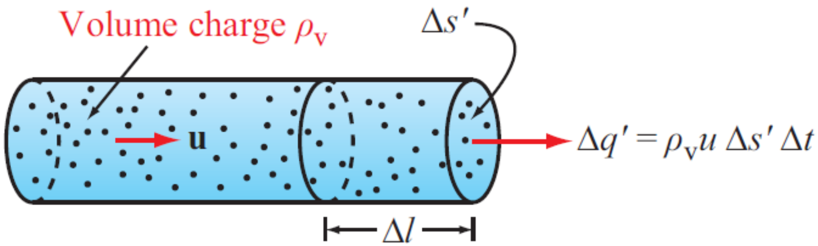
where $\Delta \mathbf{s} = \hat{\mathbf{n}}\, \Delta s$ and the corresponding total current flowing in the tube is

$$\Delta I = \frac{\Delta q}{\Delta t} = \rho_v \mathbf{u} \cdot \Delta \mathbf{s} = \mathbf{J} \cdot \Delta \mathbf{s}, \qquad (4.10)$$

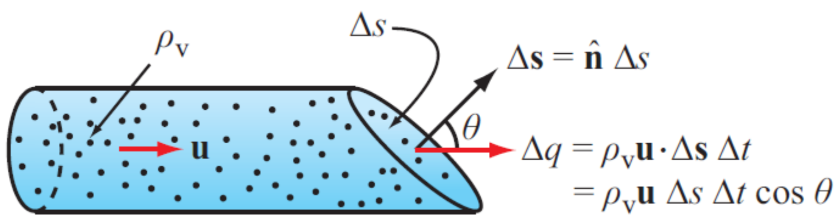
where

$$\mathbf{J} = \rho_v \mathbf{u} \qquad (\text{A/m}^2) \qquad (4.11)$$

J is called the current density



(a)



(b)

Figure 4-2: Charges with velocity **u** moving through a cross section $\Delta s'$ in (a) and Δs in (b).

$$I = \int_S \mathbf{J} \cdot d\mathbf{s} \qquad (\text{A}). \qquad (4.12)$$

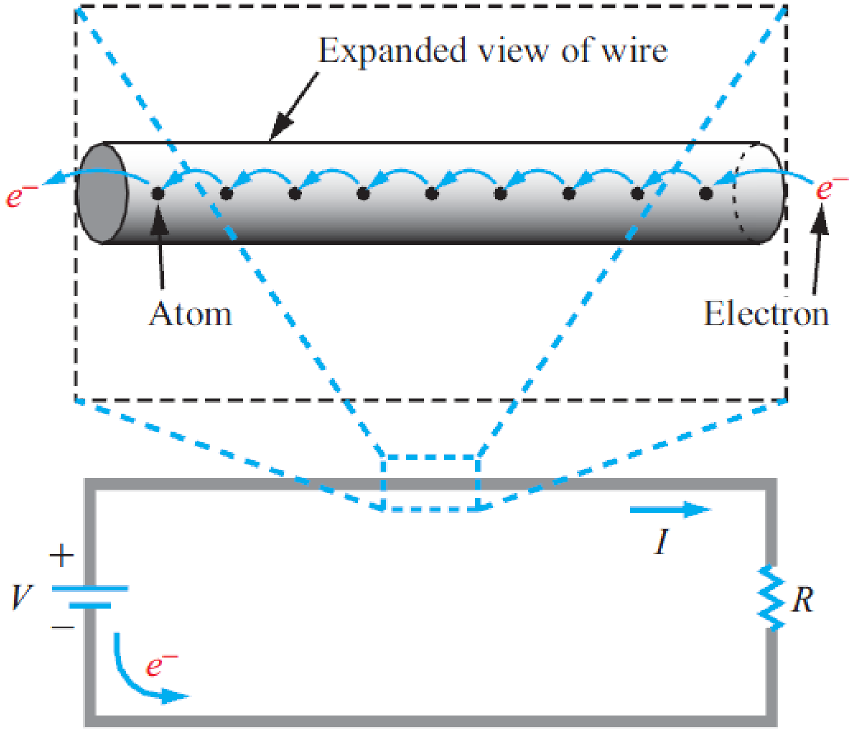
When a current is due to the actual movement of electrically charged matter, it is called a ***convection current****, and* **J** *is called a* ***convection current density****.*

Convection vs. Conduction

When a current is due to the movement of charged particles relative to their host material, **J** is called a ***conduction current density***.

This movement of electrons from atom to atom constitutes a ***conduction current****. The electrons that emerge from the wire are not necessarily the same electrons that entered the wire at the other end.*

Conduction current, which is discussed in more detail in Section 4-6, obeys Ohm's law, whereas convection current does not.



Coulomb's Law

Electric field at point P due to single charge

$$\mathbf{E} = \hat{\mathbf{R}} \frac{q}{4\pi\varepsilon R^2} \qquad \text{(V/m)}$$

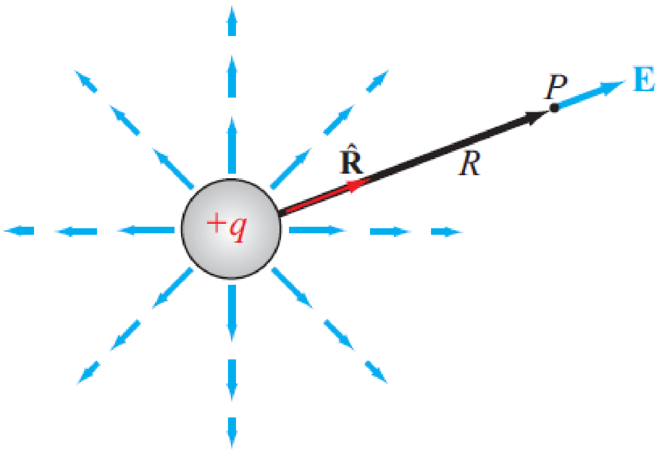


Figure 4-3: Electric-field lines due to a charge q.

Electric force on a test charge placed at *P*

$$\mathbf{F} = q'\mathbf{E} \qquad \text{(N)}$$

Electric flux density **D**

$$\mathbf{D} = \varepsilon\mathbf{E}$$

$$\varepsilon = \varepsilon_r \varepsilon_0,$$

If ε is independent of the magnitude of **E***, then the material is said to be* ***linear*** *because* **D** *and* **E** *are related linearly, and if it is independent of the direction of* **E***, the material is said to be* ***isotropic***.

$$\varepsilon_0 = 8.85 \times 10^{-12} \simeq (1/36\pi) \times 10^{-9} \qquad \text{(F/m)}$$

Electric Field Due to 2 Charges

with R, the distance between q_1 and P, replaced with $|\mathbf{R}-\mathbf{R}_1|$ and the unit vector $\hat{\mathbf{R}}$ replaced with $(\mathbf{R}-\mathbf{R}_1)/|\mathbf{R}-\mathbf{R}_1|$. Thus,

$$\mathbf{E}_1 = \frac{q_1(\mathbf{R}-\mathbf{R}_1)}{4\pi\varepsilon|\mathbf{R}-\mathbf{R}_1|^3} \quad \text{(V/m)}. \tag{4.17a}$$

Similarly, the electric field at P due to q_2 alone is

$$\mathbf{E}_2 = \frac{q_2(\mathbf{R}-\mathbf{R}_2)}{4\pi\varepsilon|\mathbf{R}-\mathbf{R}_2|^3} \quad \text{(V/m)}. \tag{4.17b}$$

The electric field obeys the principle of linear superposition.

Hence, the total electric field $\mathbf{E}$ at P due to q_1 and q_2 is

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 = \frac{1}{4\pi\varepsilon}\left[\frac{q_1(\mathbf{R}-\mathbf{R}_1)}{|\mathbf{R}-\mathbf{R}_1|^3} + \frac{q_2(\mathbf{R}-\mathbf{R}_2)}{|\mathbf{R}-\mathbf{R}_2|^3}\right]. \tag{4.18}$$

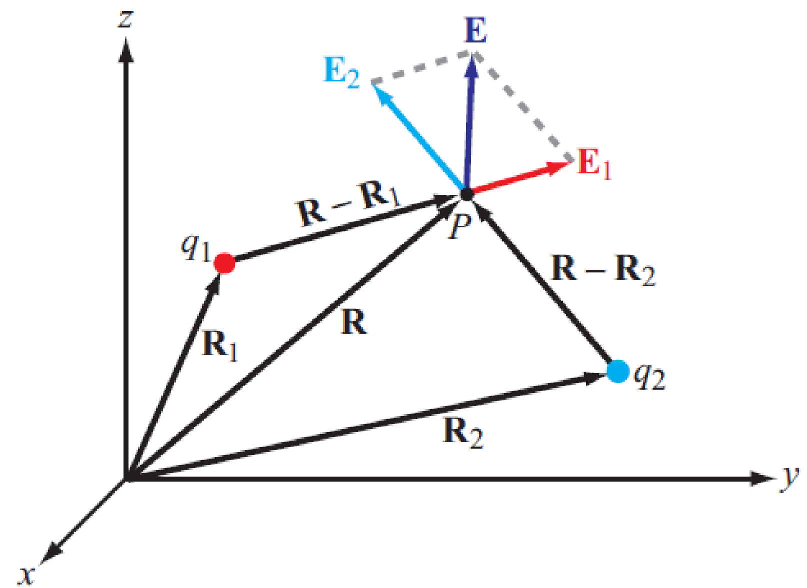


Figure 4-4: The electric field $\mathbf{E}$ at P due to two charges is equal to the vector sum of $\mathbf{E}_1$ and $\mathbf{E}_2$.

Electric Field due to Multiple Charges

$$\mathbf{E} = \frac{1}{4\pi\varepsilon}\sum_{i=1}^{N}\frac{q_i(\mathbf{R}-\mathbf{R}_i)}{|\mathbf{R}-\mathbf{R}_i|^3} \qquad \text{(V/m)}.$$

Example 4-3: Electric Field Due to Two Point Charges

Two point charges with $q_1 = 2\times10^{-5}$ C and $q_2 = -4\times10^{-5}$ C are located in free space at points with Cartesian coordinates $(1,3,-1)$ and $(-3,1,-2)$, respectively. Find (a) the electric field $\mathbf{E}$ at $(3,1,-2)$ and (b) the force on a 8×10^{-5} C charge located at that point. All distances are in meters.

Solution: (a) From Eq. (4.18), the electric field $\mathbf{E}$ with $\varepsilon = \varepsilon_0$ (free space) is

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\left[q_1\frac{(\mathbf{R}-\mathbf{R}_1)}{|\mathbf{R}-\mathbf{R}_1|^3} + q_2\frac{(\mathbf{R}-\mathbf{R}_2)}{|\mathbf{R}-\mathbf{R}_2|^3}\right] \qquad \text{(V/m)}.$$

The vectors $\mathbf{R}_1$, $\mathbf{R}_2$, and $\mathbf{R}$ are

$$\mathbf{R}_1 = \hat{\mathbf{x}} + \hat{\mathbf{y}}3 - \hat{\mathbf{z}},$$

$$\mathbf{R}_2 = -\hat{\mathbf{x}}3 + \hat{\mathbf{y}} - \hat{\mathbf{z}}2,$$

$$\mathbf{R} = \hat{\mathbf{x}}3 + \hat{\mathbf{y}} - \hat{\mathbf{z}}2.$$

Hence,

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\left[\frac{2(\hat{\mathbf{x}}2 - \hat{\mathbf{y}}2 - \hat{\mathbf{z}})}{27} - \frac{4(\hat{\mathbf{x}}6)}{216}\right]\times10^{-5}$$

$$= \frac{\hat{\mathbf{x}} - \hat{\mathbf{y}}4 - \hat{\mathbf{z}}2}{108\pi\varepsilon_0}\times10^{-5} \qquad \text{(V/m)}.$$

(b)

$$\mathbf{F} = q_3\mathbf{E} = 8\times10^{-5}\times\frac{\hat{\mathbf{x}} - \hat{\mathbf{y}}4 - \hat{\mathbf{z}}2}{108\pi\varepsilon_0}\times10^{-5}$$

$$= \frac{\hat{\mathbf{x}}2 - \hat{\mathbf{y}}8 - \hat{\mathbf{z}}4}{27\pi\varepsilon_0}\times10^{-10} \qquad \text{(N)}.$$

Electric Field Due to Charge Distributions

Field due to:

a differential amount of charge $dq = \rho_v \, d\mathcal{V}'$ contained in a differential volume $d\mathcal{V}'$ is

$$d\mathbf{E} = \hat{\mathbf{R}}' \frac{dq}{4\pi\varepsilon R'^2} = \hat{\mathbf{R}}' \frac{\rho_v \, d\mathcal{V}'}{4\pi\varepsilon R'^2}, \qquad (4.20)$$

$$\mathbf{E} = \int_{\mathcal{V}'} d\mathbf{E} = \frac{1}{4\pi\varepsilon} \int_{\mathcal{V}'} \hat{\mathbf{R}}' \frac{\rho_v \, d\mathcal{V}'}{R'^2}$$

$$\text{(volume distribution).} \qquad (4.21a)$$

$$\mathbf{E} = \frac{1}{4\pi\varepsilon} \int_{S'} \hat{\mathbf{R}}' \frac{\rho_s \, ds'}{R'^2} \qquad \text{(surface distribution),} \qquad (4.21b)$$

$$\mathbf{E} = \frac{1}{4\pi\varepsilon} \int_{l'} \hat{\mathbf{R}}' \frac{\rho_\ell \, dl'}{R'^2} \qquad \text{(line distribution).} \qquad (4.21c)$$

Example 4-4: Electric Field of a Ring of Charge

A ring of charge of radius b is characterized by a uniform line charge density of positive polarity ρ_ℓ. The ring resides in free space and is positioned in the x–y plane as shown in Fig. 4-6. Determine the electric field intensity $\mathbf{E}$ at a point $P = (0, 0, h)$ along the axis of the ring at a distance h from its center.

Solution: We start by considering the electric field generated by a differential ring segment with cylindrical coordinates $(b, \phi, 0)$ in Fig. 4-6(a). The segment has length $dl = b\,d\phi$ and contains charge $dq = \rho_\ell\,dl = \rho_\ell b\,d\phi$. The distance vector $\mathbf{R}'_1$ from segment 1 to point $P = (0, 0, h)$ is

$$\mathbf{R}'_1 = -\hat{\mathbf{r}}b + \hat{\mathbf{z}}h,$$

from which it follows that

$$R'_1 = |\mathbf{R}'_1| = \sqrt{b^2 + h^2}\,, \qquad \hat{\mathbf{R}}'_1 = \frac{\mathbf{R}'_1}{|\mathbf{R}'_1|} = \frac{-\hat{\mathbf{r}}b + \hat{\mathbf{z}}h}{\sqrt{b^2 + h^2}}\,.$$

The electric field at $P = (0, 0, h)$ due to the charge in segment 1 therefore is

$$d\mathbf{E}_1 = \frac{1}{4\pi\varepsilon_0}\,\hat{\mathbf{R}}'_1\,\frac{\rho_\ell\,dl}{{R'_1}^2} = \frac{\rho_\ell b}{4\pi\varepsilon_0}\,\frac{(-\hat{\mathbf{r}}b + \hat{\mathbf{z}}h)}{(b^2 + h^2)^{3/2}}\,d\phi.$$

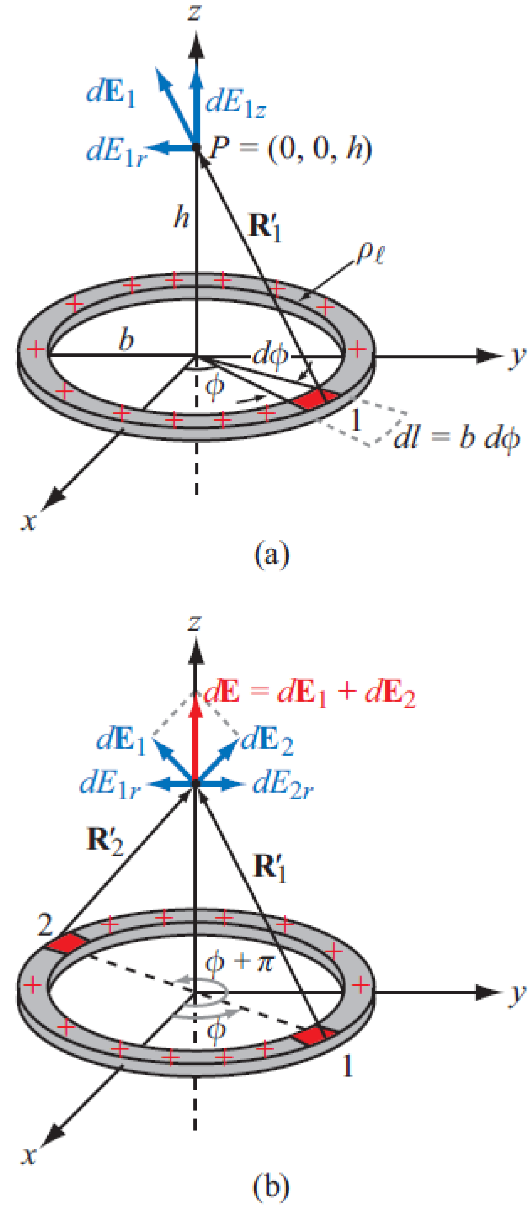


Figure 4-6: Ring of charge with line density ρ_ℓ. (a) The field $d\mathbf{E}_1$ due to infinitesimal segment 1 and (b) the fields $d\mathbf{E}_1$ and $d\mathbf{E}_2$ due to segments at diametrically opposite locations (Example 4-4).

Cont.

$$d\mathbf{E}_1 = \frac{1}{4\pi\varepsilon_0}\,\hat{\mathbf{R}}'_1\,\frac{\rho_\ell\,dl}{R_1'^2} = \frac{\rho_\ell b}{4\pi\varepsilon_0}\,\frac{(-\hat{\mathbf{r}}b + \hat{\mathbf{z}}h)}{(b^2+h^2)^{3/2}}\,d\phi.$$

The field $d\mathbf{E}_1$ has component dE_{1r} along $-\hat{\mathbf{r}}$ and component dE_{1z} along $\hat{\mathbf{z}}$. From symmetry considerations, the field $d\mathbf{E}_2$ generated by differential segment 2 in Fig. 4-6(b), which is located diametrically opposite to segment 1, is identical to $d\mathbf{E}_1$ except that the $\hat{\mathbf{r}}$-component of $d\mathbf{E}_2$ is opposite that of $d\mathbf{E}_1$. Hence, the $\hat{\mathbf{r}}$-components in the sum cancel and the $\hat{\mathbf{z}}$-contributions add. The sum of the two contributions is

$$d\mathbf{E} = d\mathbf{E}_1 + d\mathbf{E}_2 = \hat{\mathbf{z}}\,\frac{\rho_\ell bh}{2\pi\varepsilon_0}\,\frac{d\phi}{(b^2+h^2)^{3/2}}\,. \qquad (4.22)$$

Since for every ring segment in the semicircle defined over the azimuthal range $0 \le \phi \le \pi$ (the right-hand half of the circular ring) there is a corresponding segment located diametrically opposite at $(\phi + \pi)$, we can obtain the total field generated by the ring by integrating Eq. (4.22) over a semicircle as

$$\begin{aligned}\mathbf{E} &= \hat{\mathbf{z}}\,\frac{\rho_\ell bh}{2\pi\varepsilon_0(b^2+h^2)^{3/2}}\int_0^\pi d\phi \\ &= \hat{\mathbf{z}}\,\frac{\rho_\ell bh}{2\varepsilon_0(b^2+h^2)^{3/2}} \\ &= \hat{\mathbf{z}}\,\frac{h}{4\pi\varepsilon_0(b^2+h^2)^{3/2}}\,Q, \qquad (4.23)\end{aligned}$$

where $Q = 2\pi b\rho_\ell$ is the total charge on the ring.

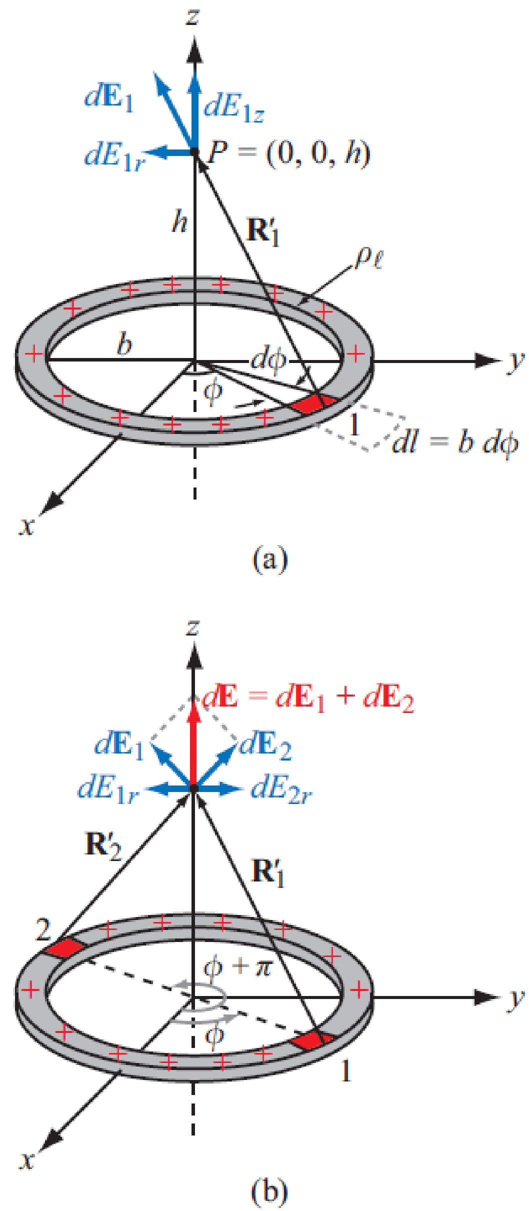


Figure 4-6: Ring of charge with line density ρ_ℓ. (a) The field $d\mathbf{E}_1$ due to infinitesimal segment 1 and (b) the fields $d\mathbf{E}_1$ and $d\mathbf{E}_2$ due to segments at diametrically opposite locations (Example 4-4).

Example 4-5: Electric Field of a Circular Disk of Charge

Find the electric field at point P with Cartesian coordinates $(0, 0, h)$ due to a circular disk of radius a and uniform charge density ρ_s residing in the x–y plane (Fig. 4-7). Also, evaluate $\mathbf{E}$ due to an infinite sheet of charge density ρ_s by letting $a \to \infty$.

Solution: Building on the expression obtained in Example 4-4 for the on-axis electric field due to a circular ring of charge, we can determine the field due to the circular disk by treating the disk as a set of concentric rings. A ring of radius r and width dr has an area $ds = 2\pi r\, dr$ and contains charge $dq = \rho_s\, ds = 2\pi\rho_s r\, dr$. Upon using this expression in Eq. (4.23) and also replacing b with r, we obtain the following expression for the field due to the ring:

$$d\mathbf{E} = \hat{\mathbf{z}}\, \frac{h}{4\pi\varepsilon_0(r^2 + h^2)^{3/2}}\, (2\pi\rho_s r\, dr).$$

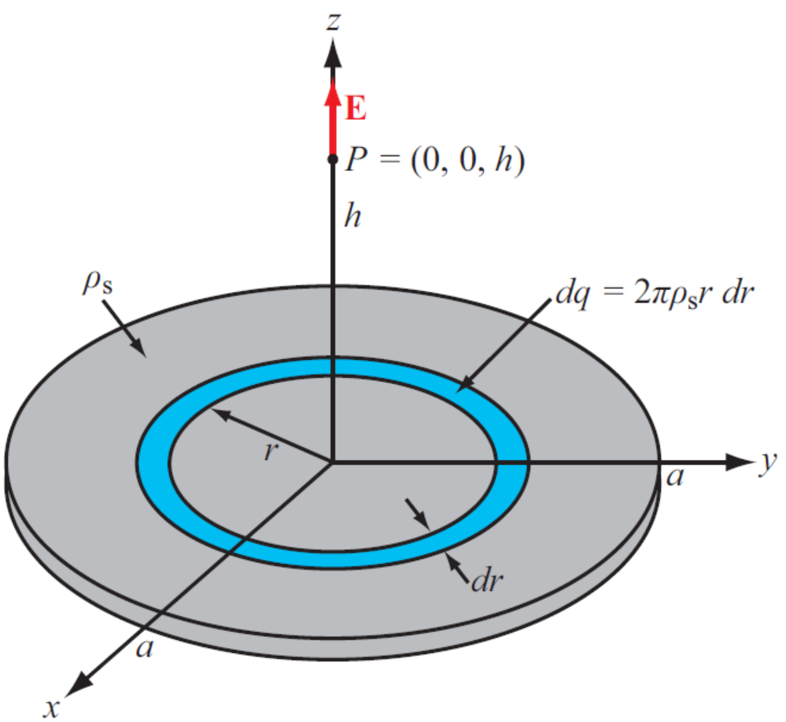


Figure 4-7: Circular disk of charge with surface charge density ρ_s. The electric field at $P = (0, 0, h)$ points along the z-direction (Example 4-5).

Cont.

Example 4-5 cont.

The total field at P is obtained by integrating the expression over the limits $r = 0$ to $r = a$:

$$\mathbf{E} = \hat{\mathbf{z}} \frac{\rho_s h}{2\varepsilon_0} \int_0^a \frac{r\, dr}{(r^2 + h^2)^{3/2}}$$

$$= \pm\hat{\mathbf{z}} \frac{\rho_s}{2\varepsilon_0} \left[1 - \frac{|h|}{\sqrt{a^2 + h^2}}\right], \qquad (4.24)$$

with the plus sign for $h > 0$ (P above the disk) and the minus sign when $h < 0$ (P below the disk).

For an infinite sheet of charge with $a = \infty$,

$$\mathbf{E} = \pm\hat{\mathbf{z}} \frac{\rho_s}{2\varepsilon_0} \quad \text{(infinite sheet of charge).} \qquad (4.25)$$

We note that for an infinite sheet of charge $\mathbf{E}$ is the same at all points above the x–y plane, and a similar statement applies for points below the x-y plane.

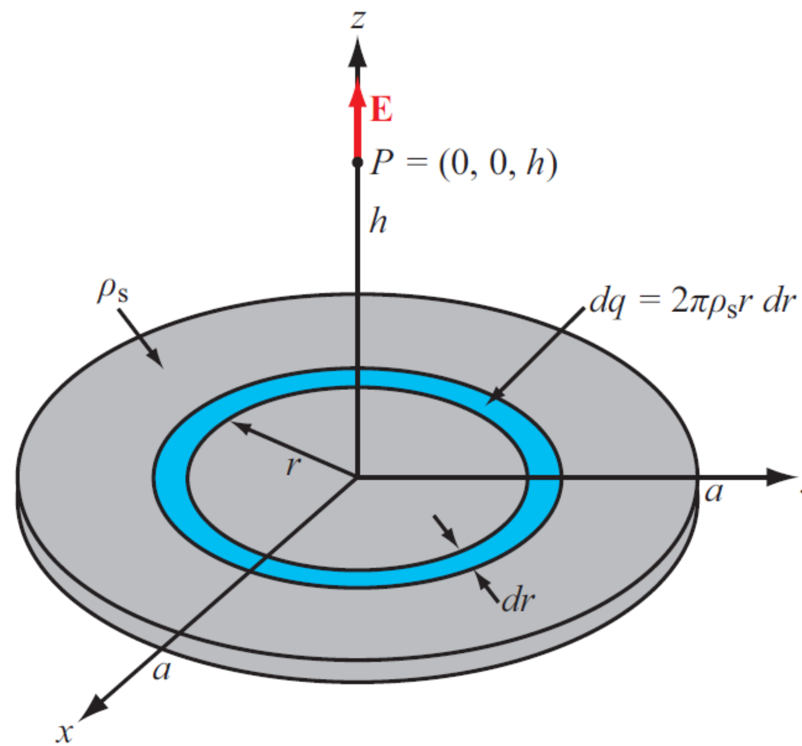


Figure 4-7: Circular disk of charge with surface charge density ρ_s. The electric field at $P = (0, 0, h)$ points along the z-direction (Example 4-5).

Gauss's Law

$$\nabla \cdot \mathbf{D} = \rho_v$$

(Differential form of Gauss's law),

$$\int_{\mathcal{V}} \nabla \cdot \mathbf{D}\, d\mathcal{V} = \int_{\mathcal{V}} \rho_v\, d\mathcal{V} = Q$$

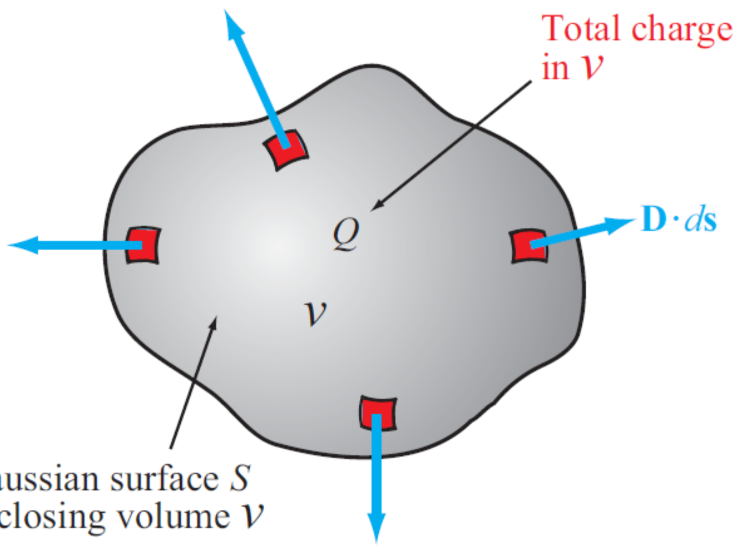


Figure 4-8: The integral form of Gauss's law states that the outward flux of **D** through a surface is proportional to the enclosed charge Q.

Application of the divergence theorem gives:

$$\int_{\mathcal{V}} \nabla \cdot \mathbf{D}\, d\mathcal{V} = \oint_S \mathbf{D} \cdot d\mathbf{s}. \qquad (4.28)$$

Comparison of Eq. (4.27) with Eq. (4.28) leads to

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q \qquad (4.29)$$

(Integral form of Gauss's law).

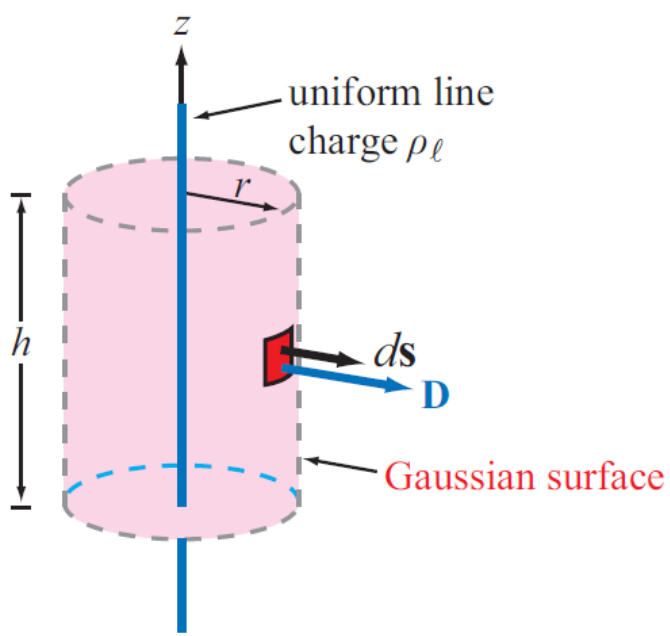
The integral form of Gauss's law is illustrated diagrammatically in Fig. 4-8; for each differential surface element $d\mathbf{s}$*,* $\mathbf{D} \cdot d\mathbf{s}$ *is the electric field flux flowing outward of* $\mathcal{V}$ *through* $d\mathbf{s}$*, and the total flux through surface* S *equals the enclosed charge* Q*. The surface* S *is called a* ***Gaussian surface****.*

Applying Gauss's Law

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q \qquad (4.29)$$

(Integral form of Gauss's law).

Gauss's law, as given by Eq. (4.29), provides a convenient method for determining the flux density **D** *when the charge distribution possesses symmetry properties that allow us to infer the variations of the magnitude and direction of* **D** *as a function of spatial location, thereby facilitating the integration of* **D** *over a cleverly chosen Gaussian surface.*



Example 4-6: Electric Field of an Infinite Line Charge

Use Gauss's law to obtain an expression for **E** due to an infinitely long line with uniform charge density ρ_ℓ that resides along the z-axis in free space.

Construct an imaginary Gaussian cylinder of radius *r* and height *h*:

$$\int_{z=0}^{h} \int_{\phi=0}^{2\pi} \hat{\mathbf{r}} D_r \cdot \hat{\mathbf{r}} r \, d\phi \, dz = \rho_\ell h$$

or

$$2\pi h D_r r = \rho_\ell h,$$

which yields

$$\mathbf{E} = \frac{\mathbf{D}}{\varepsilon_0} = \hat{\mathbf{r}} \frac{D_r}{\varepsilon_0} = \hat{\mathbf{r}} \frac{\rho_\ell}{2\pi \varepsilon_0 r} \qquad (4.33)$$

(infinite line charge).

Electric Scalar Potential

The term "voltage" is short for "voltage potential" and synonymous with ***electric potential****.*

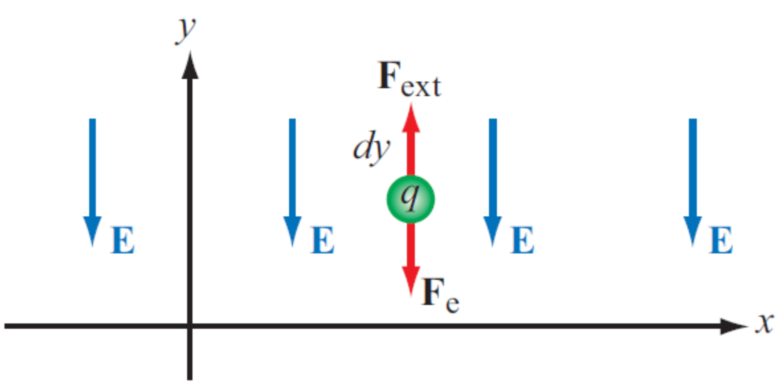


Figure 4-11: Work done in moving a charge q a distance dy against the electric field $\mathbf{E}$ is $dW = qE\,dy$.

Minimum force needed to move charge against **E** field:

$$\mathbf{F}_{ext} = -\mathbf{F}_e = -q\mathbf{E}. \qquad (4.34)$$

The work done, or energy expended, in moving any object a vector differential distance $d\mathbf{l}$ while exerting a force $\mathbf{F}_{ext}$ is

$$dW = \mathbf{F}_{ext} \cdot d\mathbf{l} = -q\mathbf{E} \cdot d\mathbf{l} \qquad \text{(J)}. \qquad (4.35)$$

Work, or energy, is measured in joules (J). If the charge is moved a distance dy along $\hat{\mathbf{y}}$, then

$$dW = -q(-\hat{\mathbf{y}}E) \cdot \hat{\mathbf{y}}\,dy = qE\,dy. \qquad (4.36)$$

The differential electric potential energy dW per unit charge is called the ***differential electric potential*** (or differential voltage) dV. That is,

$$dV = \frac{dW}{q} = -\mathbf{E} \cdot d\mathbf{l} \qquad \text{(J/C or V)}. \qquad (4.37)$$

Electric Scalar Potential

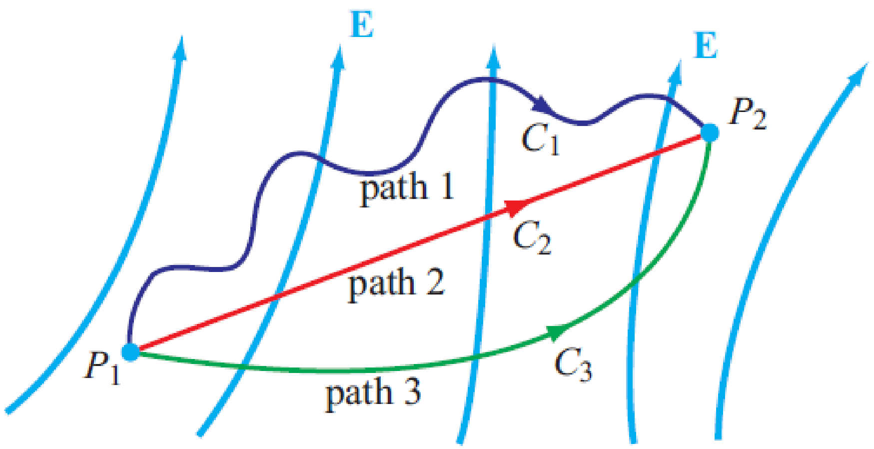


Figure 4-12: In electrostatics, the potential difference between P_2 and P_1 is the same irrespective of the path used for calculating the line integral of the electric field between them.

$$\int_{P_1}^{P_2} dV = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\mathbf{l},$$

$$V_{21} = V_2 - V_1 = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\mathbf{l}, \qquad (4.39)$$

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0 \qquad \text{(Electrostatics).} \qquad (4.40)$$

A vector field whose line integral along any closed path is zero is called a ***conservative*** *or an* ***irrotational*** *field. Hence, the electrostatic field* **E** *is conservative.*

Electric Potential Due to Charges

$$\int_{P_1}^{P_2} dV = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\mathbf{l},$$

$$V_{21} = V_2 - V_1 = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\mathbf{l}, \qquad (4.39)$$

In electric circuits, we usually select a convenient node that we call ground and assign it zero reference voltage. In free space and material media, we choose infinity as reference with $V = 0$. Hence, at a point P

$$V = -\int_{\infty}^{P} \mathbf{E} \cdot d\mathbf{l} \qquad \text{(V)}. \qquad (4.43)$$

For a point charge, V at range R is:

$$V = -\int_{\infty}^{R} \left(\hat{\mathbf{R}} \frac{q}{4\pi\varepsilon R^2} \right) \cdot \hat{\mathbf{R}}\, dR = \frac{q}{4\pi\varepsilon R} \qquad \text{(V)}. \qquad (4.45)$$

For continuous charge distributions:

$$V = \frac{1}{4\pi\varepsilon} \int_{\mathcal{V}'} \frac{\rho_{\mathrm{v}}}{R'}\, d\mathcal{V}' \qquad \text{(volume distribution)}, \qquad (4.48a)$$

$$V = \frac{1}{4\pi\varepsilon} \int_{S'} \frac{\rho_{\mathrm{s}}}{R'}\, ds' \qquad \text{(surface distribution)}, \qquad (4.48b)$$

$$V = \frac{1}{4\pi\varepsilon} \int_{l'} \frac{\rho_{\ell}}{R'}\, dl' \qquad \text{(line distribution)}. \qquad (4.48c)$$

Relating **E** to V

$$dV = -\mathbf{E} \cdot d\mathbf{l}. \tag{4.49}$$

For a scalar function V, Eq. (3.73) gives

$$dV = \nabla V \cdot d\mathbf{l}, \tag{4.50}$$

where ∇V is the gradient of V. Comparison of Eq. (4.49) with Eq. (4.50) leads to

$$\boxed{\mathbf{E} = -\nabla V.} \tag{4.51}$$

*This differential relationship between V and **E** allows us to determine **E** for any charge distribution by first calculating V and then taking the negative gradient of V to find **E**.*

Example 4-7: Electric Field of an Electric Dipole

Solution: To simplify the derivation, we align the dipole along the z-axis and center it at the origin [Fig. 4-13(a)]. For the two charges shown in Fig. 4-13(a), application of Eq. (4.47) gives

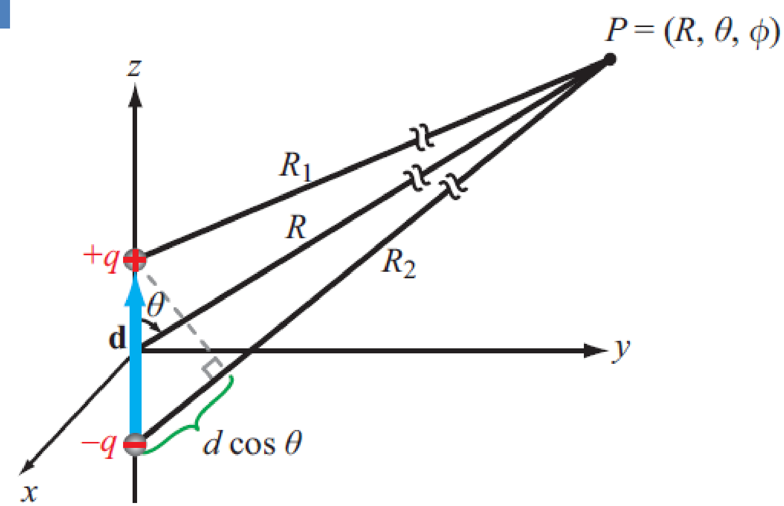
$$V = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{R_1} + \frac{-q}{R_2}\right) = \frac{q}{4\pi\varepsilon_0}\left(\frac{R_2 - R_1}{R_1 R_2}\right).$$

Since $d \ll R$, the lines labeled R_1 and R_2 in Fig. 4-13(a) are approximately parallel to each other, in which case the following approximations apply:

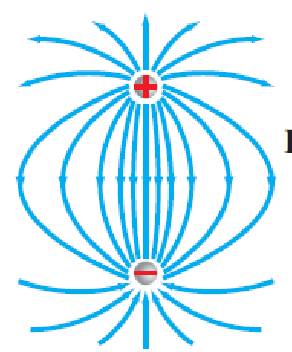
$$R_2 - R_1 \simeq d\cos\theta, \qquad R_1 R_2 \simeq R^2.$$

Hence,

$$V = \frac{qd\cos\theta}{4\pi\varepsilon_0 R^2}. \qquad (4.52)$$



(a) Electric dipole



(b) Electric-field pattern

https://www.youtube.com/watch?v=LB8Rhcb4eQM&t=68s

Cont.

Example 4-7: Electric Field of an Electric Dipole (cont.)

$$qd\cos\theta = q\mathbf{d}\cdot\hat{\mathbf{R}} = \mathbf{p}\cdot\hat{\mathbf{R}},$$

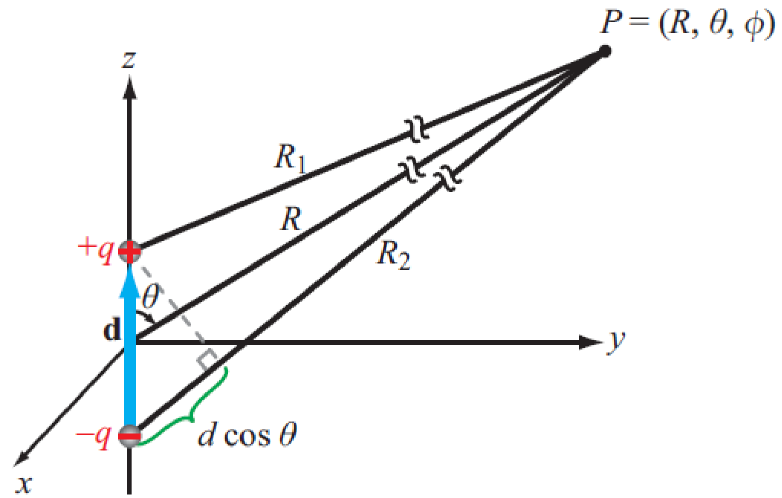
where $\mathbf{p} = q\mathbf{d}$ is called the ***dipole moment***. Using Eq. (4.53) in Eq. (4.52) then gives

$$V = \frac{\mathbf{p}\cdot\hat{\mathbf{R}}}{4\pi\varepsilon_0 R^2} \qquad \text{(electric dipole).} \qquad (4.54)$$

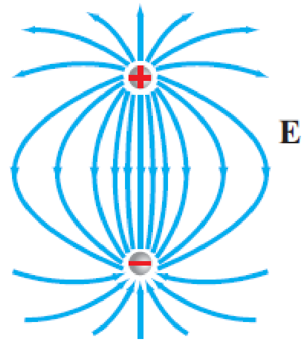
In spherical coordinates, Eq. (4.51) is given by

$$\mathbf{E} = -\nabla V$$
$$= -\left(\hat{\mathbf{R}}\frac{\partial V}{\partial R} + \hat{\boldsymbol{\theta}}\frac{1}{R}\frac{\partial V}{\partial \theta} + \hat{\boldsymbol{\phi}}\frac{1}{R\sin\theta}\frac{\partial V}{\partial \phi}\right), \qquad (4.55)$$

$$\mathbf{E} = \frac{qd}{4\pi\varepsilon_0 R^3}(\hat{\mathbf{R}}\,2\cos\theta + \hat{\boldsymbol{\theta}}\sin\theta) \quad \text{(V/m).}$$



(a) Electric dipole



(b) Electric-field pattern

Poisson's & Laplace's Equations

With $\mathbf{D} = \varepsilon\mathbf{E}$, the differential form of Gauss's law given by Eq. (4.26) may be cast as

$$\nabla \cdot \mathbf{E} = \frac{\rho_v}{\varepsilon}\,. \qquad (4.57)$$

Inserting Eq. (4.51) in Eq. (4.57) gives

$$\nabla \cdot (\nabla V) = -\frac{\rho_v}{\varepsilon}\,. \qquad (4.58)$$

Given Eq. (3.110) for the Laplacian of a scalar function V,

$$\nabla^2 V = \nabla \cdot (\nabla V) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}\,, \qquad (4.59)$$

Eq. (4.58) can be cast in the abbreviated form

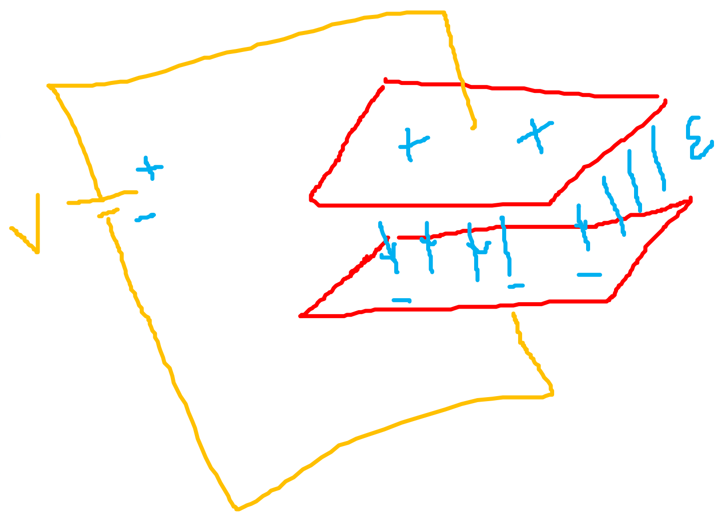
$$\nabla^2 V = -\frac{\rho_v}{\varepsilon} \qquad \text{(Poisson's equation).} \qquad (4.60)$$

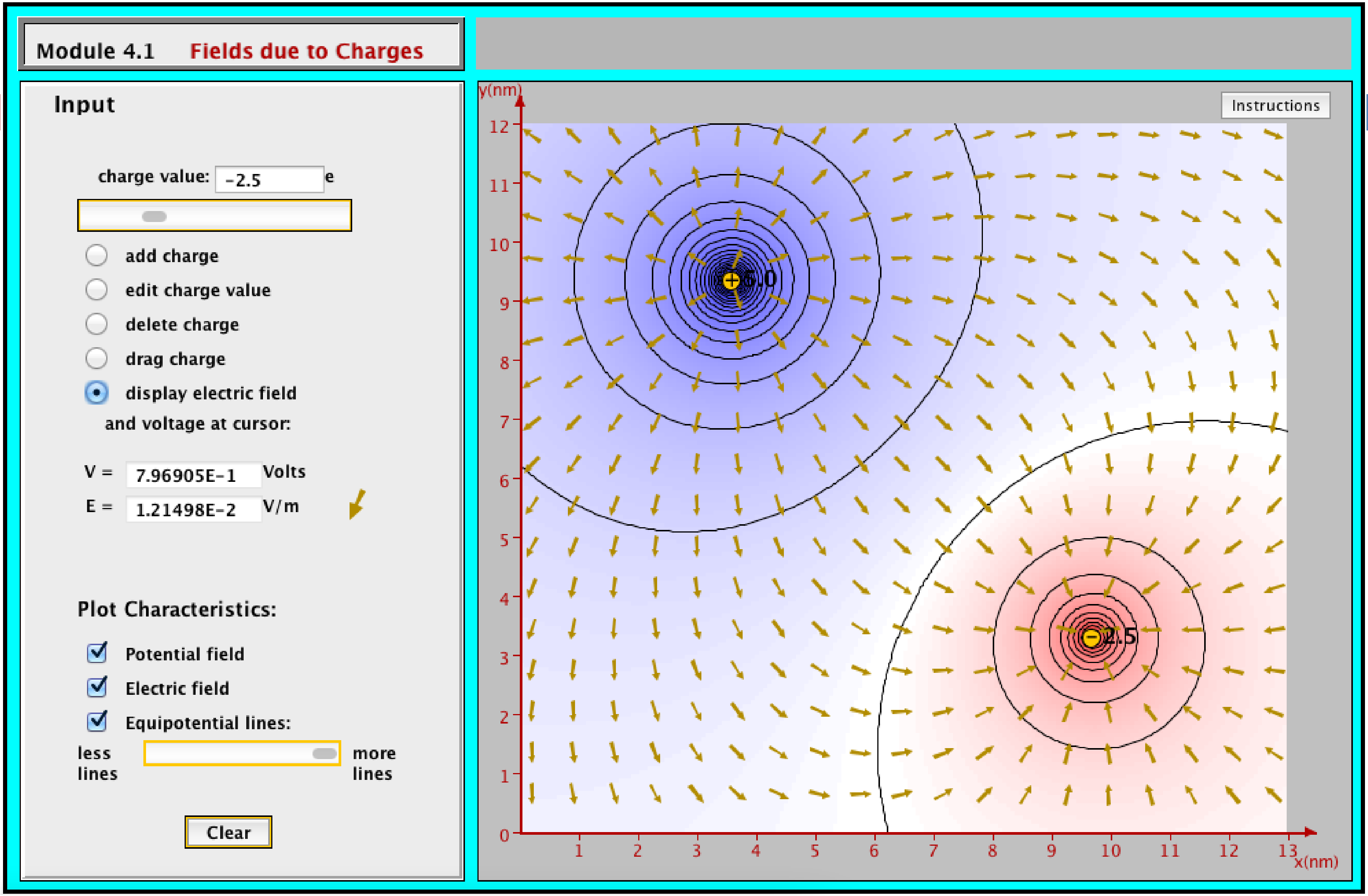
This is known as ***Poisson's equation***. For a volume $\mathcal{V}'$ containing a volume charge density distribution ρ_v, the solution for V derived previously and expressed by Eq. (4.48a) as

$$V = \frac{1}{4\pi\varepsilon}\int_{\mathcal{V}'} \frac{\rho_v}{R'}\,d\mathcal{V}' \qquad (4.61)$$

In the absence of charges:

$$\nabla^2 V = 0 \qquad \text{(Laplace's equation),}$$





Module 4.1 Fields due to Charges
Input
charge value: -2.5 e
add charge
edit charge value
delete charge
drag charge
display electric field
and voltage at cursor:
V = 7.96905E-1 Volts
E = 1.21498E-2 V/m
Plot Characteristics:
Potential field
Electric field
Equipotential lines:
less lines
more lines
Clear
Instructions
y(nm)
x(nm)
2.5

Conduction Current

The conductivity of a material is a measure of how easily electrons can travel through the material under the influence of an externally applied electric field.

Conduction current density:

$$\mathbf{J} = \sigma \mathbf{E} \qquad (\text{A/m}^2) \qquad (\text{Ohm's law}),$$

original

*A **perfect dielectric** is a material with $\sigma = 0$. In contrast, a **perfect conductor** is a material with $\sigma = \infty$. Some materials, called superconductors, exhibit such a behavior.*

Table 4-1: Conductivity of some common materials at 20°C.

Material	Conductivity, σ (S/m)
Conductors	
Silver	6.2×10^7
Copper	5.8×10^7
Gold	4.1×10^7
Aluminum	3.5×10^7
Iron	10^7
Mercury	10^6
Carbon	3×10^4
Semiconductors	
Pure germanium	2.2
Pure silicon	4.4×10^{-4}
Insulators	
Glass	10^{-12}
Paraffin	10^{-15}
Mica	10^{-15}
Fused quartz	10^{-17}

Note how wide the range is, over 24 orders of magnitude

Conductivity

$$\sigma = -\rho_{\text{ve}}\mu_{\text{e}} + \rho_{\text{vh}}\mu_{\text{h}}$$
$$= (N_{\text{e}}\mu_{\text{e}} + N_{\text{h}}\mu_{\text{h}})e \qquad \text{(S/m)} \quad \text{(semiconductor)}, \qquad (4.67\text{a})$$

and its unit is siemens per meter (S/m). For a good conductor, $N_{\text{h}}\mu_{\text{h}} \ll N_{\text{e}}\mu_{\text{e}}$, and Eq. (4.67a) reduces to

$$\sigma = -\rho_{\text{ve}}\mu_{\text{e}} = N_{\text{e}}\mu_{\text{e}}e \qquad \text{(S/m)} \quad \text{(conductor)}. \qquad (4.67\text{b})$$

In view of Eq. (4.66), in a perfect dielectric with $\sigma = 0$, $\mathbf{J} = 0$ regardless of $\mathbf{E}$, and in a perfect conductor with $\sigma = \infty$, $\mathbf{E} = \mathbf{J}/\sigma = 0$ regardless of $\mathbf{J}$.

$$\mathbf{J} = \sigma\mathbf{E} \qquad (\text{A/m}^2) \quad \text{(Ohm's law)},$$

That is,

Perfect dielectric:	$\mathbf{J} = 0$,
Perfect conductor:	$\mathbf{E} = 0$.

ρ_{ve} = volume charge density of electrons

ρ_{he} = volume charge density of holes

μ_e = electron mobility

μ_h = hole mobility

N_e = number of electrons per unit volume

N_h = number of holes per unit volume

Example 4-8: Conduction Current in a Copper Wire

A 2-mm-diameter copper wire with conductivity of 5.8×10^7 S/m and electron mobility of 0.0032 (m^2/V·s) is subjected to an electric field of 20 (mV/m). Find (a) the volume charge density of the free electrons, (b) the current density, (c) the current flowing in the wire, (d) the electron drift velocity, and (e) the volume density of the free electrons.

Solution:

(a)

$$\rho_{\text{ve}} = -\frac{\sigma}{\mu_{\text{e}}} = -\frac{5.8 \times 10^7}{0.0032} = -1.81 \times 10^{10} \text{ (C/m}^3\text{)}.$$

(b)

$$J = \sigma E = 5.8 \times 10^7 \times 20 \times 10^{-3} = 1.16 \times 10^6 \text{ (A/m}^2\text{)}.$$

(c)

$$I = JA$$

$$= J\left(\frac{\pi d^2}{4}\right) = 1.16 \times 10^6 \left(\frac{\pi \times 4 \times 10^{-6}}{4}\right) = 3.64 \text{ A}.$$

(d)

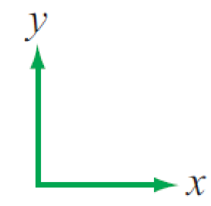
$$u_{\text{e}} = -\mu_{\text{e}} E = -0.0032 \times 20 \times 10^{-3} = -6.4 \times 10^{-5} \text{ m/s}.$$

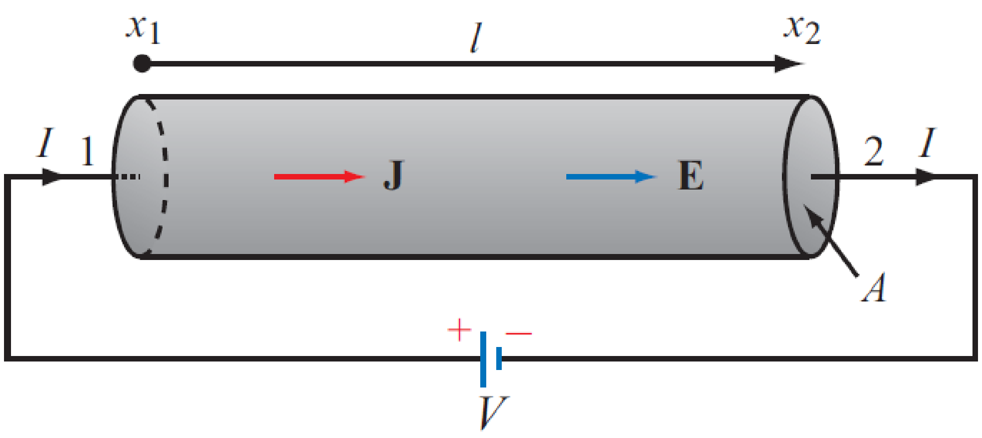
The minus sign indicates that $\mathbf{u}_{\text{e}}$ is in the opposite direction of $\mathbf{E}$.

(e)

$$N_{\text{e}} = -\frac{\rho_{\text{ve}}}{e} = \frac{1.81 \times 10^{10}}{1.6 \times 10^{-19}} = 1.13 \times 10^{29} \text{ electrons/m}^3.$$

Resistance





Longitudinal Resistor

$$V = V_1 - V_2 = -\int_{x_2}^{x_1} \mathbf{E} \cdot d\mathbf{l}$$

$$= -\int_{x_2}^{x_1} \hat{\mathbf{x}} E_x \cdot \hat{\mathbf{x}}\, dl = E_x l \qquad \text{(V)}. \qquad (4.68)$$

Using Eq. (4.63), the current flowing through the cross section A at x_2 is

$$I = \int_A \mathbf{J} \cdot d\mathbf{s} = \int_A \sigma \mathbf{E} \cdot d\mathbf{s} = \sigma E_x A \qquad \text{(A)}. \qquad (4.69)$$

From $R = V/I$, the ratio of Eq. (4.68) to Eq. (4.69) gives

$$R = \frac{l}{\sigma A} \qquad (\Omega). \qquad (4.70)$$

For any conductor:

$$R = \frac{V}{I} = \frac{-\int_l \mathbf{E} \cdot d\mathbf{l}}{\int_S \mathbf{J} \cdot d\mathbf{s}} = \frac{-\int_l \mathbf{E} \cdot d\mathbf{l}}{\int_S \sigma \mathbf{E} \cdot d\mathbf{s}}\,.$$

Example 4-9: Conductance of Coaxial Cable

The radii of the inner and outer conductors of a coaxial cable of length l are a and b, respectively (Fig. 4-15). The insulation material has conductivity σ. Obtain an expression for G', the conductance per unit length of the insulation layer.

Solution: Let I be the total current flowing radially (along $\hat{\mathbf{r}}$) from the inner conductor to the outer conductor through the insulation material. At any radial distance r from the axis of the center conductor, the area through which the current flows is $A = 2\pi r l$. Hence,

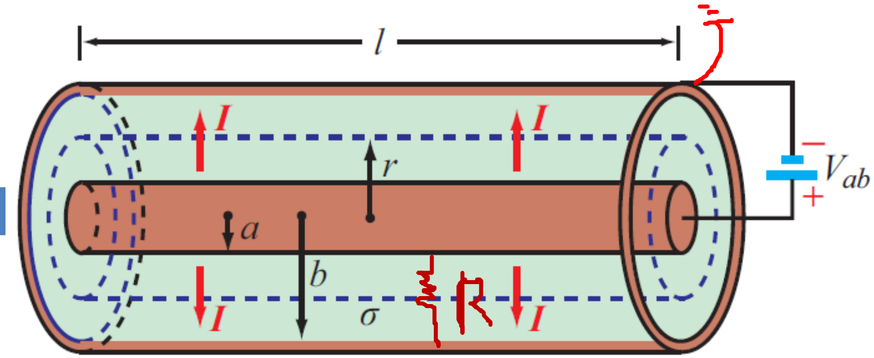
$$\mathbf{J} = \hat{\mathbf{r}}\,\frac{I}{A} = \hat{\mathbf{r}}\,\frac{I}{2\pi r l}\,, \qquad (4.73)$$

and from $\mathbf{J} = \sigma\mathbf{E}$,

$$\mathbf{E} = \hat{\mathbf{r}}\,\frac{I}{2\pi\sigma r l}\,. \qquad (4.74)$$

In a resistor, the current flows from higher electric potential to lower potential. Hence, if $\mathbf{J}$ is in the $\hat{\mathbf{r}}$-direction, the inner

G = 1/R



conductor must be at a higher potential than the outer conductor. Accordingly, the voltage difference between the conductors is

$$V_{ab} = -\int_b^a \mathbf{E}\cdot d\mathbf{l} = -\int_b^a \frac{I}{2\pi\sigma l}\,\frac{\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}\,dr}{r}$$

$$= \frac{I}{2\pi\sigma l}\ln\left(\frac{b}{a}\right). \qquad (4.75)$$

The conductance per unit length is then

$$G' = \frac{G}{l} = \frac{1}{Rl} = \frac{I}{V_{ab}l} = \frac{2\pi\sigma}{\ln(b/a)} \quad \text{(S/m)}. \qquad (4.76)$$

G'=0 if the insulating material is air or a perfect dielectric with zero conductivity.

Joule's Law

The power dissipated in a volume containing electric field **E** and current density **J** is:

$$P = \int_{\mathcal{V}} \mathbf{E} \cdot \mathbf{J} \, d\mathcal{V} \qquad \text{(W)} \qquad \text{(Joule's law)}$$

For a coaxial cable:

$$P = I^2 \ln(b/a)/(2\pi\sigma l)$$

Derive

~~Find~~

For a resistor, Joule's law reduces to:

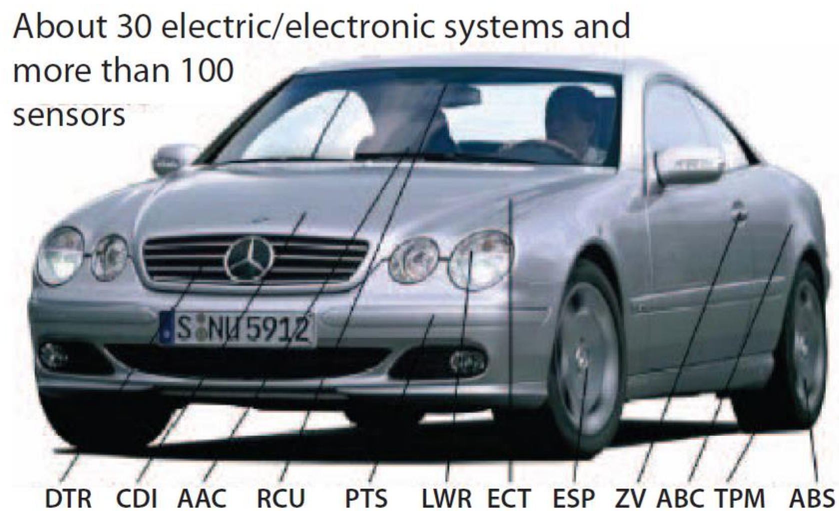
$$P = I^2 R \qquad \text{(W)}$$

Tech Brief 7: Resistive Sensors

An ***electrical sensor*** is a device capable of responding to an applied stimulus by generating an electrical signal whose voltage, current, or some other attribute is related to the intensity of the stimulus.

Typical stimuli : temperature, pressure, position, distance, motion, velocity, acceleration, concentration (of a gas or liquid), blood flow, etc.

Sensing process relies on measuring resistance, capacitance, inductance, induced electromotive force (emf), oscillation frequency or time delay, etc.



System	Abbrev.	Sensors	System	Abbrev.	Sensors
Distronic	DTR	3	Common-rail diesel injection	CDI	11
Electronic controlled transmission	ECT	9	Automatic air condition	AAC	13
Roof control unit	RCU	7	Active body control	ABC	12
Antilock braking system	ABS	4	Tire pressure monitoring	TPM	11
Central locking system	ZV	3	Elektron. stability program	ESP	14
Dyn. beam levelling	LWR	6	Parktronic system	PTS	12

Figure TF7-1: Most cars use on the order of 100 sensors. (Courtesy Mercedes-Benz.)

Piezoresistivity

The Greek word ***piezein*** means to press

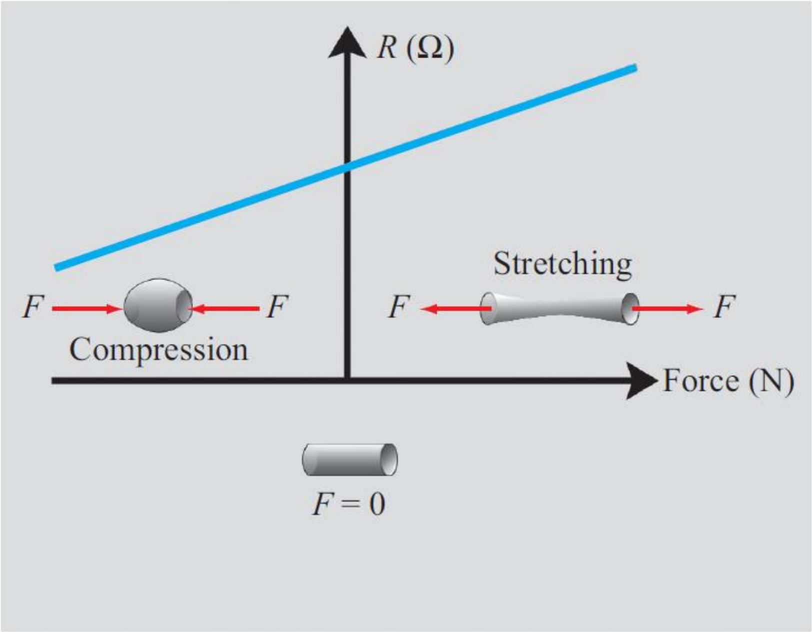


Figure TF7-2: Piezoresistance varies with applied force.

$$R = R_0 \left(1 + \frac{\alpha F}{A_0}\right)$$

R_0 = resistance when $F = 0$
F = applied force
A_0 = cross-section when $F = 0$
α = piezoresistive coefficient of material

Piezoresistors

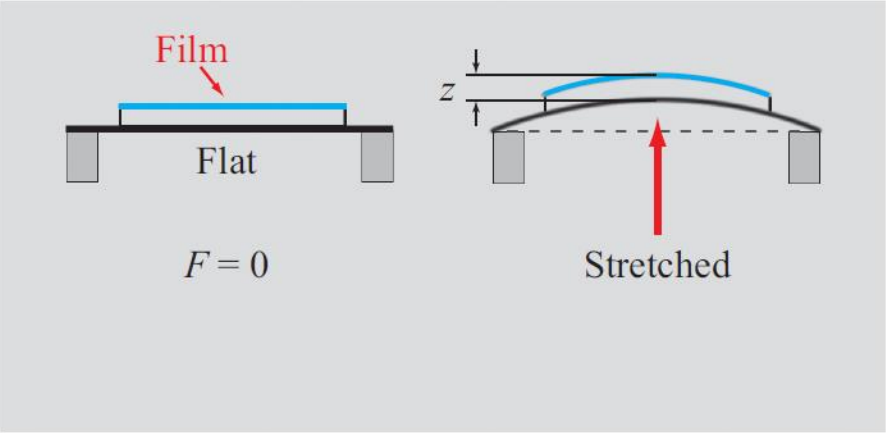


Figure TF7-3: Piezoresistor films.

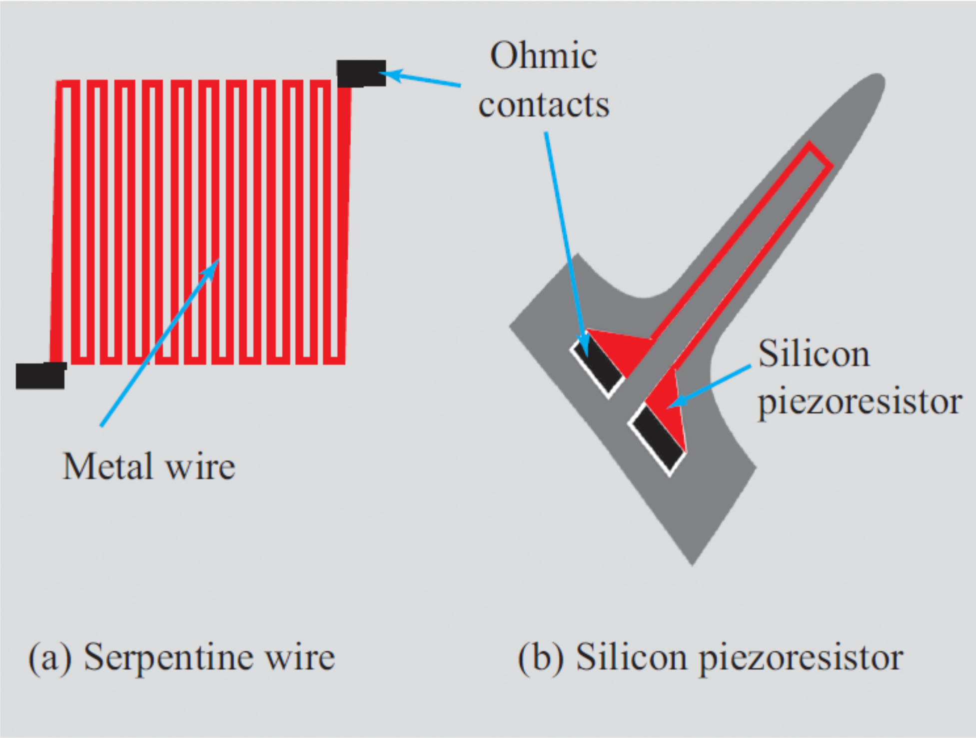
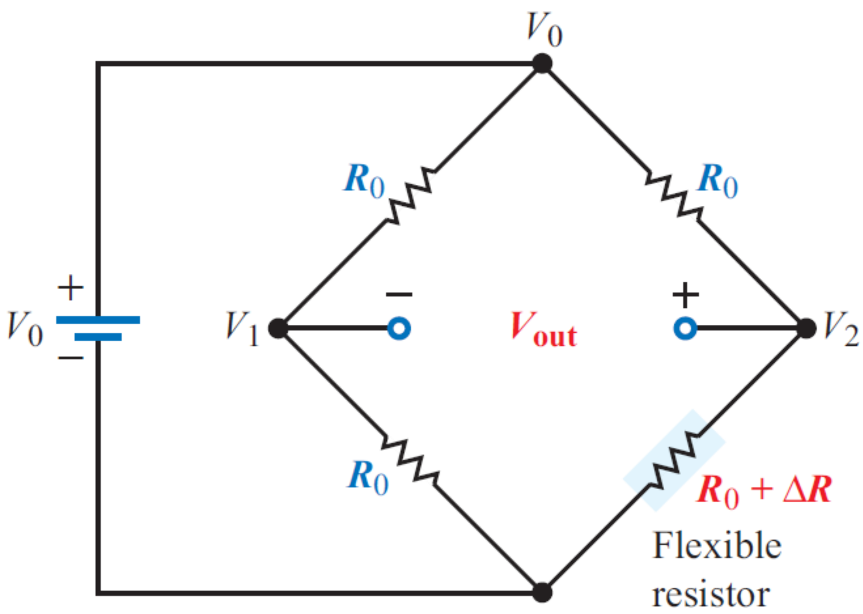


Figure TF7-4: Metal and silicon piezoresistors.

Wheatstone Bridge



Wheatstone bridge is a high sensitivity circuit for measuring small changes in resistance

$$V_{out} = \frac{V_0}{4}\left(\frac{\Delta R}{R_0}\right)$$

Figure TF7-5: Wheatstone bridge circuit with piezoresistor.

Dielectric Materials

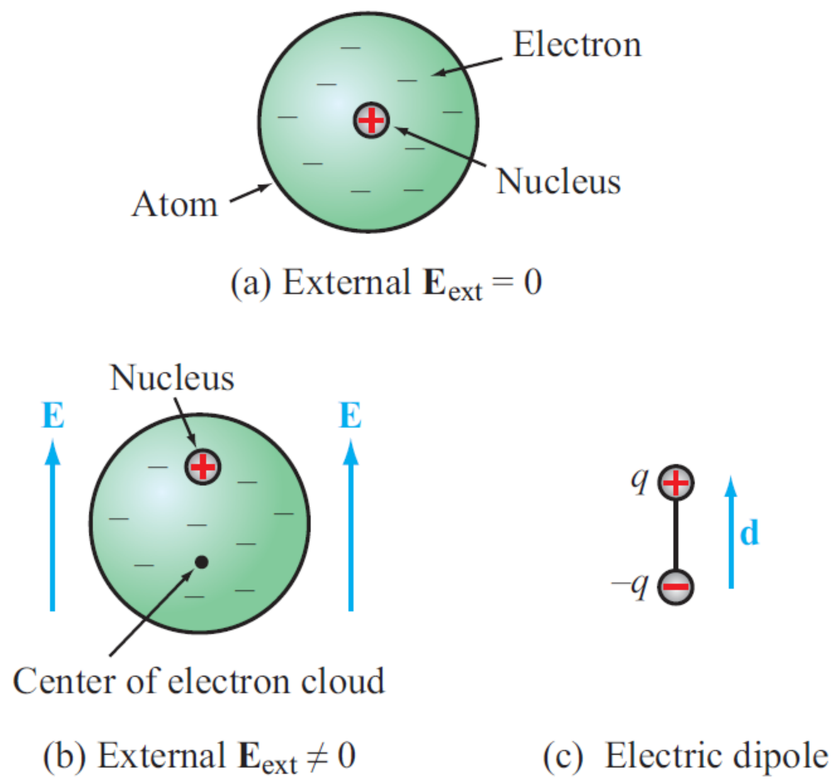


Figure 4-16: In the absence of an external electric field **E**, the center of the electron cloud is co-located with the center of the nucleus, but when a field is applied, the two centers are separated by a distance d.

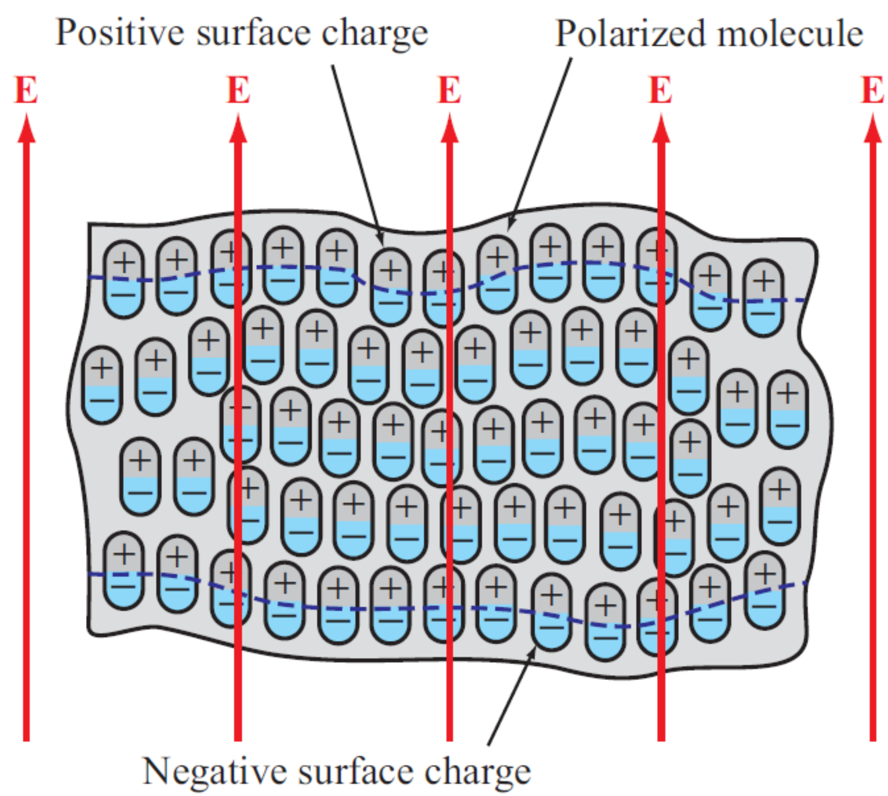


Figure 4-17: A dielectric medium polarized by an external electric field **E**.

Polarization Field

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$$

P = electric flux density induced by **E**

$$\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E}, \quad (4.84)$$

where χ_e is called the ***electric susceptibility*** of the material. Inserting Eq. (4.84) into Eq. (4.83), we have

$$\begin{aligned} \mathbf{D} &= \varepsilon_0 \mathbf{E} + \varepsilon_0 \chi_e \mathbf{E} \\ &= \varepsilon_0 (1 + \chi_e) \mathbf{E} = \varepsilon \mathbf{E}, \end{aligned} \quad (4.85)$$

Electric Breakdown

The dielectric strength E_{ds} is the largest magnitude of **E** *that the material can sustain without breakdown.*

Table 4-2: Relative permittivity (dielectric constant) and dielectric strength of common materials.

Material	Relative Permittivity, ε_r	Dielectric Strength, E_{ds} (MV/m)
Air (at sea level)	1.0006	3
Petroleum oil	2.1	12
Polystyrene	2.6	20
Glass	4.5–10	25–40
Quartz	3.8–5	30
Bakelite	5	20
Mica	5.4–6	200

$\varepsilon = \varepsilon_r \varepsilon_0$ and $\varepsilon_0 = 8.854 \times 10^{-12}$ F/m.

Boundary Conditions

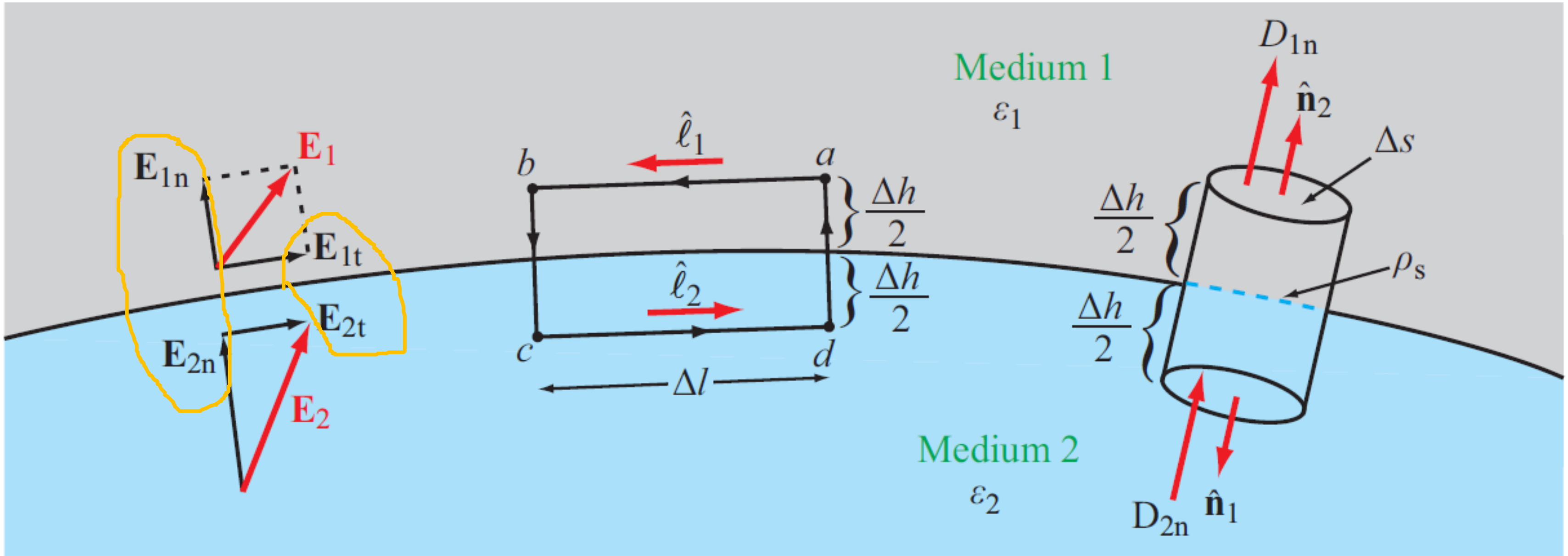


Figure 4-18: Interface between two dielectric media.

$$\mathbf{E}_{1t} = \mathbf{E}_{2t} \qquad \text{(V/m)}. \qquad (4.90)$$

$$\frac{\mathbf{D}_{1t}}{\varepsilon_1} = \frac{\mathbf{D}_{2t}}{\varepsilon_2}. \qquad (4.91)$$

$$\hat{\mathbf{n}}_2 \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s \qquad (\text{C/m}^2).$$

$$D_{1n} - D_{2n} = \rho_s \qquad (\text{C/m}^2). \qquad (4.94)$$

The normal component of **D** *changes abruptly at a charged boundary between two different media in an amount equal to the surface charge density.*

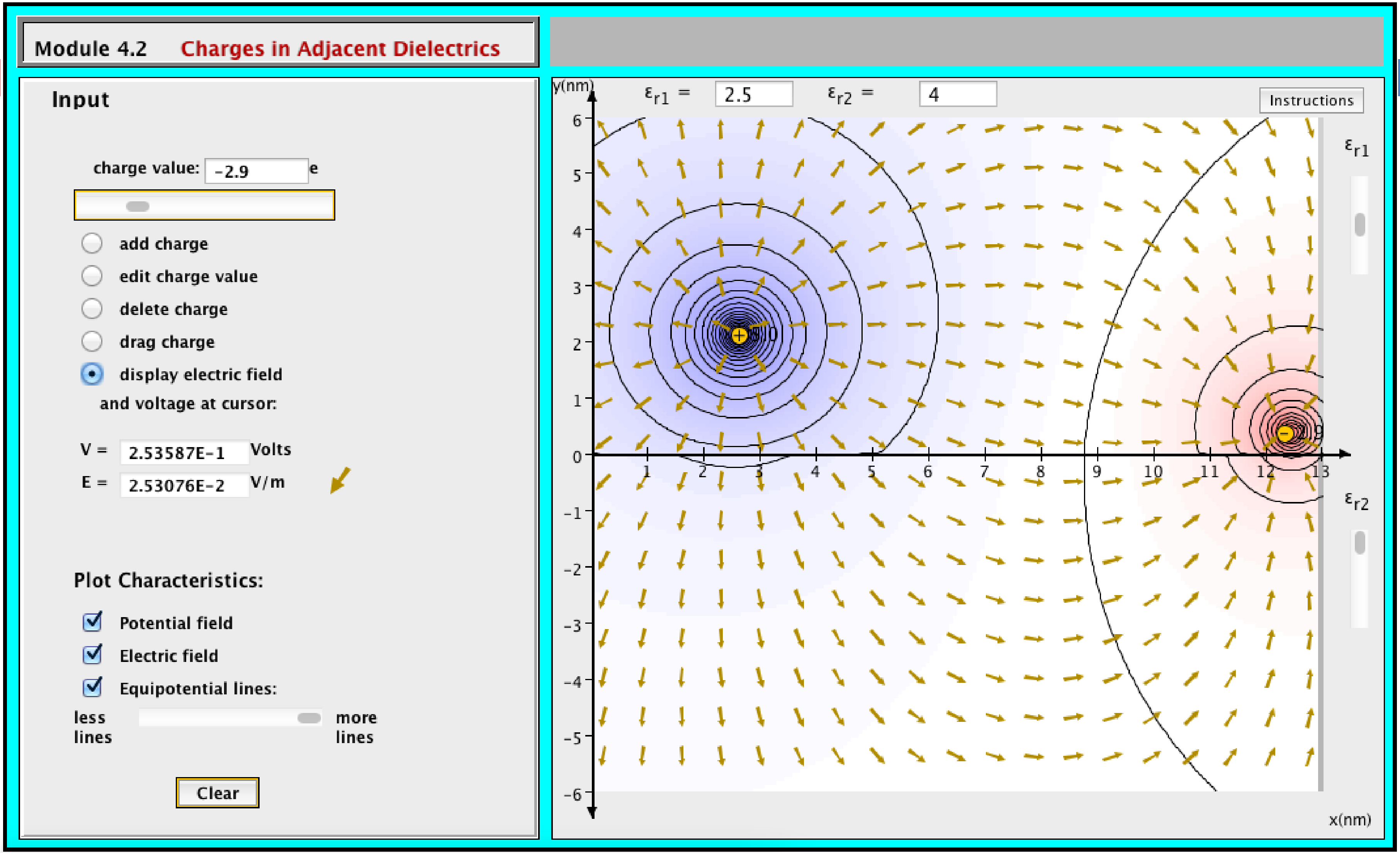
Summary of Boundary Conditions

Table 4-3: Boundary conditions for the electric fields.

Field Component	Any Two Media	Medium 1 Dielectric ε_1	Medium 2 Conductor
Tangential E	$\mathbf{E}_{1t} = \mathbf{E}_{2t}$	$\mathbf{E}_{1t} = \mathbf{E}_{2t} = 0$	
Tangential D	$\mathbf{D}_{1t}/\varepsilon_1 = \mathbf{D}_{2t}/\varepsilon_2$	$\mathbf{D}_{1t} = \mathbf{D}_{2t} = 0$	
Normal E	$\varepsilon_1 E_{1n} - \varepsilon_2 E_{2n} = \rho_s$	$E_{1n} = \rho_s/\varepsilon_1$	$E_{2n} = 0$
Normal D	$D_{1n} - D_{2n} = \rho_s$	$D_{1n} = \rho_s$	$D_{2n} = 0$

Notes: (1) ρ_s is the surface charge density at the boundary; (2) normal components of $\mathbf{E}_1$, $\mathbf{D}_1$, $\mathbf{E}_2$, and $\mathbf{D}_2$ are along $\hat{\mathbf{n}}_2$, the outward normal unit vector of medium 2.

Remember $\mathbf{E} = 0$ in a good conductor


Module 4.2
Charges in Adjacent Dielectrics
Input
charge value: -2.9 e
add charge
edit charge value
delete charge
drag charge
display electric field
and voltage at cursor:
V = 2.53587E-1 Volts
E = 2.53076E-2 V/m
Plot Characteristics:
Potential field
Electric field
Equipotential lines:
less lines
more lines
Clear
y(nm)
εr1 = 2.5
εr2 = 4
Instructions
εr1
εr2
x(nm)

Conductors

Net electric field inside a conductor is zero

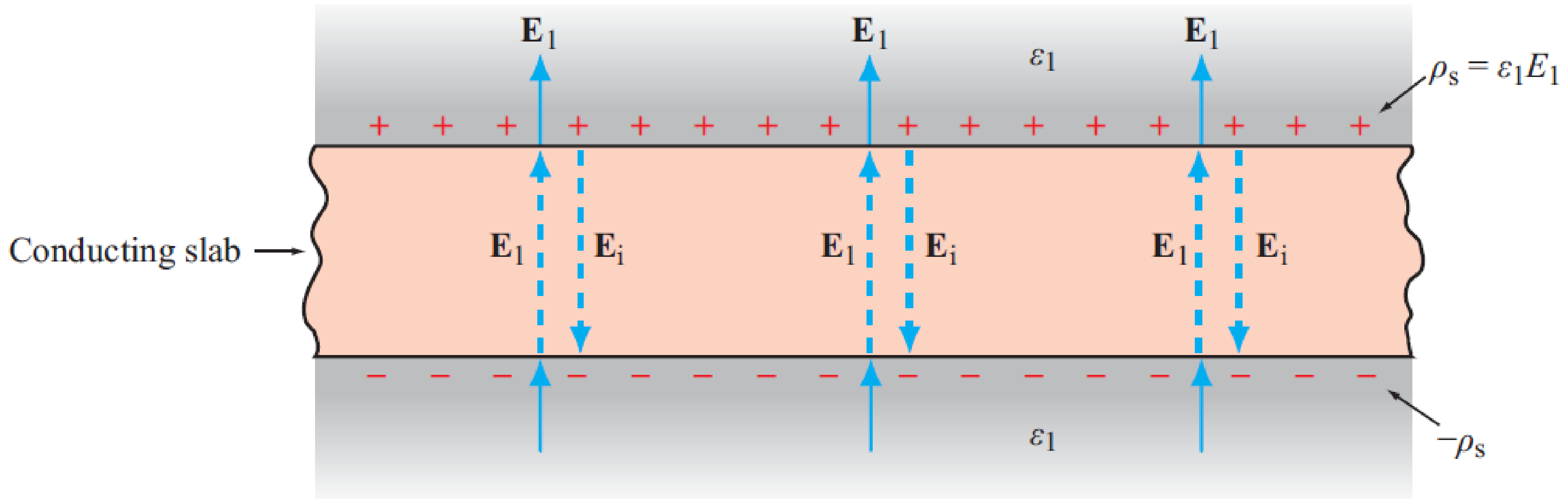


Figure 4-20: When a conducting slab is placed in an external electric field $\mathbf{E}_1$, charges that accumulate on the conductor surfaces induce an internal electric field $\mathbf{E}_i = -\mathbf{E}_1$. Consequently, the total field inside the conductor is zero.

Field Lines at Conductor Boundary

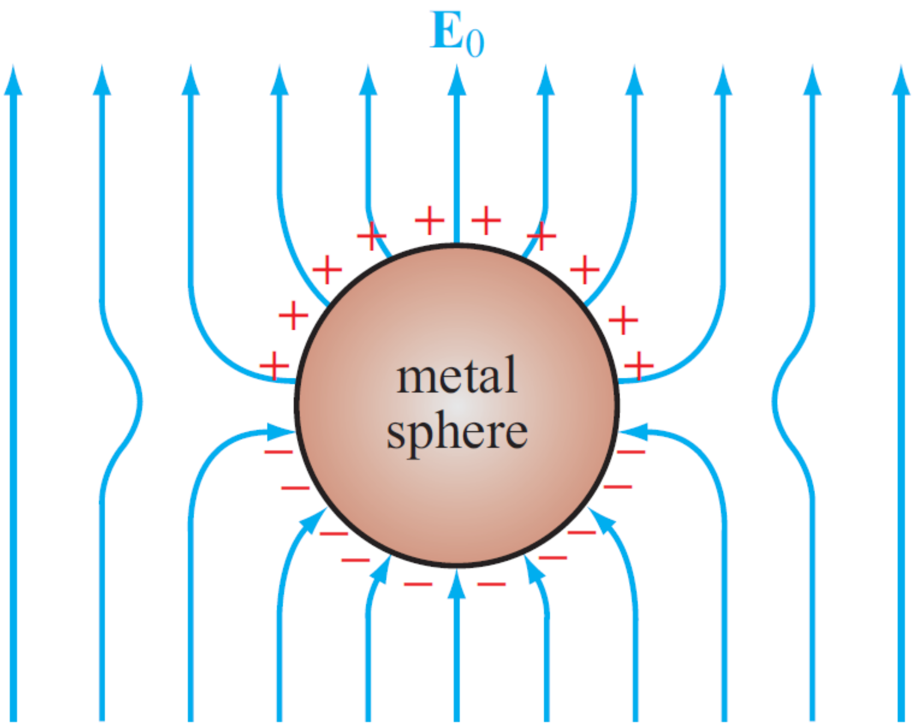
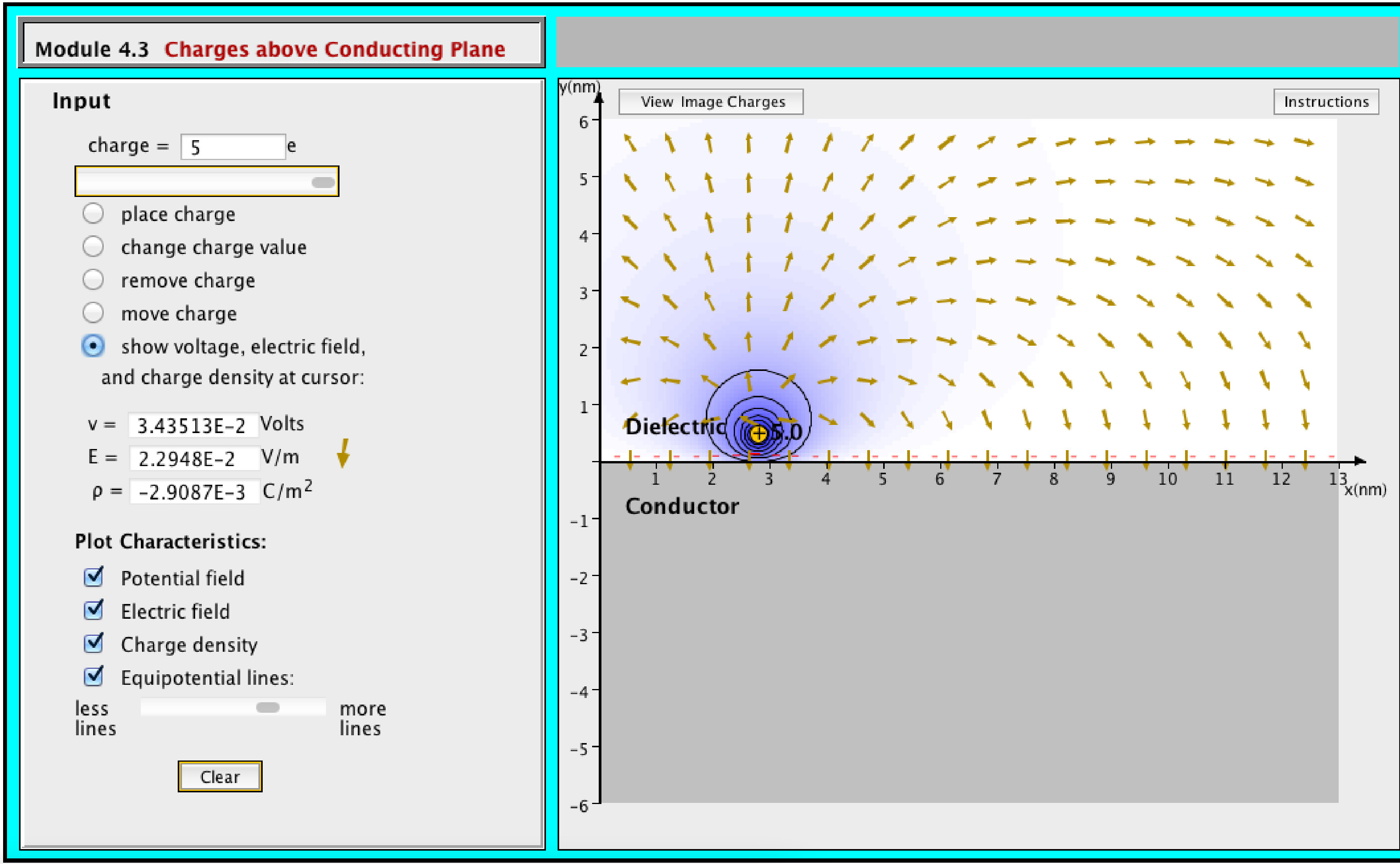
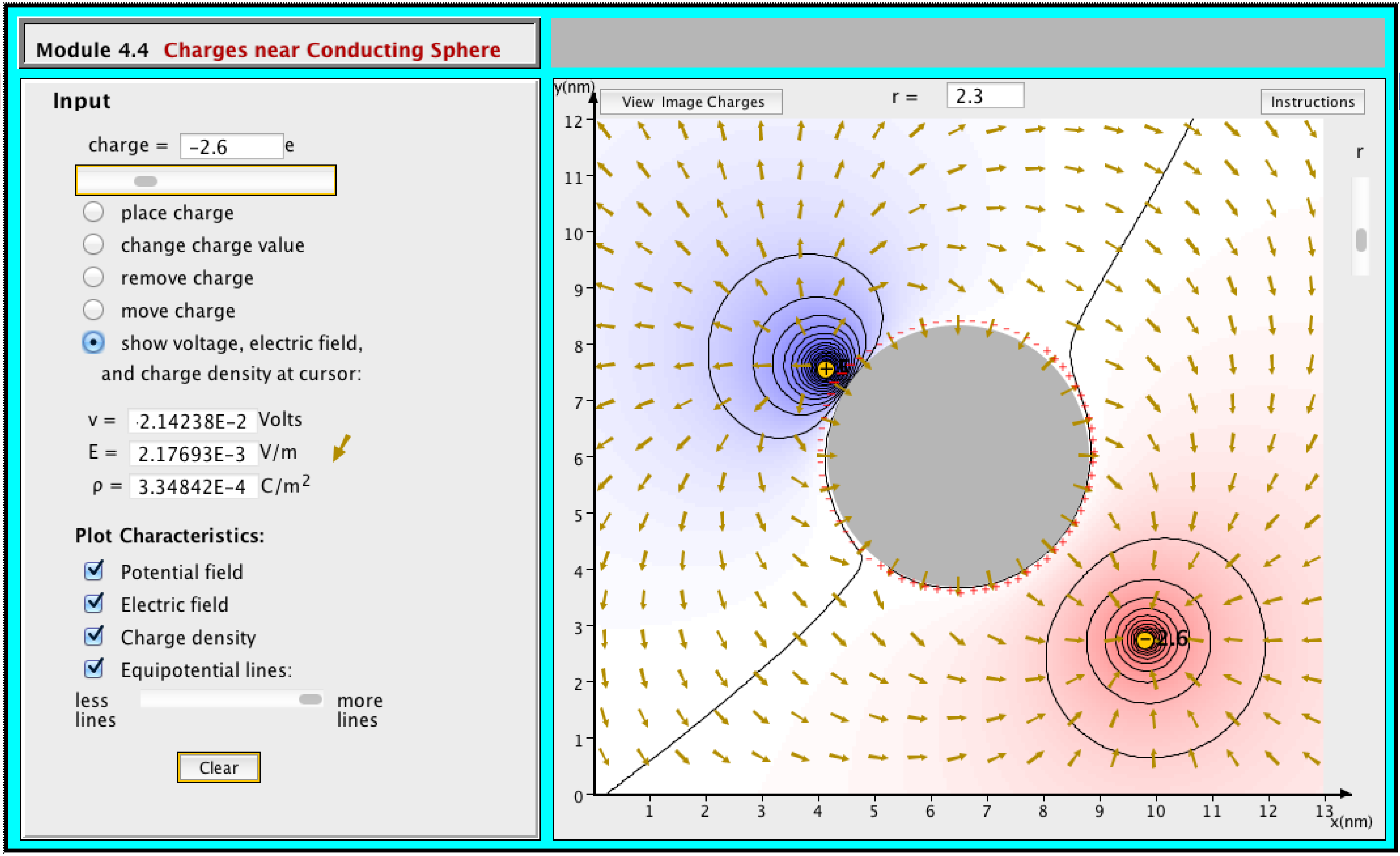


Figure 4-21: Metal sphere placed in an external electric field $\mathbf{E}_0$.

At conductor boundary, **E** field direction is always perpendicular to conductor surface


Module 4.3 Charges above Conducting Plane
Input
charge = 5 e
place charge
change charge value
remove charge
move charge
show voltage, electric field, and charge density at cursor:
v = 3.43513E-2 Volts
E = 2.2948E-2 V/m
ρ = -2.9087E-3 C/m2
Plot Characteristics:
Potential field
Electric field
Charge density
Equipotential lines:
less lines
more lines
Clear
View Image Charges
Instructions
y(nm)
x(nm)
Dielectric
5.0
Conductor



Module 4.4 Charges near Conducting Sphere
Input
charge = -2.6 e
place charge
change charge value
remove charge
move charge
show voltage, electric field, and charge density at cursor:
v = 2.14238E-2 Volts
E = 2.17693E-3 V/m
ρ = 3.34842E-4 C/m2
Plot Characteristics:
Potential field
Electric field
Charge density
Equipotential lines:
less lines
more lines
Clear
View Image Charges
r = 2.3
Instructions
y(nm)
x(nm)
r
-2.6

Capacitance

When a conductor has excess charge, it distributes the charge on its surface in such a manner as to maintain a zero electric field everywhere within the conductor, thereby ensuring that the electric potential is the same at every point in the conductor.

The ***capacitance*** of a two-conductor configuration is defined as

$$C = \frac{Q}{V} \qquad \text{(C/V or F)}, \qquad (4.105)$$

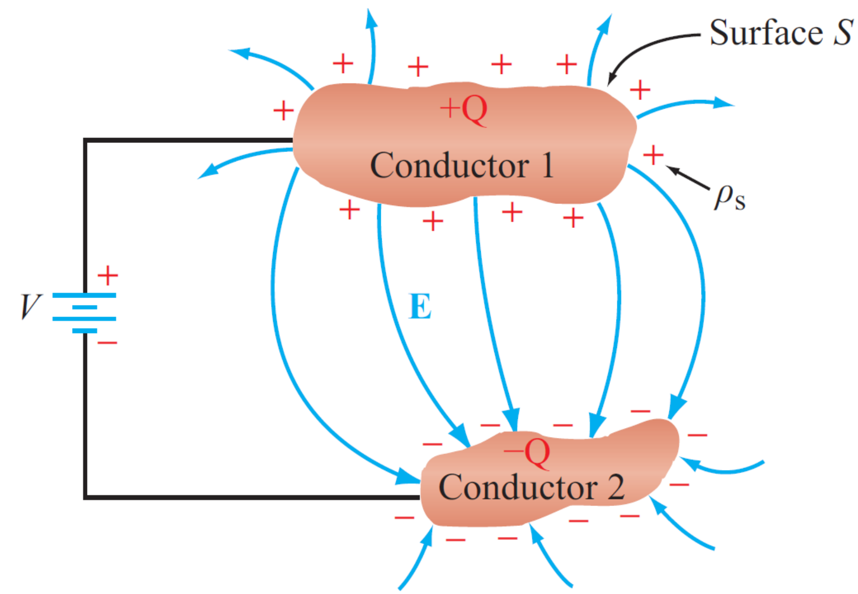


Figure 4-23: A dc voltage source connected to a capacitor composed of two conducting bodies.

Capacitance

For any two-conductor configuration:

$$C = \frac{\int_S \varepsilon \mathbf{E} \cdot d\mathbf{s}}{-\int_l \mathbf{E} \cdot d\mathbf{l}} \quad \text{(F)},$$

For any resistor:

$$R = \frac{-\int_l \mathbf{E} \cdot d\mathbf{l}}{\int_S \sigma \mathbf{E} \cdot d\mathbf{s}} \quad (\Omega). \qquad (4.110)$$

For a medium with uniform σ and ε, the product of Eqs. (4.109) and (4.110) gives

$$RC = \frac{\varepsilon}{\sigma}. \qquad (4.111)$$

This simple relation allows us to find R if C is known, or vice versa.

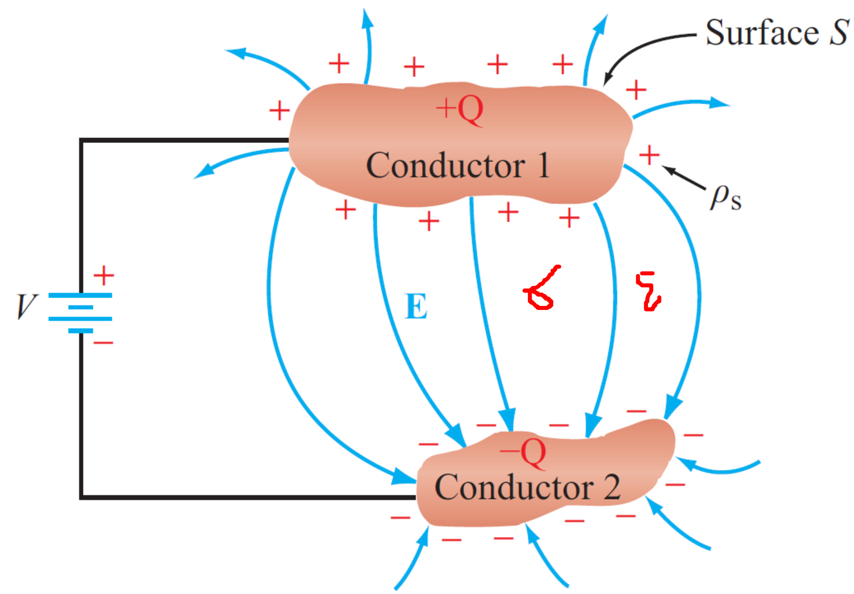


Figure 4-23: A dc voltage source connected to a capacitor composed of two conducting bodies.

Example 4-11: Capacitance and Breakdown Voltage of Parallel-Plate Capacitor

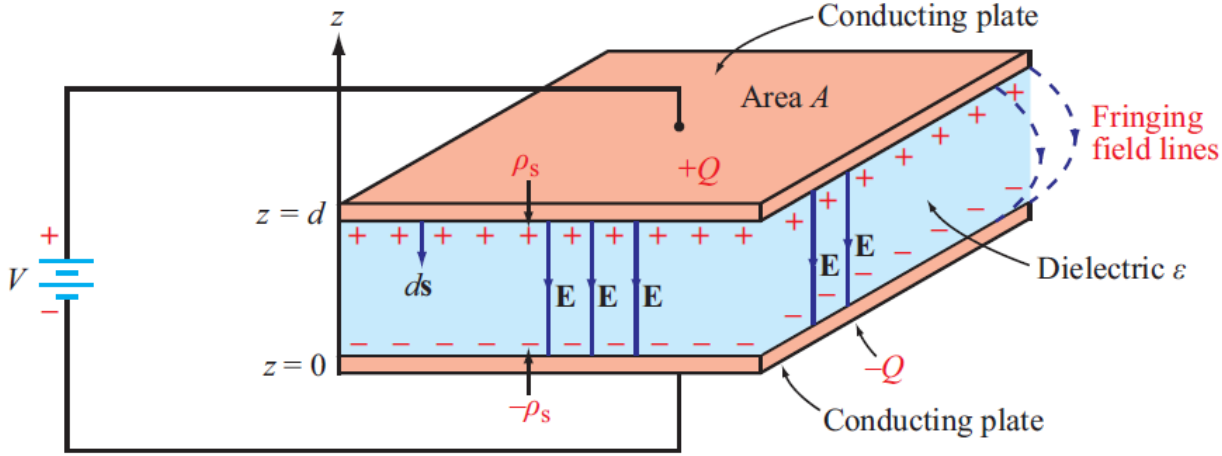


Figure 4-24: A dc voltage source connected to a parallel-plate capacitor (Example 4-11).

$$V = -\int_0^d \mathbf{E} \cdot d\mathbf{l} = -\int_0^d (-\hat{\mathbf{z}}E) \cdot \hat{\mathbf{z}}\, dz = Ed, \qquad (4.112)$$

and the capacitance is

$$C = \frac{Q}{V} = \frac{Q}{Ed} = \frac{\varepsilon A}{d}, \qquad (4.113)$$

where use was made of the relation $E = Q/\varepsilon A$.

From $V = Ed$, as given by Eq. (4.112), $V = V_{br}$ when $E = E_{ds}$, the dielectric strength of the material. According to Table 4-2, $E_{ds} = 30$ (MV/m) for quartz. Hence, the breakdown voltage is

$$V_{br} = E_{ds}d = 30 \times 10^6 \times 10^{-2} = 3 \times 10^5 \text{ V}.$$

Example 4-12: Capacitance Per Unit Length of Coaxial Line

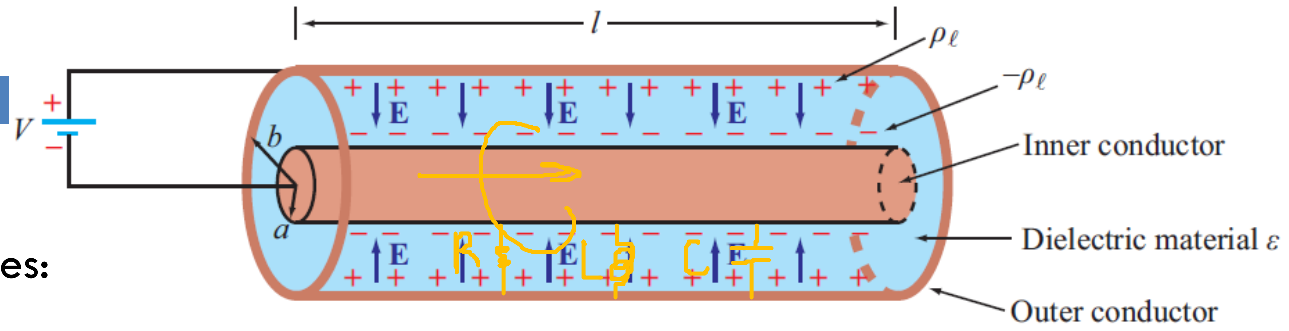


Figure 4-25: Coaxial capacitor filled with insulating material of permittivity ε (Example 4-12).

Application of Gauss's law gives:

$$\mathbf{E} = -\hat{\mathbf{r}} \frac{Q}{2\pi \varepsilon r l} .$$

The potential difference V between the outer and inner conductors is

$$V = -\int_a^b \mathbf{E} \cdot d\mathbf{l} = -\int_a^b \left(-\hat{\mathbf{r}} \frac{Q}{2\pi \varepsilon r l}\right) \cdot (\hat{\mathbf{r}}\, dr)$$

$$= \frac{Q}{2\pi \varepsilon l} \ln\left(\frac{b}{a}\right) . \qquad (4.115)$$

Q is total charge on inside of outer cylinder, and –Q is on outside surface of inner cylinder

The capacitance C is then given by

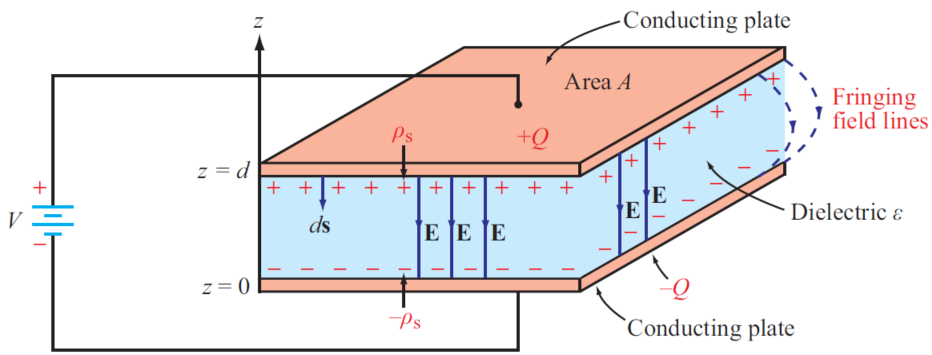
$$C = \frac{Q}{V} = \frac{2\pi \varepsilon l}{\ln(b/a)} , \qquad (4.116)$$

and the capacitance per unit length of the coaxial line is

$$C' = \frac{C}{l} = \frac{2\pi \varepsilon}{\ln(b/a)} \quad \text{(F/m)}. \qquad (4.117)$$

Tech Brief 8: Supercapacitors

For a traditional parallel-plate capacitor, what is the maximum attainable energy density?



Energy density is given by:

$$W' = \frac{\varepsilon V^2}{2\rho d^2} \qquad \text{(J/kg)}$$

ε = permittivity of insulation material

V = applied voltage

ρ = density of insulation material

d = separation between plates

Mica has one of the highest dielectric strengths ~2 x 10**8 V/m.

If we select a voltage rating of 1 V and a breakdown voltage of 2 V (50% safety), this will require that d be no smaller than 10 nm.

For mica, $\varepsilon = 6\varepsilon_0$ and $\rho = 3 \times 10^{3}$ kg/m^3 .

Hence:

W′ = 90 J/kg = 2.5 x10**–2 Wh/kg.

By comparison, a lithium-ion battery has W′ = 1.5 x 10**2 Wh/kg, almost 4 orders of magnitude greater

A supercapacitor is a "hybrid" battery/capacitor

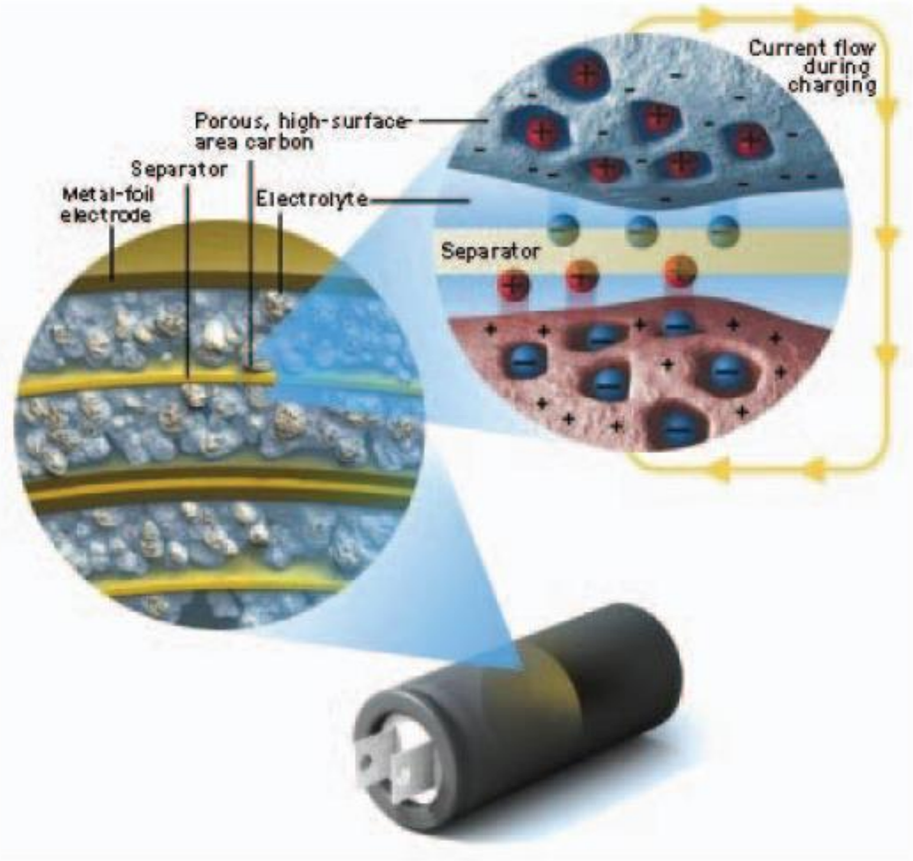


Figure TF8-1: Cross-sectional view of an electrochemical double-layer capacitor (EDLC), otherwise known as a supercapacitor. (Courtesy of Ultracapacitor.org.)

Users of Supercapacitors

Figure TF8-2: Examples of systems that use supercapacitors. (Courtesy of Railway Gazette International; BMW; NASA; Applied Innovative Technologies.)

Energy Comparison

Energy Storage Devices

Feature	Traditional Capacitor	Supercapacitor	Battery
Energy density W' (Wh/kg)	$\sim 10^{-2}$	1 to 10	5 to 150
Power density P' (W/kg)	1,000 to 10,000	1,000 to 5,000	10 to 500
Charge and discharge rate T	10^{-3} sec	$\sim$ 1 sec to 1 min	$\sim$ 1 to 5 hrs
Cycle life N_c	∞	$\sim 10^6$	$\sim 10^3$

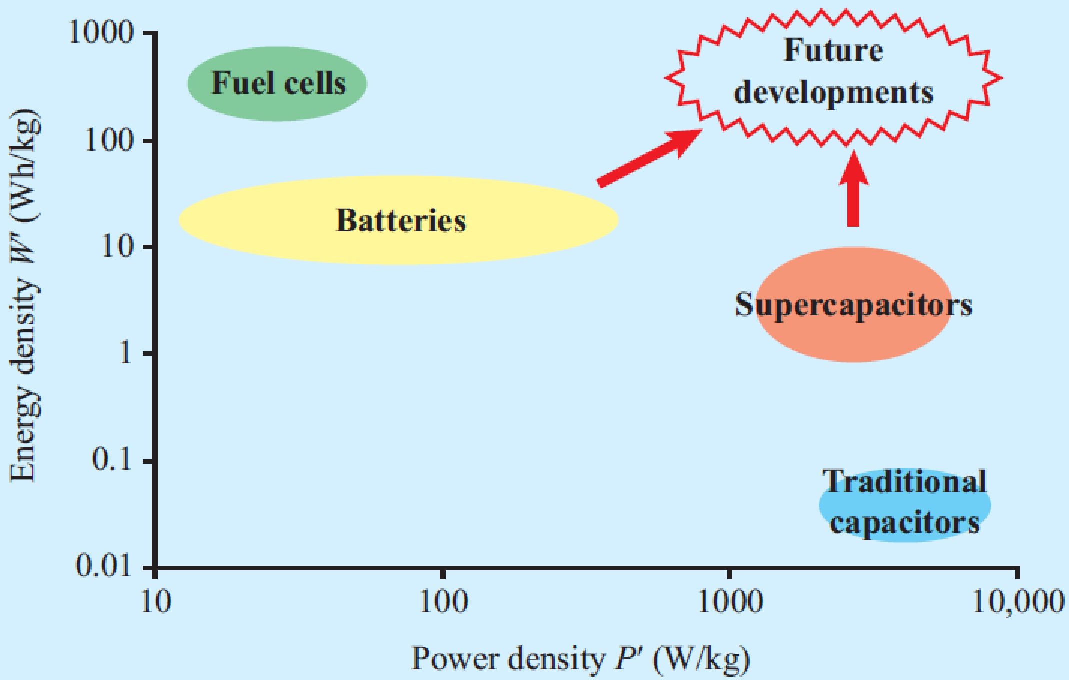


Figure TF8-3: Comparison of energy storage devices.

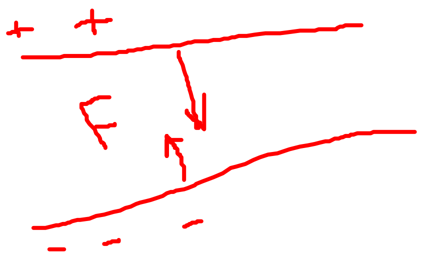
Electrostatic Potential Energy

Electrostatic potential energy density (Joules/volume)

$$w_e = \frac{W_e}{\mathcal{V}} = \frac{1}{2}\,\varepsilon E^2 \qquad (\text{J/m}^3).$$

Energy stored in a capacitor

$$W_e = \tfrac{1}{2} C V^2 \qquad (\text{J}).$$



Total electrostatic energy stored in a volume

$$W_e = \frac{1}{2} \int_{\mathcal{V}} \varepsilon E^2 \, d\mathcal{V} \qquad (\text{J})$$

Image Method

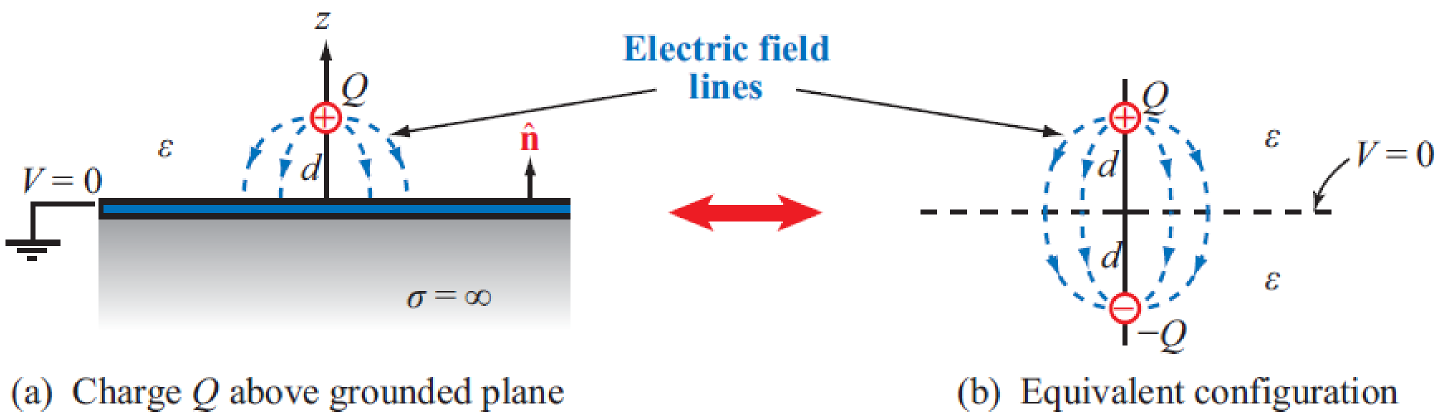


Figure 4-26: By image theory, a charge Q above a grounded perfectly conducting plane is equivalent to Q and its image $-Q$ with the ground plane removed.

Image method simplifies calculation for **E** and *V* due to charges near conducting planes.

1. For each charge Q, add an image charge –Q
2. Remove conducting plane
3. Calculate field due to all charges

Example 4-13: Image Method for Charge Above Conducting Plane

Use image theory to determine **E** at an arbitrary point $P = (x, y, z)$ in the region $z > 0$ due to a charge Q in free space at a distance d above a grounded conducting plate residing in the $z = 0$ plane.

Solution: In Fig. 4-28, charge Q is at $(0, 0, d)$ and its image $-Q$ is at $(0, 0, -d)$. From Eq. (4.19), the electric field at point $P = (x, y, z)$ due to the two charges is given by

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q\mathbf{R}_1}{R_1^3} + \frac{-Q\mathbf{R}_2}{R_2^3}\right)$$

$$= \frac{Q}{4\pi\varepsilon_0}\left[\frac{\hat{\mathbf{x}}x + \hat{\mathbf{y}}y + \hat{\mathbf{z}}(z-d)}{[x^2 + y^2 + (z-d)^2]^{3/2}} - \frac{\hat{\mathbf{x}}x + \hat{\mathbf{y}}y + \hat{\mathbf{z}}(z+d)}{[x^2 + y^2 + (z+d)^2]^{3/2}}\right]$$

for $z \geq 0$.

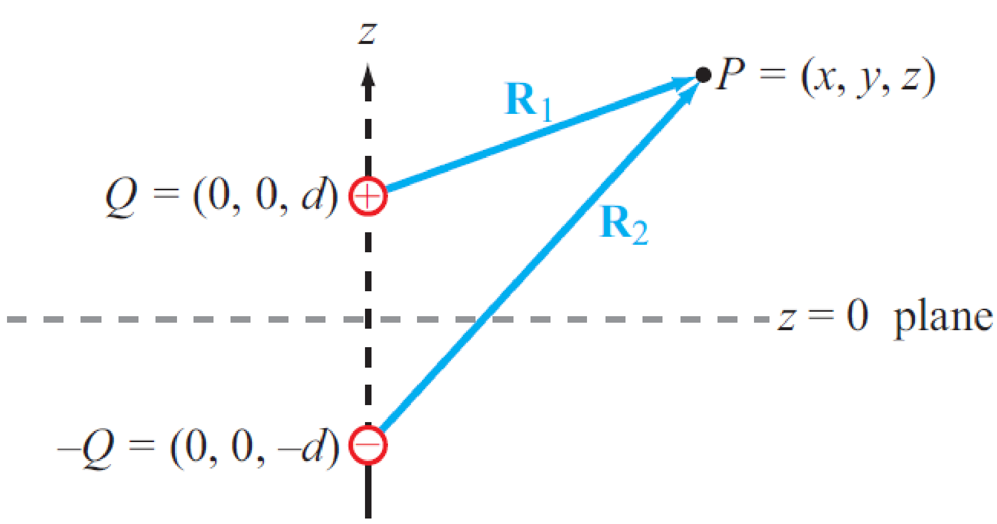


Figure 4-28: Application of the image method for finding **E** a point P (Example 4-13).

Tech Brief 9:
Capacitive Sensors

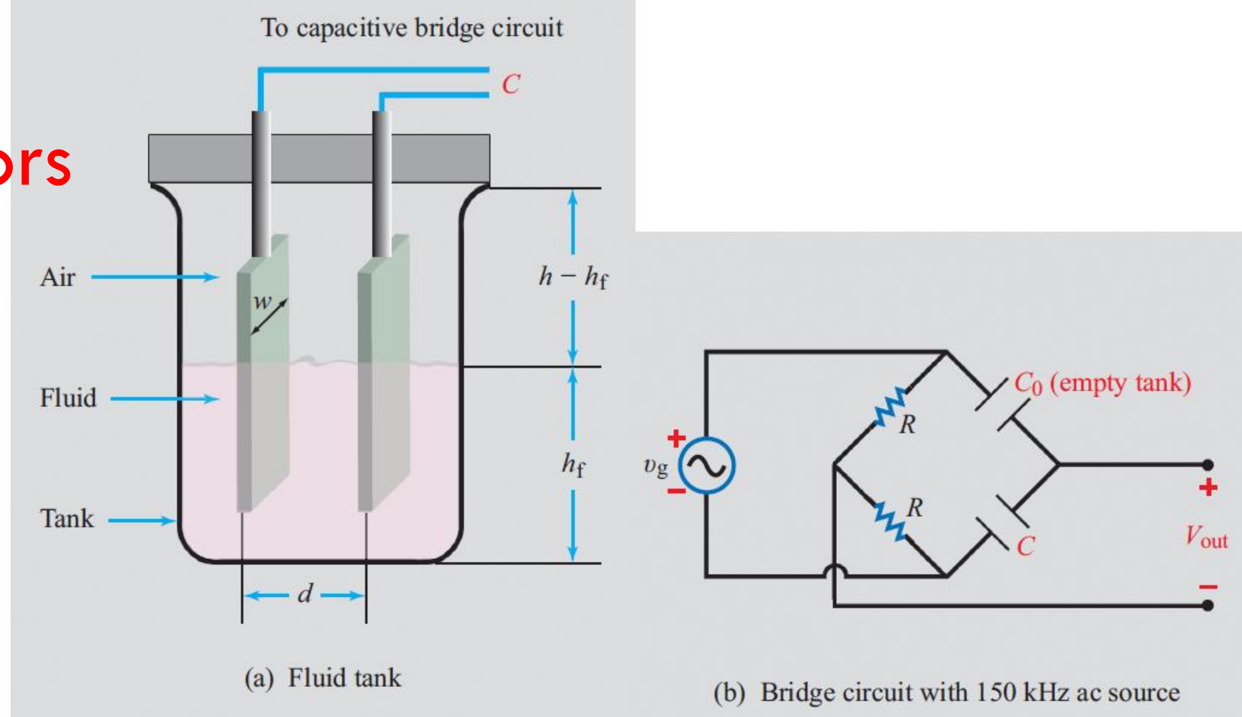


Figure TF9-1: Fluid gauge and associated bridge circuit, with C_0 being the capacitance that an empty tank would have and C the capacitance of the tank under test.

Fluid Gauge

The two metal electrodes in Fig. TF9-1(a), usually rods or plates, form a capacitor whose capacitance is directly proportional to the ***permittivity*** of the material between them. If the fluid section is of height h_f and the height of the empty space above it is $(h - h_f)$, then the overall capacitance is equivalent to two capacitors in parallel, or

$$C = C_f + C_a = \varepsilon_f w \frac{h_f}{d} + \varepsilon_a w \frac{(h - h_f)}{d},$$

where w is the electrode plate width, d is the spacing between electrodes, and ε_f and ε_a are the permittivities of the fluid and air, respectively. Rearranging the expression as a linear equation yields

$$C = kh_f + C_0,$$

Humidity Sensor

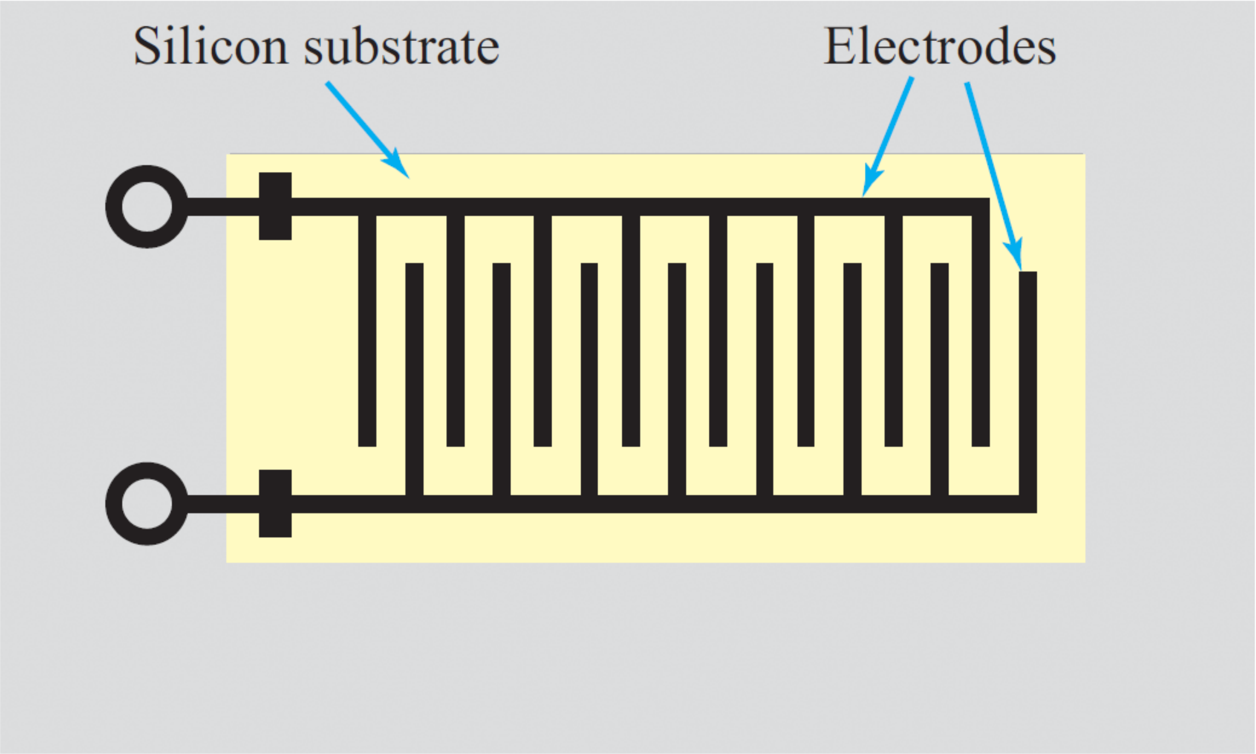


Figure TF9-2: Interdigital capacitor used as a humidity sensor.

Pressure Sensor

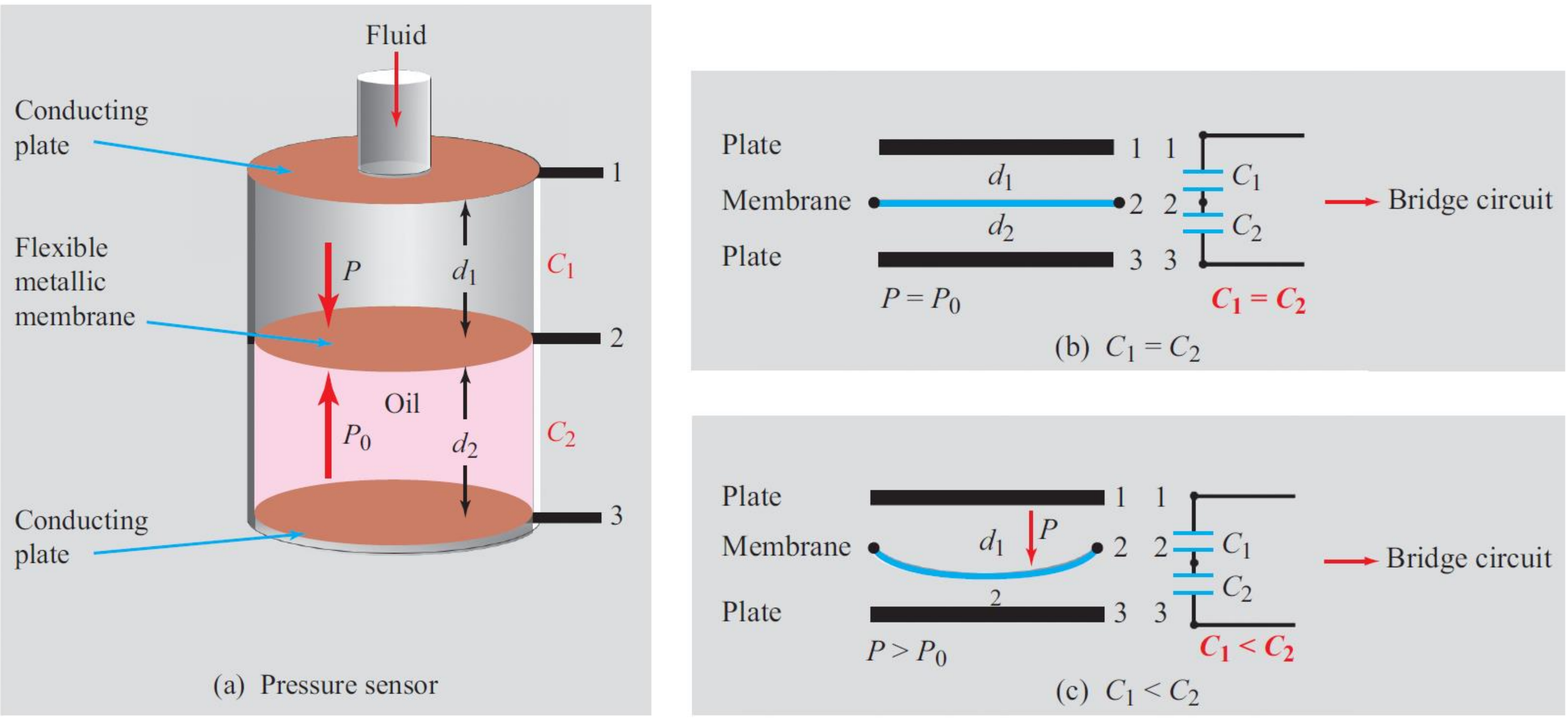


Figure TF9-3: Pressure sensor responds to deflection of metallic membrane.

Planar capacitors

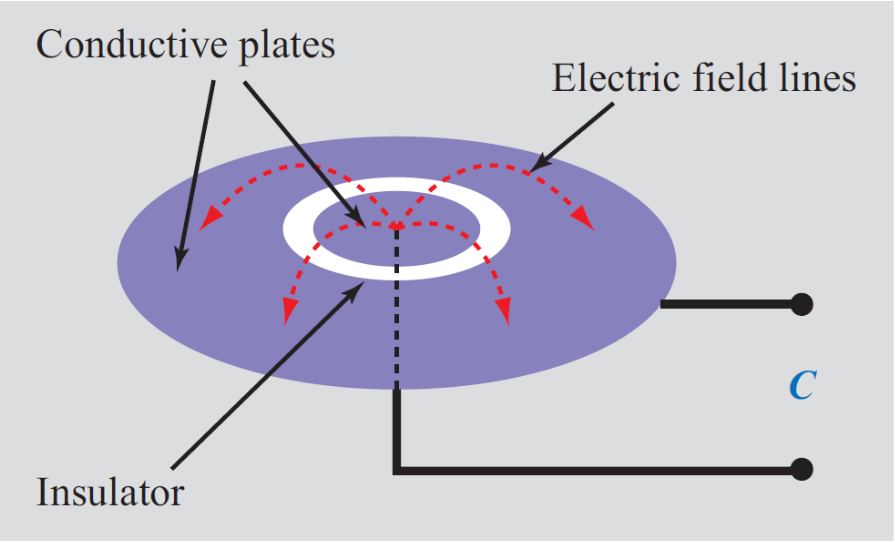


Figure TF9-4: Concentric-plate capacitor.

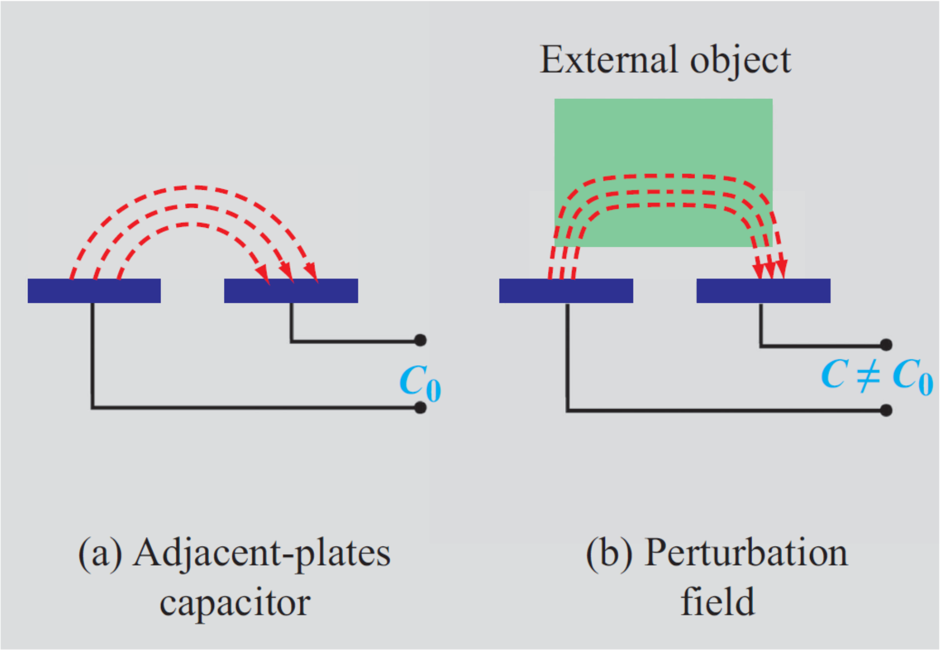


Figure TF9-5: (a) Adjacent-plates capacitor; (b) perturbation field.

Fingerprint Imager

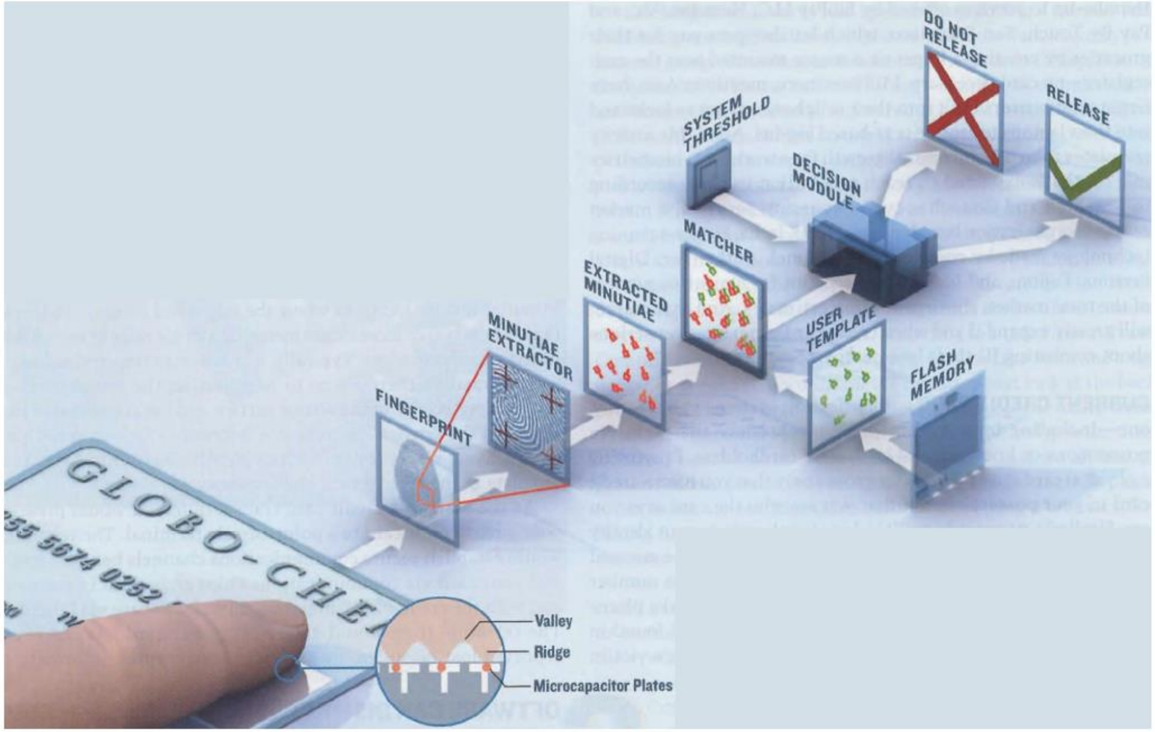


Figure TF9-6: Elements of a fingerprint matching system. (Courtesy of IEEE Spectrum.)

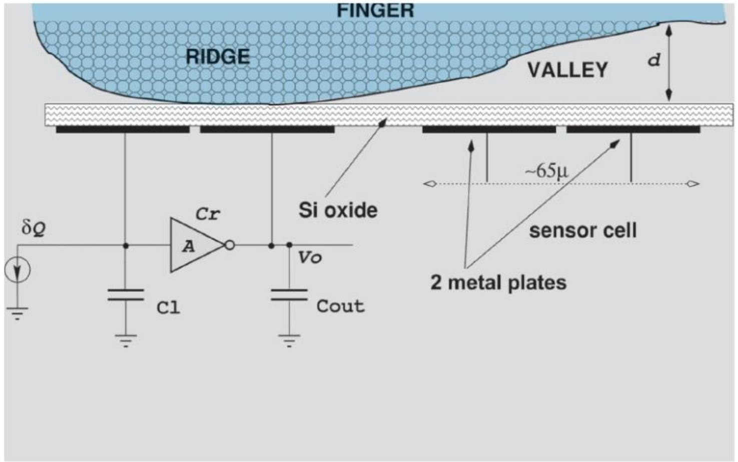


Figure TF9-7: Fingerprint representation. (Courtesy of Dr. M. Tartagni, University of Bologna, Italy.)

Chapter 4 Relationships

Maxwell's Equations for Electrostatics

Name	Differential Form	Integral Form
Gauss's law	$\nabla \cdot \mathbf{D} = \rho_v$	$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q$
Kirchhoff's law	$\nabla \times \mathbf{E} = 0$	$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$

Current density	$\mathbf{J} = \rho_v \mathbf{u}$
Poisson's equation	$\nabla^2 V = -\frac{\rho_v}{\varepsilon}$
Laplace's equation	$\nabla^2 V = 0$
Resistance	$R = \frac{-\int_l \mathbf{E} \cdot d\mathbf{l}}{\int_S \sigma \mathbf{E} \cdot d\mathbf{s}}$
Boundary conditions	Table 4-3
Capacitance	$C = \frac{\int_S \varepsilon \mathbf{E} \cdot d\mathbf{s}}{-\int_l \mathbf{E} \cdot d\mathbf{l}}$
RC relation	$RC = \frac{\varepsilon}{\sigma}$
Energy density	$w_e = \frac{1}{2} \varepsilon E^2$

Electric Field

Point charge	$\mathbf{E} = \hat{\mathbf{R}} \frac{q}{4\pi\varepsilon R^2}$
Many point charges	$\mathbf{E} = \frac{1}{4\pi\varepsilon} \sum_{i=1}^{N} \frac{q_i (\mathbf{R} - \mathbf{R}_i)}{\lvert\mathbf{R} - \mathbf{R}_i\rvert^3}$
Volume distribution	$\mathbf{E} = \frac{1}{4\pi\varepsilon} \int_{V'} \hat{\mathbf{R}}' \frac{\rho_v \, d\mathcal{V}'}{R'^2}$
Surface distribution	$\mathbf{E} = \frac{1}{4\pi\varepsilon} \int_{S'} \hat{\mathbf{R}}' \frac{\rho_s \, ds'}{R'^2}$
Line distribution	$\mathbf{E} = \frac{1}{4\pi\varepsilon} \int_{l'} \hat{\mathbf{R}}' \frac{\rho_\ell \, dl'}{R'^2}$
Infinite sheet of charge	$\mathbf{E} = \hat{\mathbf{z}} \frac{\rho_s}{2\varepsilon_0}$
Infinite line of charge	$\mathbf{E} = \frac{\mathbf{D}}{\varepsilon_0} = \hat{\mathbf{r}} \frac{D_r}{\varepsilon_0} = \hat{\mathbf{r}} \frac{\rho_\ell}{2\pi\varepsilon_0 r}$
Dipole	$\mathbf{E} = \frac{qd}{4\pi\varepsilon_0 R^3} (\hat{\mathbf{R}}\, 2\cos\theta + \hat{\boldsymbol{\theta}} \sin\theta)$
Relation to V	$\mathbf{E} = -\nabla V$

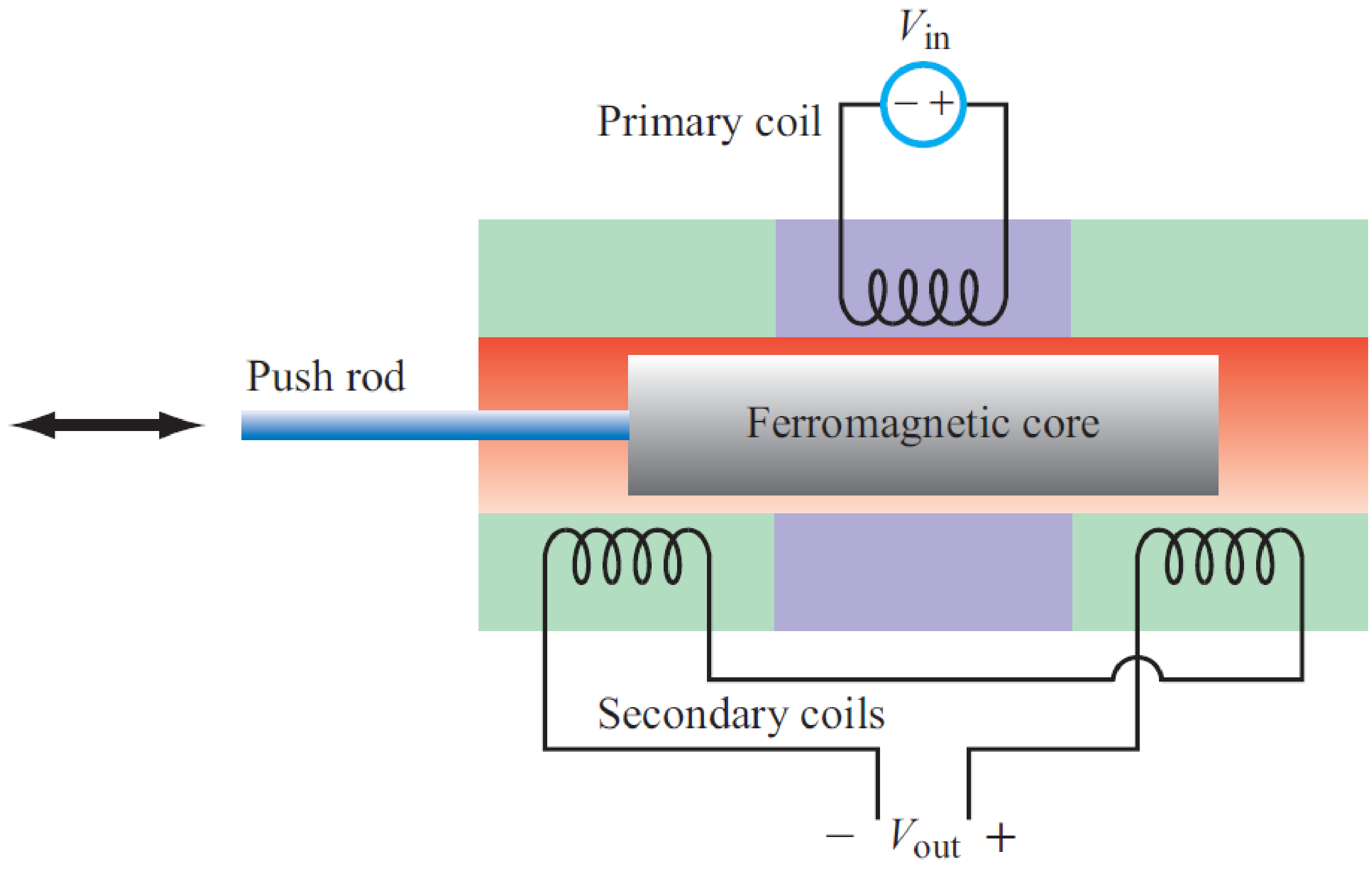


Figure TF11-1: Linear variable differential transformer (LVDT) circuit.

5. MAGNETOSTATICS

7e Applied EM by Ulaby and Ravaioli

Chapter 5 Overview

Chapter Contents

Objectives

Upon learning the material presented in this chapter, you should be able to:

1. Calculate the magnetic force on a current-carrying wire placed in a magnetic field and the torque exerted on a current loop.
2. Apply the Biot–Savart law to calculate the magnetic field due to current distributions.
3. Apply Ampère's law to configurations with appropriate symmetry.
4. Explain magnetic hysteresis in ferromagnetic materials.
5. Calculate the inductance of a solenoid, a coaxial transmission line, or other configurations.
6. Relate the magnetic energy stored in a region to the magnetic field distribution in that region.

Electric vs Magnetic Comparison

Table 5-1: Attributes of electrostatics and magnetostatics.

Attribute	Electrostatics	Magnetostatics
Sources	Stationary charges ρ_v	Steady currents $\mathbf{J}$
Fields and Fluxes	$\mathbf{E}$ and $\mathbf{D}$	$\mathbf{H}$ and $\mathbf{B}$
Constitutive parameter(s)	ε and σ	μ
Governing equations		
• Differential form	$\nabla \cdot \mathbf{D} = \rho_v$ $\nabla \times \mathbf{E} = 0$	$\nabla \cdot \mathbf{B} = 0$ $\nabla \times \mathbf{H} = \mathbf{J}$
• Integral form	$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q$ $\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{s} = 0$ $\oint_C \mathbf{H} \cdot d\mathbf{l} = I$
Potential	Scalar V, with $\mathbf{E} = -\nabla V$	Vector $\mathbf{A}$, with $\mathbf{B} = \nabla \times \mathbf{A}$
Energy density	$w_e = \frac{1}{2}\varepsilon E^2$	$w_m = \frac{1}{2}\mu H^2$
Force on charge q	$\mathbf{F}_e = q\mathbf{E}$	$\mathbf{F}_m = q\mathbf{u} \times \mathbf{B}$
Circuit element(s)	C and R	L

Electric & Magnetic Forces

Magnetic force

$$\mathbf{F}_\mathrm{m} = q\mathbf{u} \times \mathbf{B} \qquad \text{(N)}$$

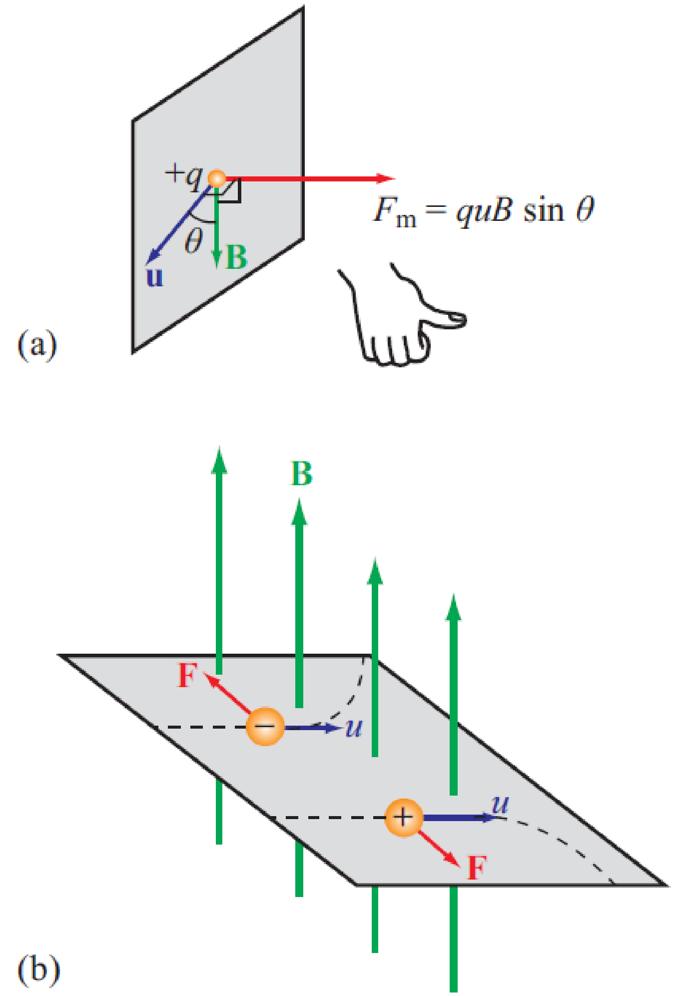


Figure 5-1: The direction of the magnetic force exerted on a charged particle moving in a magnetic field is (a) perpendicular to both **B** and **u** and (b) depends on the charge polarity (positive or negative).

Electromagnetic (Lorentz) force

$$\mathbf{F} = \mathbf{F}_\mathrm{e} + \mathbf{F}_\mathrm{m} = q\mathbf{E} + q\mathbf{u} \times \mathbf{B} = q(\mathbf{E} + \mathbf{u} \times \mathbf{B}).$$

Magnetic Force on a Current Element

Differential force dFm on a differential current I dl:

$$d\mathbf{F}_m = I\, d\mathbf{l} \times \mathbf{B} \quad \text{(N)}. \tag{5.9}$$

For a closed circuit of contour C carrying a current I, the total magnetic force is

$$\mathbf{F}_m = I \oint_C d\mathbf{l} \times \mathbf{B} \quad \text{(N)}. \tag{5.10}$$

If the closed wire shown in Fig. 5-3(a) resides in a uniform external magnetic field **B**, then **B** can be taken outside the integral in Eq. (5.10), in which case

$$\mathbf{F}_m = I \left(\oint_C d\mathbf{l} \right) \times \mathbf{B} = 0. \tag{5.11}$$

*This result, which is a consequence of the fact that the vector sum of the infinitesimal vectors d***l** *over a closed path equals zero, states that the total magnetic force on any closed current loop in a uniform magnetic field is zero.*

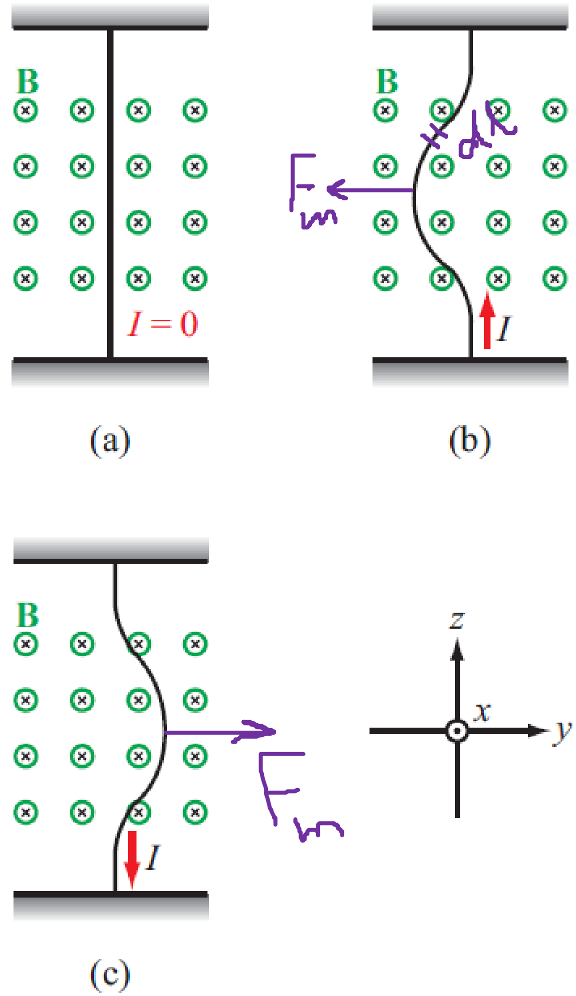


Figure 5-2: When a slightly flexible vertical wire is placed in a magnetic field directed into the page (as denoted by the crosses), it is (a) not deflected when the current through it is zero, (b) deflected to the left when I is upward, and (c) deflected to the right when I is downward.

Torque

$$\mathbf{T} = \mathbf{d} \times \mathbf{F} \qquad (\mathrm{N{\cdot}m})$$

d = moment arm
F = force
T = torque

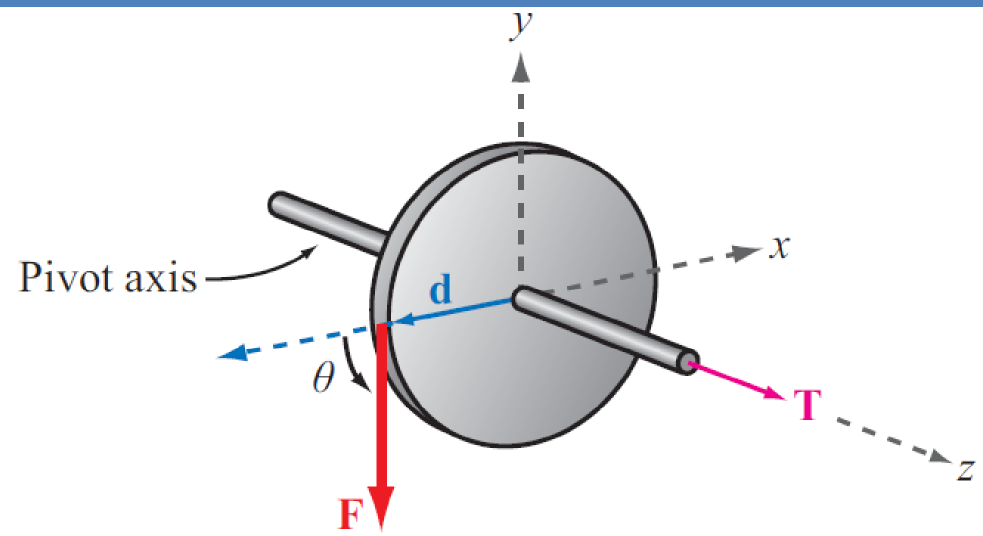


Figure 5-5: The force **F** acting on a circular disk that can pivot along the *z*-axis generates a torque $\mathbf{T} = \mathbf{d} \times \mathbf{F}$ that causes the disk to rotate.

These directions are governed by the following ***right-hand rule:*** *when the thumb of the right hand points along the direction of the torque, the four fingers indicate the direction that the torque tries to rotate the body.*

Magnetic Torque on Current Loop

$$\mathbf{F}_1 = I(-\hat{\mathbf{y}}b) \times (\hat{\mathbf{x}}B_0) = \hat{\mathbf{z}}IbB_0,$$

$$\mathbf{F}_3 = I(\hat{\mathbf{y}}b) \times (\hat{\mathbf{x}}B_0) = -\hat{\mathbf{z}}IbB_0.$$

No forces on arms 2 and 4 (because I and B are parallel, or anti-parallel)

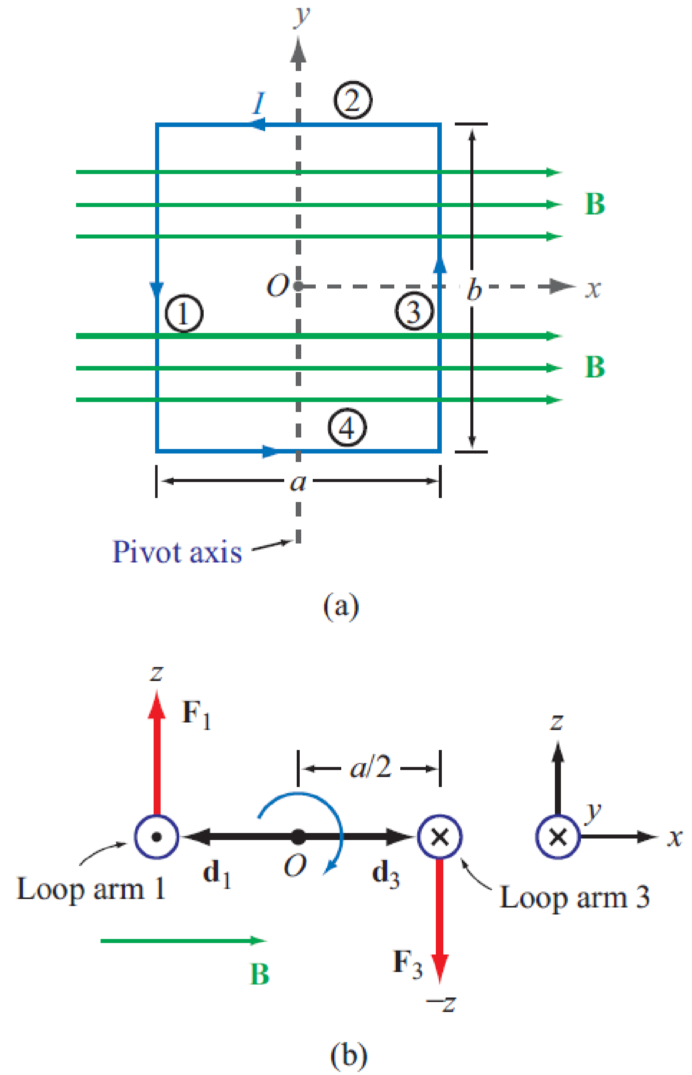


Figure 5-6: Rectangular loop pivoted along the y-axis: (a) front view and (b) bottom view. The combination of forces $\mathbf{F}_1$ and $\mathbf{F}_3$ on the loop generates a torque that tends to rotate the loop in a clockwise direction as shown in (b).

Magnetic torque:

$$\mathbf{T} = \mathbf{d}_1 \times \mathbf{F}_1 + \mathbf{d}_3 \times \mathbf{F}_3$$

$$= \left(-\hat{\mathbf{x}}\,\frac{a}{2}\right) \times (\hat{\mathbf{z}}IbB_0) + \left(\hat{\mathbf{x}}\,\frac{a}{2}\right) \times (-\hat{\mathbf{z}}IbB_0)$$

$$= \hat{\mathbf{y}}IabB_0 = \hat{\mathbf{y}}IAB_0,$$

Area of Loop

Inclined Loop

For a loop with *N* turns and whose surface normal is at angle theta relative to *B* direction:

$$T = NIAB_0 \sin\theta. \qquad (5.18)$$

The quantity NIA is called the ***magnetic moment*** m of the loop. Now, consider the vector

$$\mathbf{m} = \hat{\mathbf{n}}\, NIA = \hat{\mathbf{n}}\, m \qquad (\text{A}\cdot\text{m}^2), \qquad (5.19)$$

where $\hat{\mathbf{n}}$ is the surface normal of the loop and governed by the following ***right-hand rule: when the four fingers of the right hand advance in the direction of the current*** I, ***the direction of the thumb specifies the direction of*** $\hat{\mathbf{n}}$. In terms of $\mathbf{m}$, the torque vector $\mathbf{T}$ can be written as

$$\mathbf{T} = \mathbf{m} \times \mathbf{B} \qquad (\text{N}\cdot\text{m}). \qquad (5.20)$$

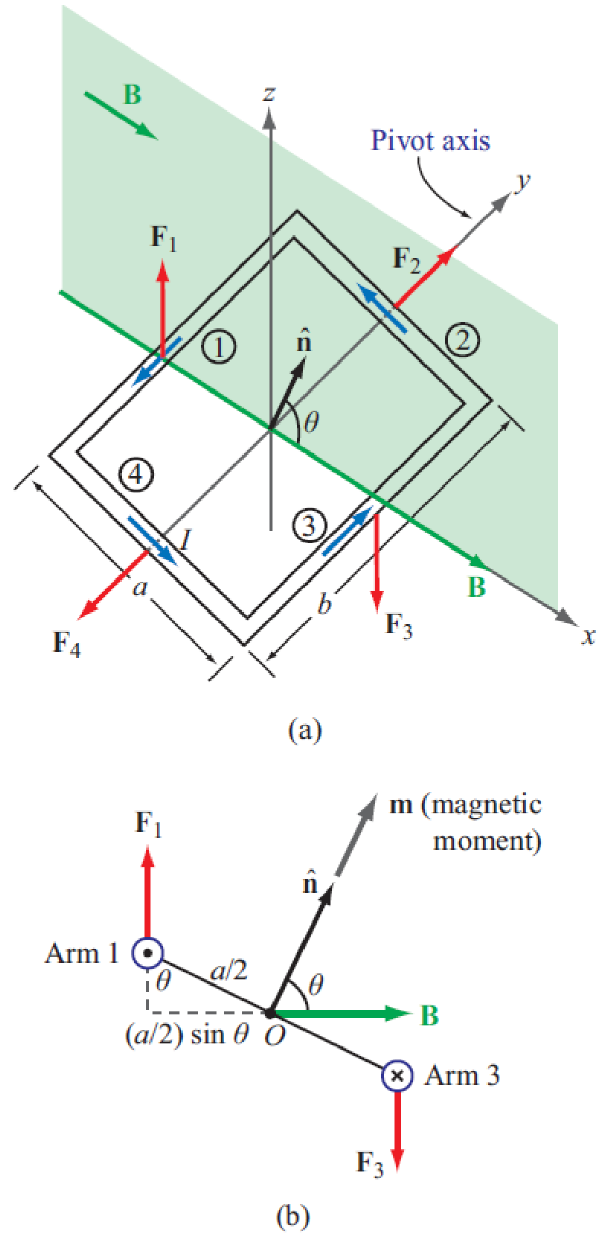


Figure 5-7: Rectangular loop in a uniform magnetic field with flux density **B** whose direction is perpendicular to the rotation axis of the loop, but makes an angle θ with the loop's surface normal $\hat{\mathbf{n}}$.

Biot-Savart Law

Magnetic field induced by a differential current:

$$d\mathbf{H} = \frac{I}{4\pi} \frac{d\mathbf{l} \times \hat{\mathbf{R}}}{R^2} \qquad \text{(A/m)}$$

For the entire length:

$$\mathbf{H} = \frac{I}{4\pi} \int_l \frac{d\mathbf{l} \times \hat{\mathbf{R}}}{R^2} \qquad \text{(A/m)}, \qquad (5.22)$$

where l is the line path along which I exists.

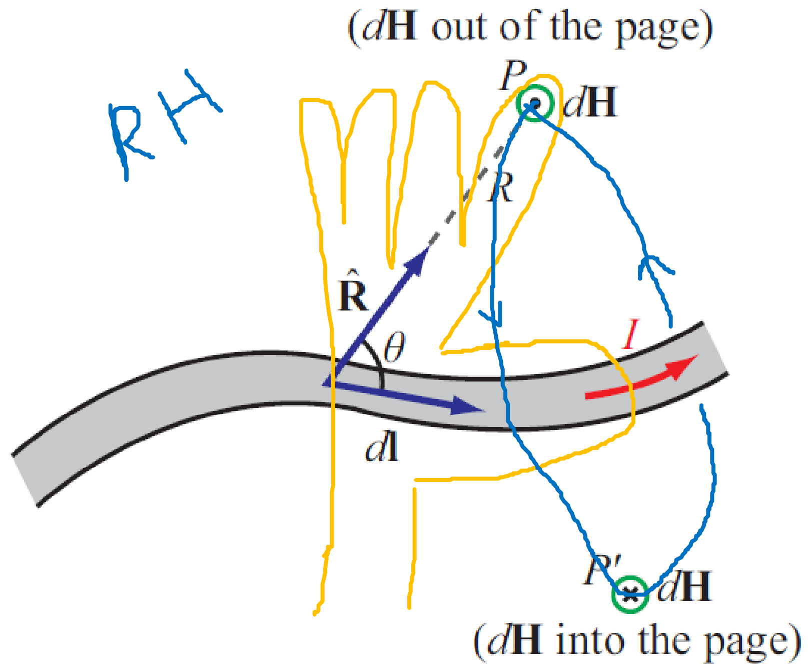
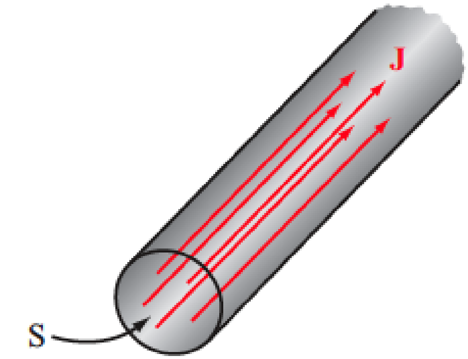


Figure 5-8: Magnetic field $d\mathbf{H}$ generated by a current element $I\ d\mathbf{l}$. The direction of the field induced at point P is opposite to that induced at point P'.

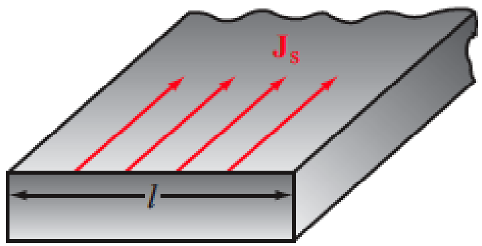
Magnetic Field due to Current Densities

$$\mathbf{H} = \frac{1}{4\pi} \int_S \frac{\mathbf{J}_s \times \hat{\mathbf{R}}}{R^2} \, ds \quad \text{(surface current)},$$

$$\mathbf{H} = \frac{1}{4\pi} \int_{\mathcal{V}} \frac{\mathbf{J} \times \hat{\mathbf{R}}}{R^2} \, d\mathcal{V} \quad \text{(volume current)}.$$



(a) Volume current density **J** in A/m²



(b) Surface current density $\mathbf{J}_s$ in A/m

Figure 5-9: (a) The total current crossing the cross section S of the cylinder is $I = \int_S \mathbf{J} \cdot d\mathbf{s}$. (b) The total current flowing across the surface of the conductor is $I = \int_l J_s \, dl$.

Example 5-2: Magnetic Field of Linear Conductor

Solution: From Fig. 5-10, the differential length vector $d\mathbf{l} = \hat{\mathbf{z}}\, dz$. Hence, $d\mathbf{l} \times \hat{\mathbf{R}} = dz\, (\hat{\mathbf{z}} \times \hat{\mathbf{R}}) = \hat{\boldsymbol{\phi}} \sin\theta\, dz$, where $\hat{\boldsymbol{\phi}}$ is the azimuth direction and θ is the angle between $d\mathbf{l}$ and $\hat{\mathbf{R}}$. Application of Eq. (5.22) gives

$$\mathbf{H} = \frac{I}{4\pi} \int_{z=-l/2}^{z=l/2} \frac{d\mathbf{l} \times \hat{\mathbf{R}}}{R^2} = \hat{\boldsymbol{\phi}} \frac{I}{4\pi} \int_{-l/2}^{l/2} \frac{\sin\theta}{R^2}\, dz. \quad (5.25)$$

Both R and θ are dependent on the integration variable z, but the radial distance r is not. For convenience, we will convert the integration variable from z to θ by using the transformations

$$R = r \csc\theta, \quad (5.26a)$$

$$z = -r \cot\theta, \quad (5.26b)$$

$$dz = r \csc^2\theta\, d\theta. \quad (5.26c)$$

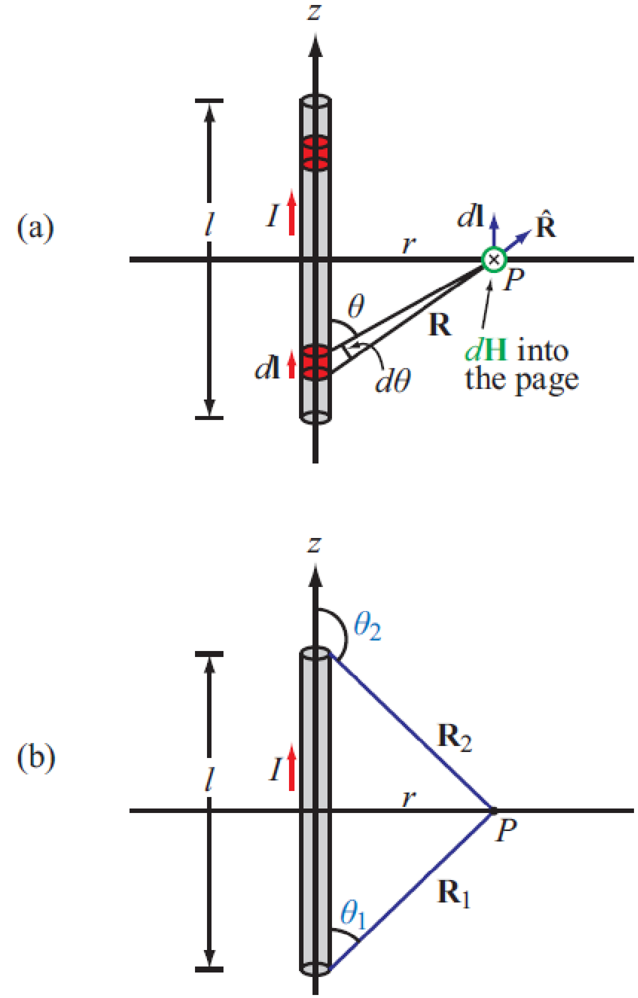


Figure 5-10: Linear conductor of length l carrying a current I. (a) The field $d\mathbf{H}$ at point P due to incremental current element $d\mathbf{l}$. (b) Limiting angles θ_1 and θ_2, each measured between vector $I\, d\mathbf{l}$ and the vector connecting the end of the conductor associated with that angle to point P (Example 5-2).

Cont.

Example 5-2: Magnetic Field of Linear Conductor

Upon inserting Eqs. (5.26a) and (5.26c) into Eq. (5.25), we have

$$\mathbf{H} = \hat{\boldsymbol{\phi}} \frac{I}{4\pi} \int_{\theta_1}^{\theta_2} \frac{\sin\theta \; r\csc^2\theta \; d\theta}{r^2\csc^2\theta}$$

$$= \hat{\boldsymbol{\phi}} \frac{I}{4\pi r} \int_{\theta_1}^{\theta_2} \sin\theta \; d\theta$$

$$= \hat{\boldsymbol{\phi}} \frac{I}{4\pi r}(\cos\theta_1 - \cos\theta_2), \qquad (5.27)$$

where θ_1 and θ_2 are the limiting angles at $z = -l/2$ and $z = l/2$, respectively. From the right triangle in Fig. 5-10(b), it follows that

$$\cos\theta_1 = \frac{l/2}{\sqrt{r^2 + (l/2)^2}}, \qquad (5.28a)$$

$$\cos\theta_2 = -\cos\theta_1 = \frac{-l/2}{\sqrt{r^2 + (l/2)^2}}. \qquad (5.28b)$$

Hence,

$$\mathbf{B} = \mu_0 \mathbf{H} = \hat{\boldsymbol{\phi}} \frac{\mu_0 I l}{2\pi r\sqrt{4r^2 + l^2}} \quad \text{(T)}. \qquad (5.29)$$

For an infinitely long wire with $l \gg r$, Eq. (5.29) reduces to

$$\mathbf{B} = \hat{\boldsymbol{\phi}} \frac{\mu_0 I}{2\pi r} \quad \text{(infinitely long wire)}. \qquad (5.30)$$

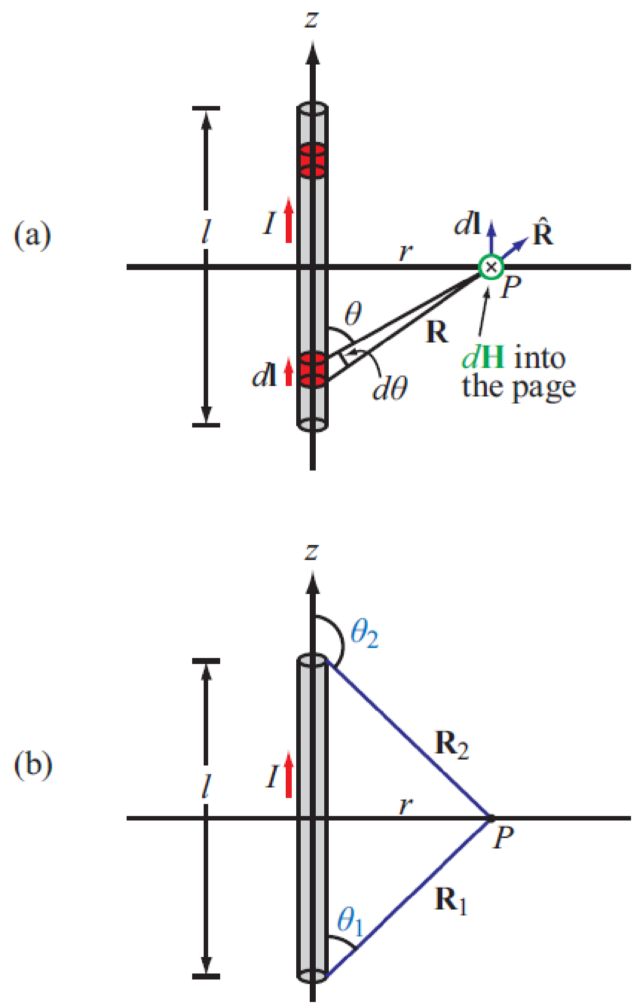
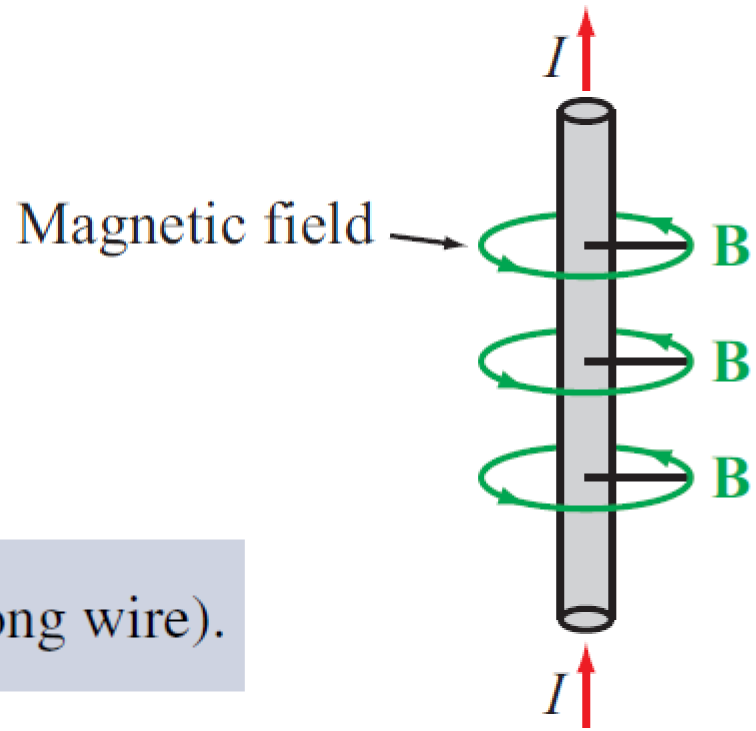
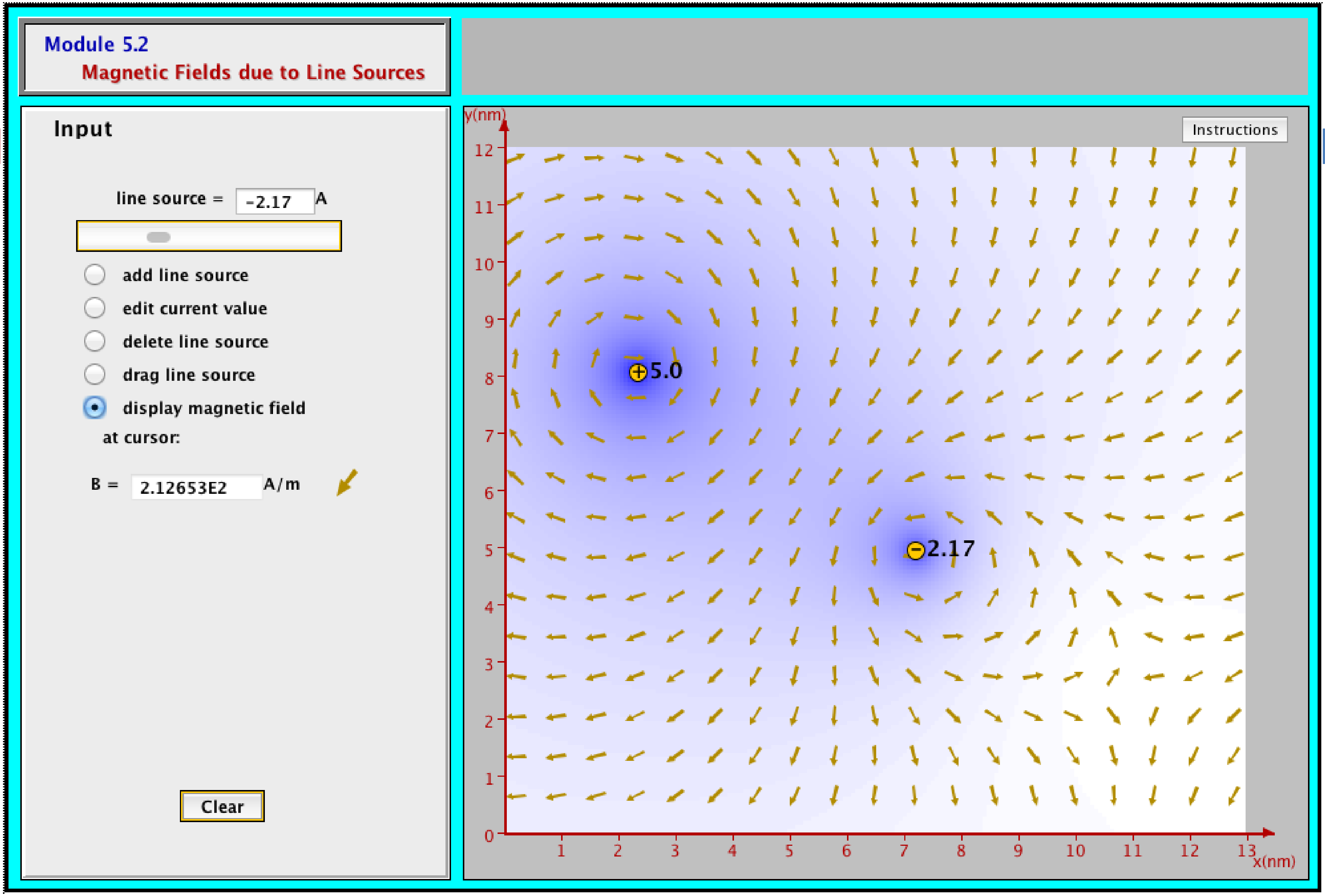


Figure 5-10: Linear conductor of length l carrying a current I. (a) The field $d\mathbf{H}$ at point P due to incremental current element $d\mathbf{l}$. (b) Limiting angles θ_1 and θ_2, each measured between vector $I\,d\mathbf{l}$ and the vector connecting the end of the conductor associated with that angle to point P (Example 5-2).

Magnetic Field of Long Conductor



$$\mathbf{B} = \hat{\boldsymbol{\phi}} \frac{\mu_0 I}{2\pi r} \qquad \text{(infinitely long wire).}$$



Module 5.2
Magnetic Fields due to Line Sources
Input
line source = -2.17 A
add line source
edit current value
delete line source
drag line source
display magnetic field
at cursor:
B = 2.12653E2 A/m
Clear
Instructions
y(nm)
x(nm)
5.0
2.17

Example 5-3: Magnetic Field of a Loop

Magnitude of field due to dl is

$$dH = \frac{I}{4\pi R^2} |d\mathbf{l} \times \hat{\mathbf{R}}| = \frac{I\, dl}{4\pi (a^2 + z^2)}$$

d**H** is in the r–z plane , and therefore it has components dHr and dHz

z-components of the magnetic fields due to d**l** and dl' add because they are in the same direction, but their r-components cancel

Hence for element dl:

$$d\mathbf{H} = \hat{\mathbf{z}}\, dH_z = \hat{\mathbf{z}}\, dH \cos\theta = \hat{\mathbf{z}}\, \frac{I \cos\theta}{4\pi(a^2 + z^2)}\, dl$$

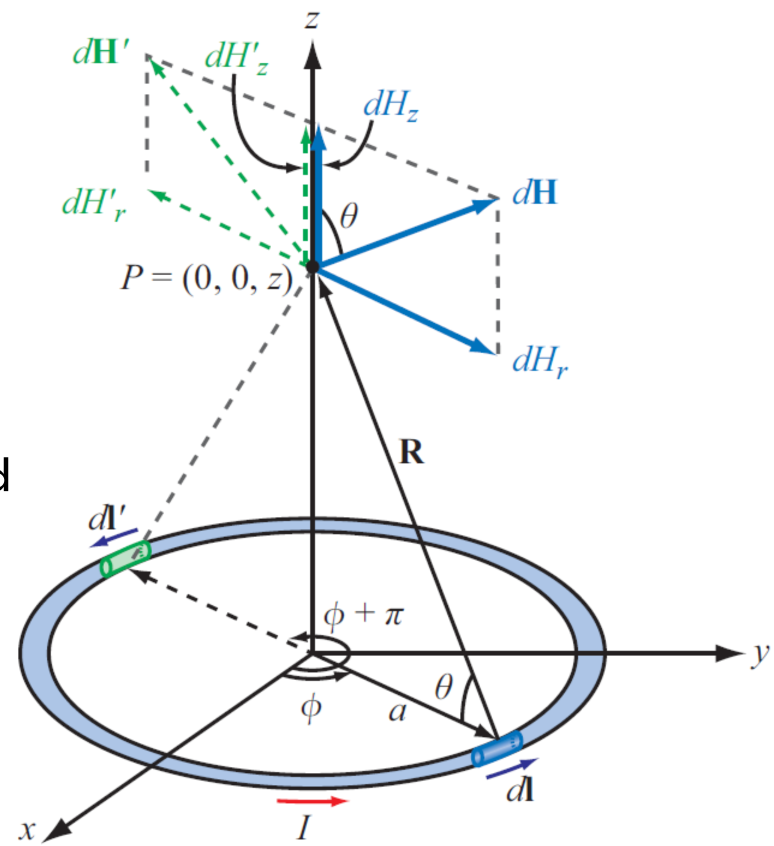


Figure 5-12: Circular loop carrying a current I (Example 5-3).

Cont.

Example 5-3: Magnetic Field of a Loop (cont.)

For the entire loop:

$$\mathbf{H} = \hat{\mathbf{z}} \frac{I \cos\theta}{4\pi(a^2 + z^2)} \oint dl = \hat{\mathbf{z}} \frac{I \cos\theta}{4\pi(a^2 + z^2)} (2\pi a). \quad (5.33)$$

Upon using the relation $\cos\theta = a/(a^2 + z^2)^{1/2}$, we obtain

$$\mathbf{H} = \hat{\mathbf{z}} \frac{I a^2}{2(a^2 + z^2)^{3/2}} \quad \text{(A/m)}. \quad (5.34)$$

At the center of the loop ($z = 0$), Eq. (5.34) reduces to

$$\mathbf{H} = \hat{\mathbf{z}} \frac{I}{2a} \quad (\text{at } z = 0), \quad (5.35)$$

and at points very far away from the loop such that $z^2 \gg a^2$, Eq. (5.34) simplifies to

$$\mathbf{H} = \hat{\mathbf{z}} \frac{I a^2}{2|z|^3} \quad (\text{at } |z| \gg a). \quad (5.36)$$

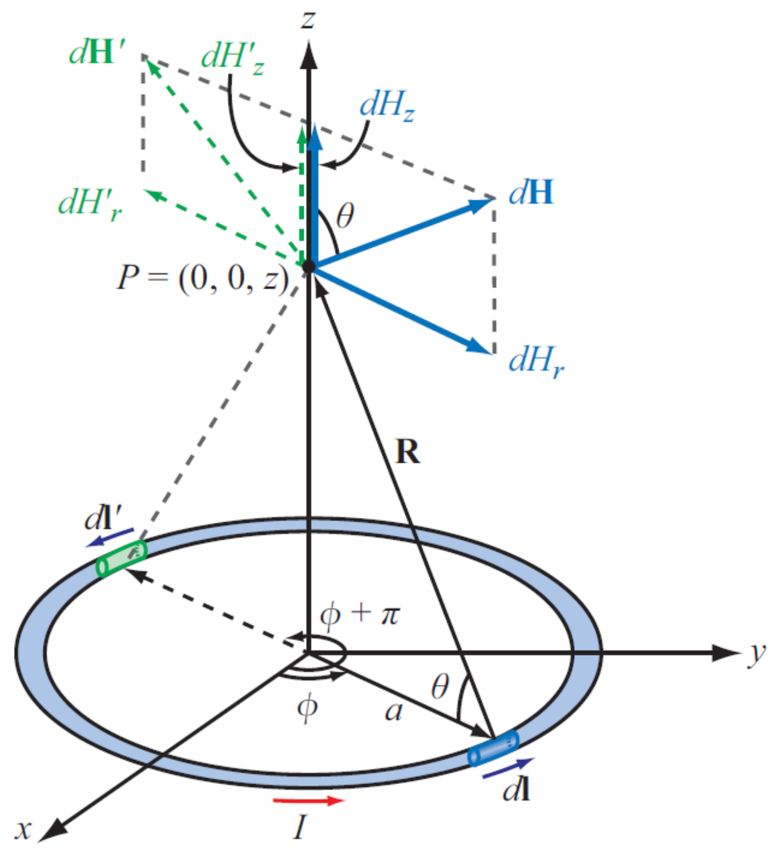
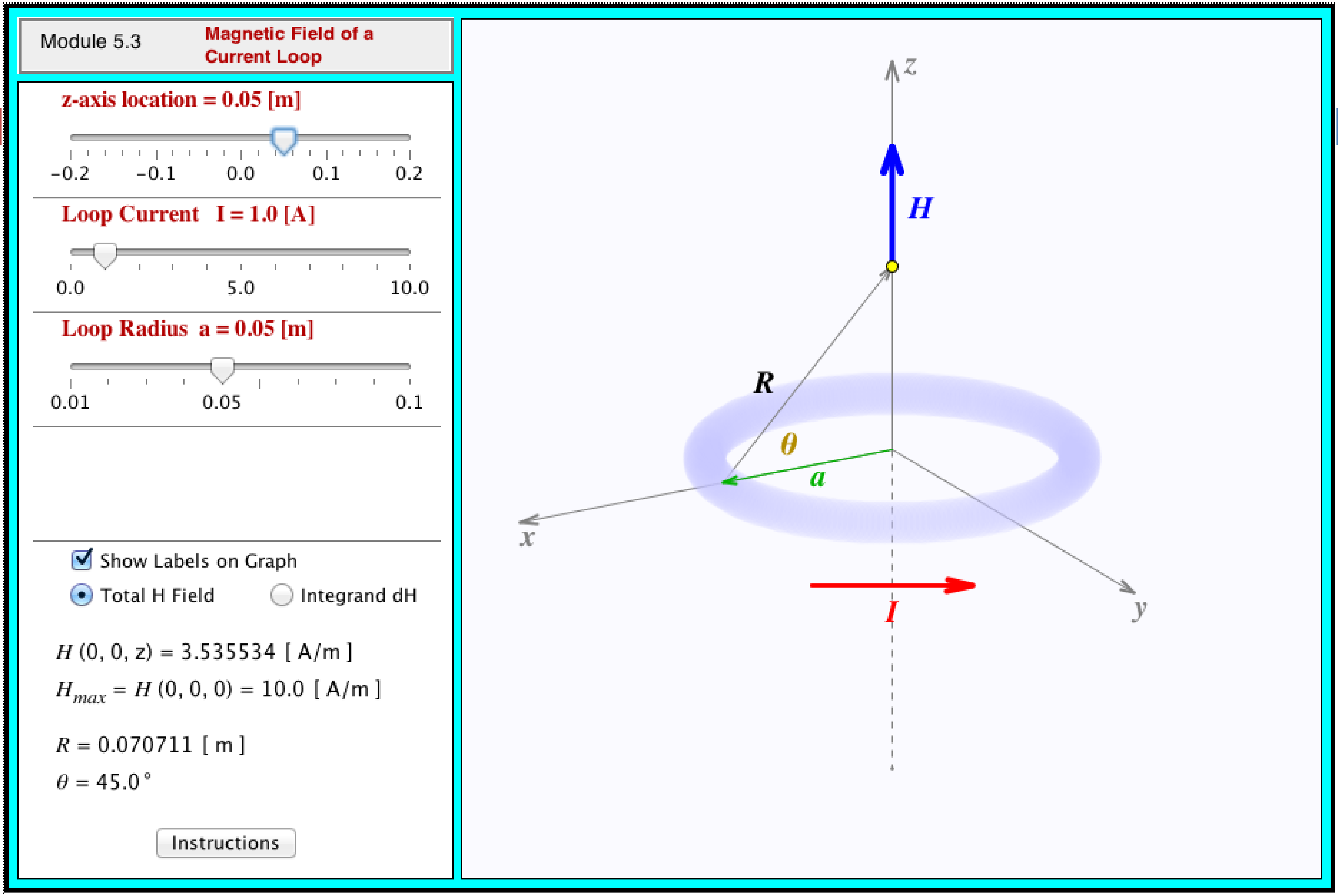


Figure 5-12: Circular loop carrying a current I (Example 5-3).



Module 5.3
Magnetic Field of a Current Loop
z-axis location = 0.05 [m]
-0.2 -0.1 0.0 0.1 0.2
Loop Current I = 1.0 [A]
0.0 5.0 10.0
Loop Radius a = 0.05 [m]
0.01 0.05 0.1
Show Labels on Graph
Total H Field
Integrand dH
H (0, 0, z) = 3.535534 [A/m]
H max = H (0, 0, 0) = 10.0 [A/m]
R = 0.070711 [m]
θ = 45.0 °
Instructions
z
H
R
θ
a
x
y
I

Magnetic Dipole

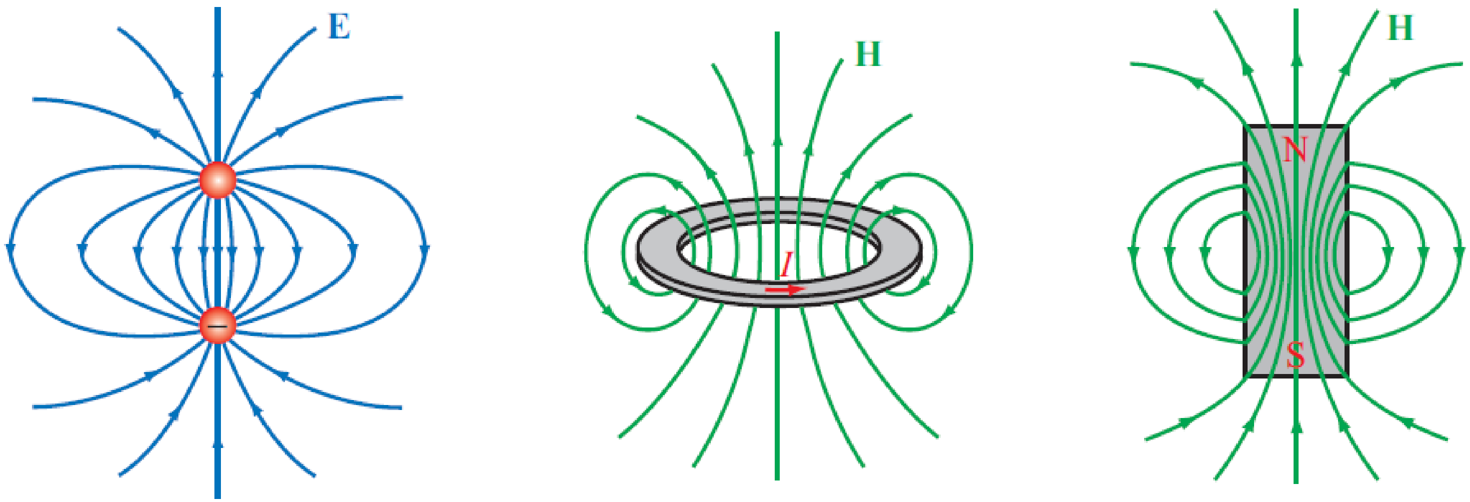


Figure 5-13: Patterns of (a) the electric field of an electric dipole, (b) the magnetic field of a magnetic dipole, and (c) the magnetic field of a bar magnet. Far away from the sources, the field patterns are similar in all three cases.

Because a circular loop exhibits a magnetic field pattern similar to the electric field of an electric dipole, it is called a *magnetic dipole*

Forces on Parallel Conductors

$$\mathbf{B}_1 = -\hat{\mathbf{x}} \frac{\mu_0 I_1}{2\pi d} . \qquad (5.39)$$

The force $\mathbf{F}_2$ exerted on a length l of wire I_2 due to its presence in field $\mathbf{B}_1$ may be obtained by applying Eq. (5.12):

$$\mathbf{F}_2 = I_2 l \hat{\mathbf{z}} \times \mathbf{B}_1 = I_2 l \hat{\mathbf{z}} \times (-\hat{\mathbf{x}}) \frac{\mu_0 I_1}{2\pi d}$$

$$= -\hat{\mathbf{y}} \frac{\mu_0 I_1 I_2 l}{2\pi d} , \qquad (5.40)$$

and the corresponding force per unit length is

$$\mathbf{F}'_2 = \frac{\mathbf{F}_2}{l} = -\hat{\mathbf{y}} \frac{\mu_0 I_1 I_2}{2\pi d} . \qquad (5.41)$$

A similar analysis performed for the force per unit length exerted on the wire carrying I_1 leads to

$$\mathbf{F}'_1 = \hat{\mathbf{y}} \frac{\mu_0 I_1 I_2}{2\pi d} . \qquad (5.42)$$

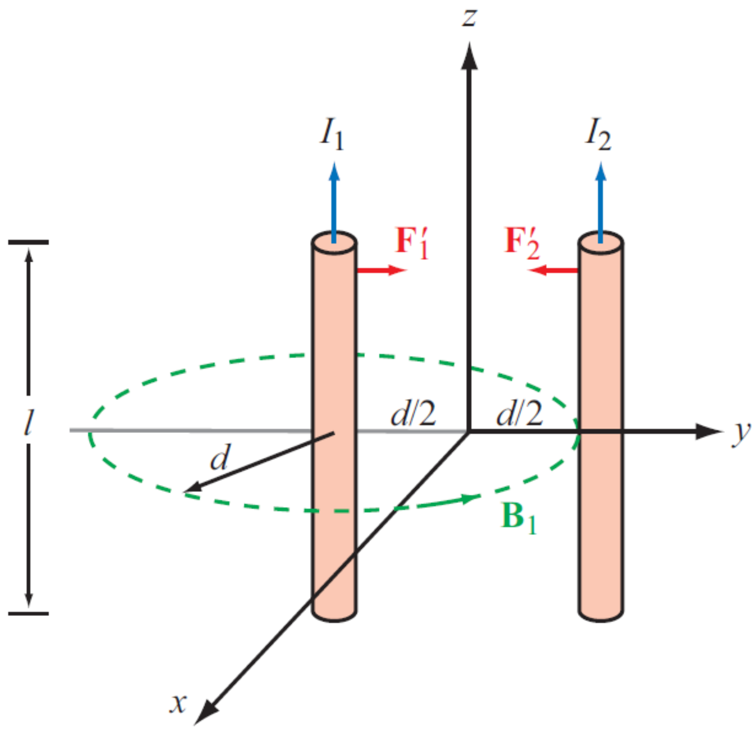
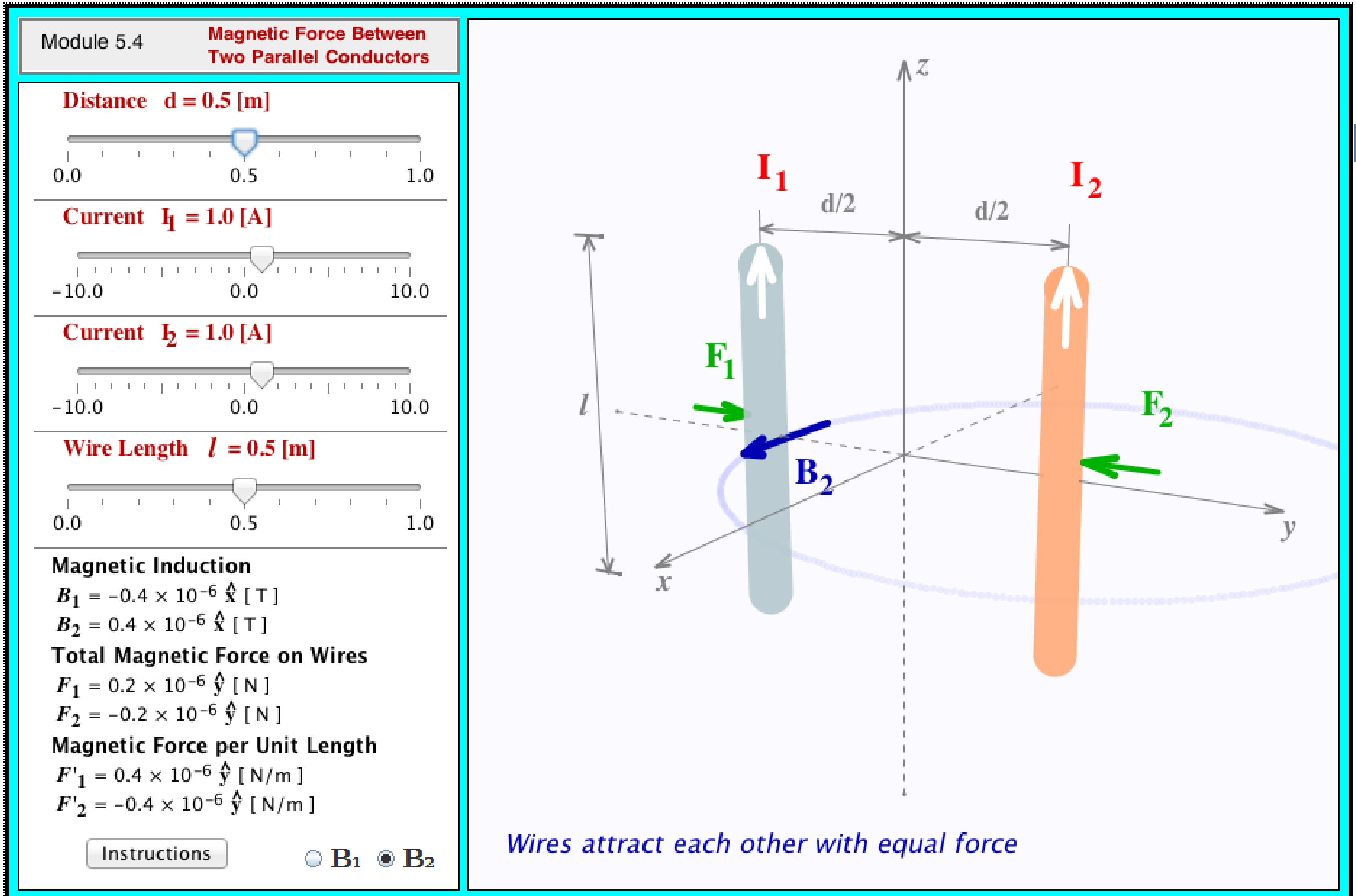


Figure 5-14: Magnetic forces on parallel current-carrying conductors.

Parallel wires attract if their currents are in the same direction, and repel if currents are in opposite directions



Module 5.4
Magnetic Force Between Two Parallel Conductors
Distance d = 0.5 [m]
0.0 0.5 1.0
Current I1 = 1.0 [A]
-10.0 0.0 10.0
Current I2 = 1.0 [A]
-10.0 0.0 10.0
Wire Length l = 0.5 [m]
0.0 0.5 1.0
Magnetic Induction
B1 = -0.4 × 10^-6 x̂ [T]
B2 = 0.4 × 10^-6 x̂ [T]
Total Magnetic Force on Wires
F1 = 0.2 × 10^-6 ŷ [N]
F2 = -0.2 × 10^-6 ŷ [N]
Magnetic Force per Unit Length
F'1 = 0.4 × 10^-6 ŷ [N/m]
F'2 = -0.4 × 10^-6 ŷ [N/m]
Instructions
B1 B2
z
I1
I2
d/2
d/2
l
F1
F2
B2
x
y
Wires attract each other with equal force

Tech Brief 10: Electromagnets

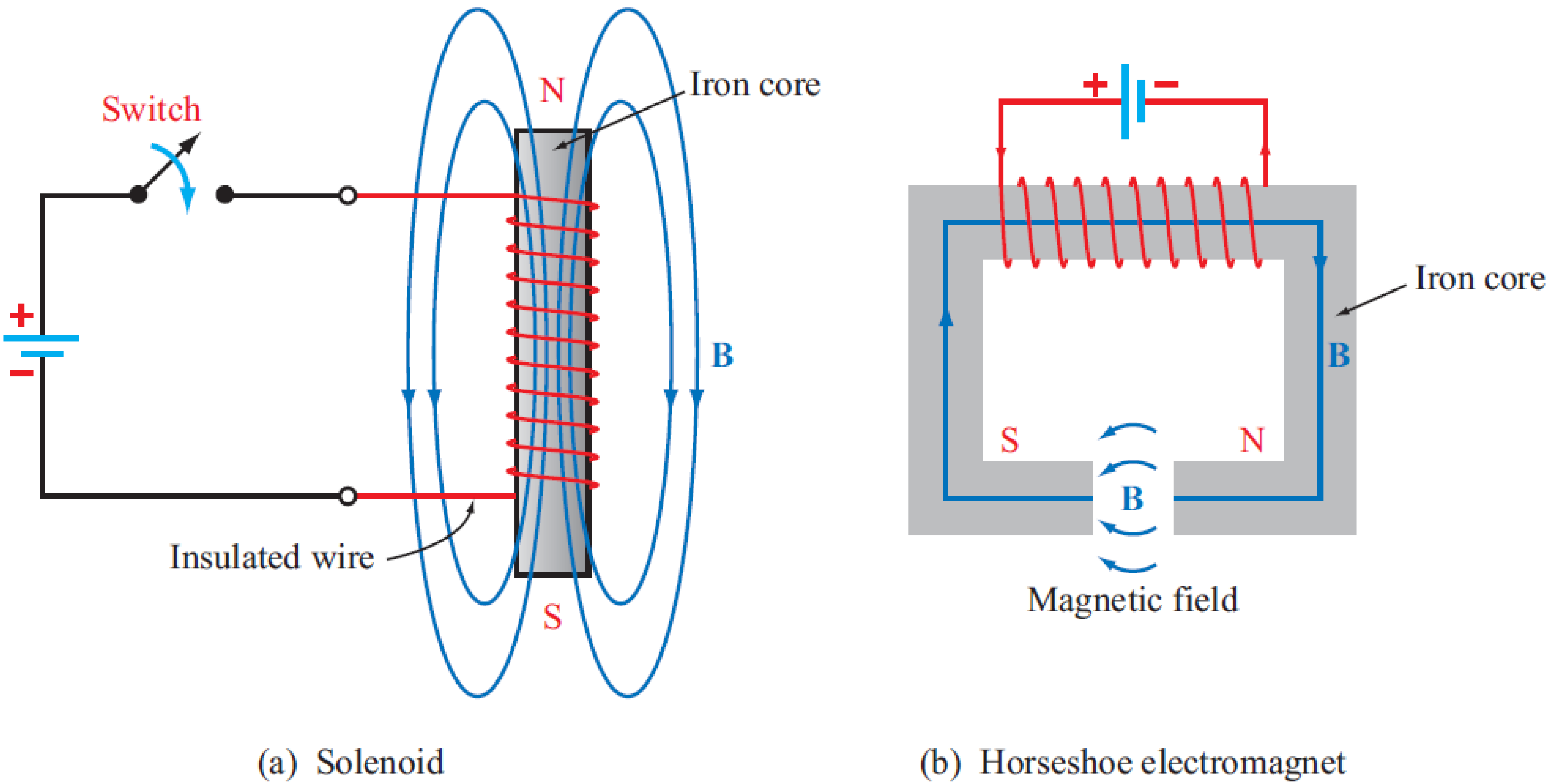
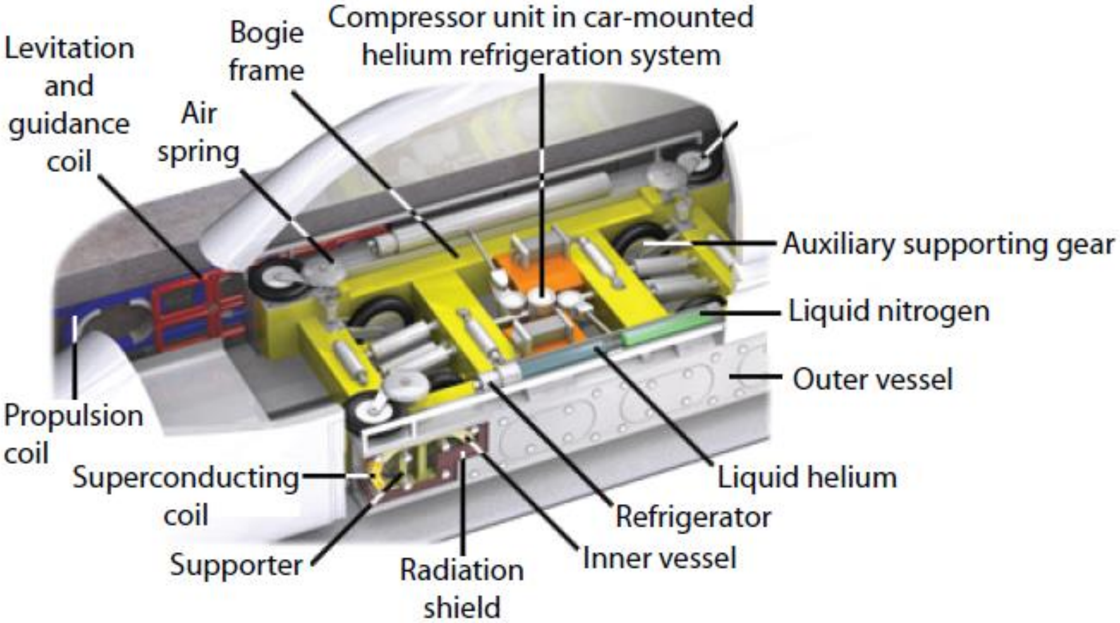


Figure TF10-1: Solenoid and horseshoe magnets.

Magnetic Levitation

(a) Maglev train



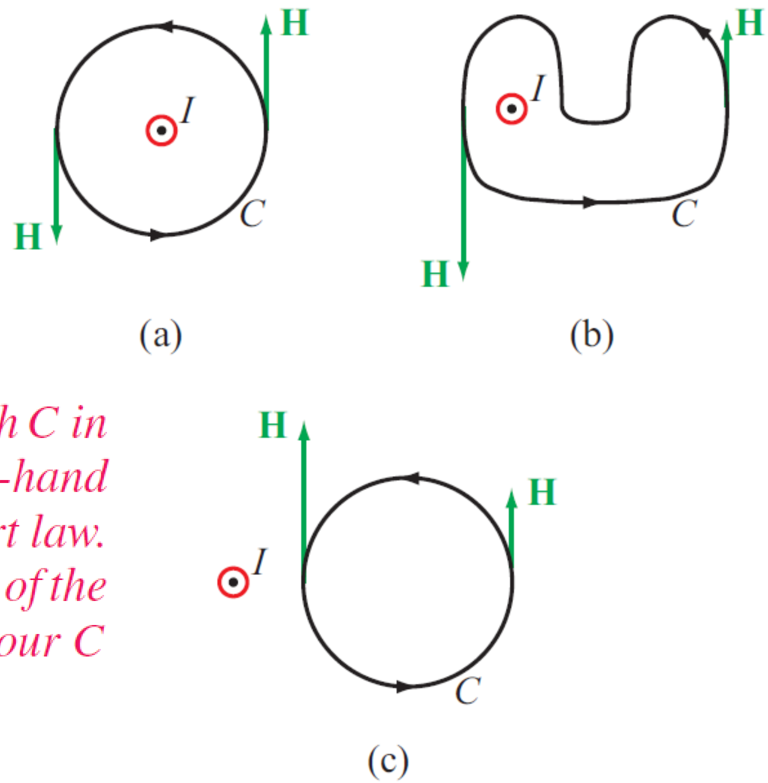
(b) Internal workings of the Maglev train

Figure TF10-5: Magnetic trains. (Courtesy Shanghai.com.)

https://www.youtube.com/watch?v=Wor8C3ZIAu8

Ampère's Law

$$\nabla \times \mathbf{H} = \mathbf{J} \quad \longleftrightarrow \quad \oint_C \mathbf{H} \cdot d\boldsymbol{\ell} = I$$



The sign convention for the direction of the contour path C in Ampère's law is taken so that I and **H** *satisfy the right-hand rule defined earlier in connection with the Biot–Savart law. That is, if the direction of I is aligned with the direction of the thumb of the right hand, then the direction of the contour C should be chosen along that of the other four fingers.*

Figure 5-16: Ampère's law states that the line integral of **H** around a closed contour C is equal to the current traversing the surface bounded by the contour. This is true for contours (a) and (b), but the line integral of **H** is zero for the contour in (c) because the current I (denoted by the symbol ⊙) is not enclosed by the contour C.

Internal Magnetic Field of Long Conductor

For $r < a$

$$\oint \mathbf{H}_1 \cdot d\mathbf{l}_1 = I_1,$$

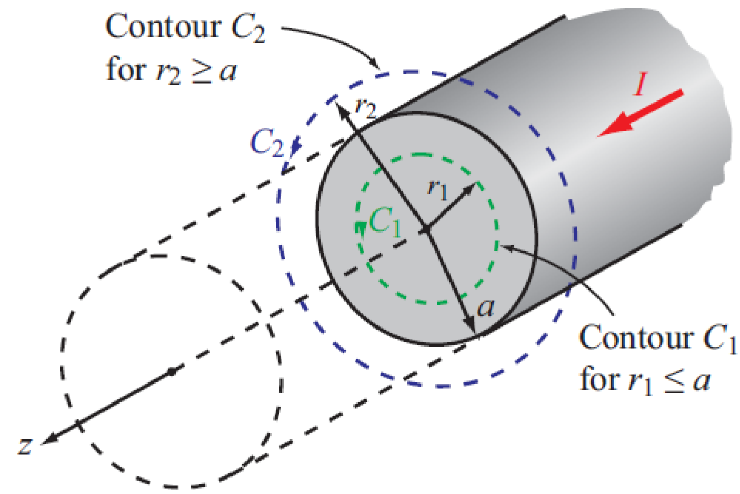
$$\oint_{C_1} \mathbf{H}_1 \cdot d\mathbf{l}_1 = \int_0^{2\pi} H_1(\hat{\boldsymbol{\phi}} \cdot \hat{\boldsymbol{\phi}}) r_1 \, d\phi = 2\pi r_1 H_1.$$

The current I_1 flowing through the area enclosed by C_1 is equa
to the total current I multiplied by the ratio of the area enclose
by C_1 to the total cross-sectional area of the wire:

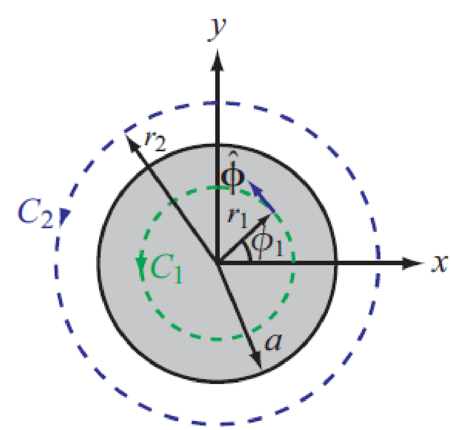
$$I_1 = \left(\frac{\pi r_1^2}{\pi a^2}\right) I = \left(\frac{r_1}{a}\right)^2 I.$$

Equating both sides of Eq. (5.48) and then solving for $\mathbf{H}_1$ yielc

$$\mathbf{H}_1 = \hat{\boldsymbol{\phi}} H_1 = \hat{\boldsymbol{\phi}} \frac{r_1}{2\pi a^2} I \qquad (\text{for } r_1 \leq a). \qquad (5.49a$$



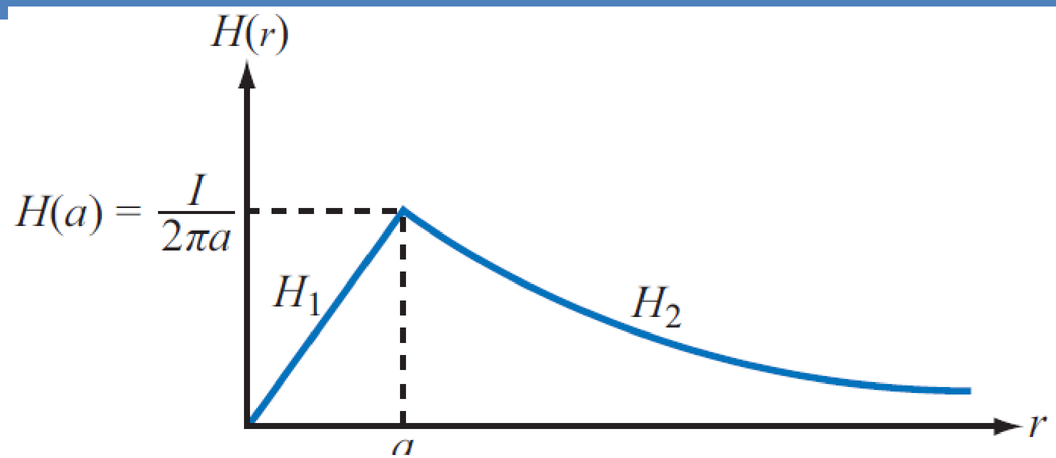
(a) Cylindrical wire



(b) Wire cross section

Cont.

External Magnetic Field of Long Conductor



For $r > a$

(b) For $r = r_2 \geq a$, we choose path C_2, which encloses all the current I. Hence, $\mathbf{H}_2 = \hat{\boldsymbol{\phi}} H_2$, $d\boldsymbol{\ell}_2 = \hat{\boldsymbol{\phi}} r_2 \, d\phi$, and

$$\oint_{C_2} \mathbf{H}_2 \cdot d\mathbf{l}_2 = 2\pi r_2 H_2 = I,$$

which yields

$$\mathbf{H}_2 = \hat{\boldsymbol{\phi}} H_2 = \hat{\boldsymbol{\phi}} \frac{I}{2\pi r_2} \qquad (\text{for } r_2 \geq a). \qquad (5.49\text{b})$$

Magnetic Field of Toroid

Applying Ampere's law over contour C:

$$\oint_C \mathbf{H} \cdot d\boldsymbol{\ell} = I$$

Ampere's law states that the line integral of **H** around a closed contour C *is equal to the current traversing* the surface bounded by the contour.

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_0^{2\pi} (-\hat{\phi} H) \cdot \hat{\phi} r \, d\phi = -2\pi r H = -NI.$$

Hence, $H = NI/(2\pi r)$ and

$$\mathbf{H} = -\hat{\phi} H = -\hat{\phi} \frac{NI}{2\pi r} \qquad (\text{for } a < r < b).$$

The magnetic field outside the toroid is zero. Why?

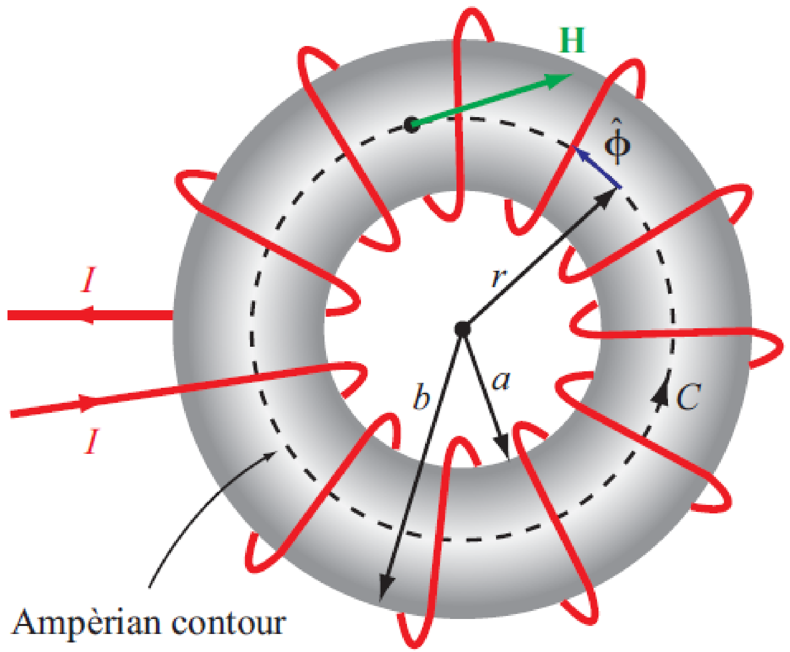


Figure 5-18: Toroidal coil with inner radius a and outer radius b. The wire loops usually are much more closely spaced than shown in the figure (Example 5-5).

Magnetic Vector Potential A

Electrostatics

$$\mathbf{E} = -\nabla V$$

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon}$$

$$V = \frac{1}{4\pi\varepsilon} \int_{v'} \frac{\rho_v}{R'} \, dv'$$

Magnetostatics

$$\mathbf{B} = \nabla \times \mathbf{A} \qquad (\text{Wb/m}^2),$$

$$\nabla^2 \mathbf{A} = -\mu \mathbf{J}.$$

$$\mathbf{A} = \frac{\mu}{4\pi} \int_{v'} \frac{\mathbf{J}}{R'} \, dv' \qquad (\text{Wb/m}).$$

Magnetic Properties of Materials

The magnetic behavior of a material is governed by the interaction of the magnetic dipole moments of its atoms with an external magnetic field. The nature of the behavior depends on the crystalline structure of the material and is used as a basis for classifying materials as ***diamagnetic****,* ***paramagnetic****, or* ***ferromagnetic****.*

$$\mathbf{B} = \mu_0 \mathbf{H} + \mu_0 \mathbf{M} = \mu_0 (\mathbf{H} + \mathbf{M})$$

$$\mathbf{M} = \chi_m \mathbf{H}$$

$$\mathbf{B} = \mu_0 (\mathbf{H} + \chi_m \mathbf{H}) = \mu_0 (1 + \chi_m) \mathbf{H},$$

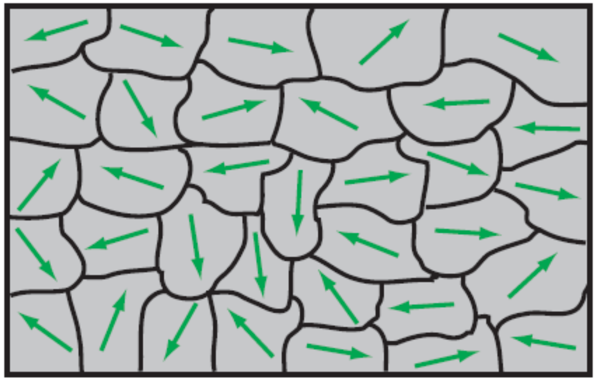
$$\mathbf{B} = \mu \mathbf{H},$$

Table 5-2: Properties of magnetic materials.

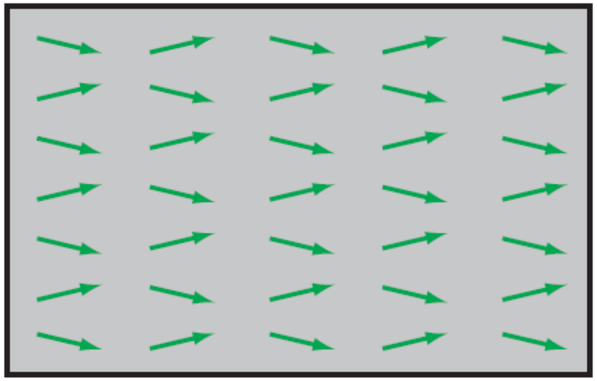
	Diamagnetism	Paramagnetism	Ferromagnetism
Permanent magnetic dipole moment	No	Yes, but weak	Yes, and strong
Primary magnetization mechanism	Electron orbital magnetic moment	Electron spin magnetic moment	Magnetized domains
Direction of induced magnetic field (relative to external field)	Opposite	Same	Hysteresis (see Fig. 5-22)
Common substances	Bismuth, copper, diamond, gold, lead, mercury, silver, silicon	Aluminum, calcium, chromium, magnesium, niobium, platinum, tungsten	Iron, nickel, cobalt
Typical value of χ_m **Typical value of μ_r**	$\approx -10^{-5}$ ≈ 1	$\approx 10^{-5}$ ≈ 1	$\lvert\chi_m\rvert \gg 1$ and hysteretic $\lvert\mu_r\rvert \gg 1$ and hysteretic

Thus, $\mu_r \simeq 1$ or $\mu \simeq \mu_0$ for diamagnetic and paramagnetic substances, which include dielectric materials and most metals. In contrast, $|\mu_r| \gg 1$ for ferromagnetic materials; $|\mu_r|$ of purified iron, for example, is on the order of 2×10^5.

Magnetic Hysteresis



(a) Unmagnetized domains



(b) Magnetized domains

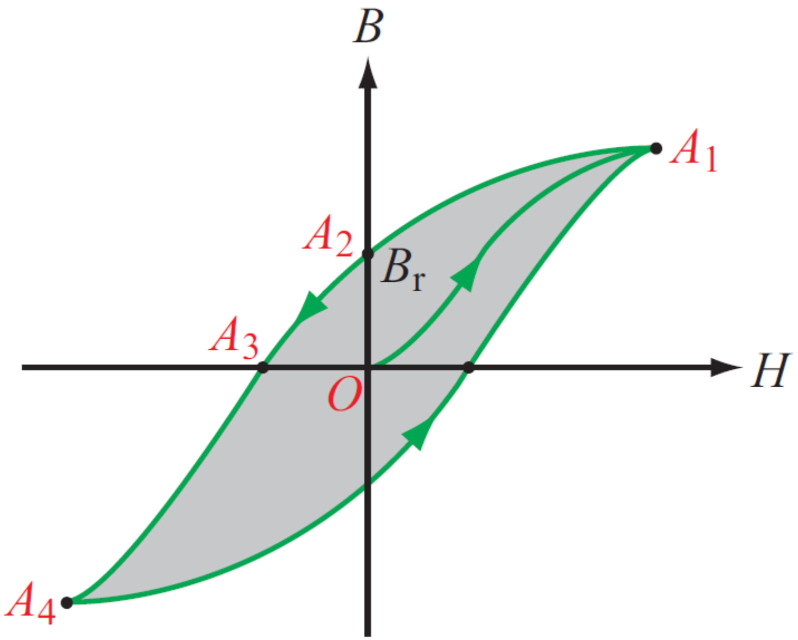
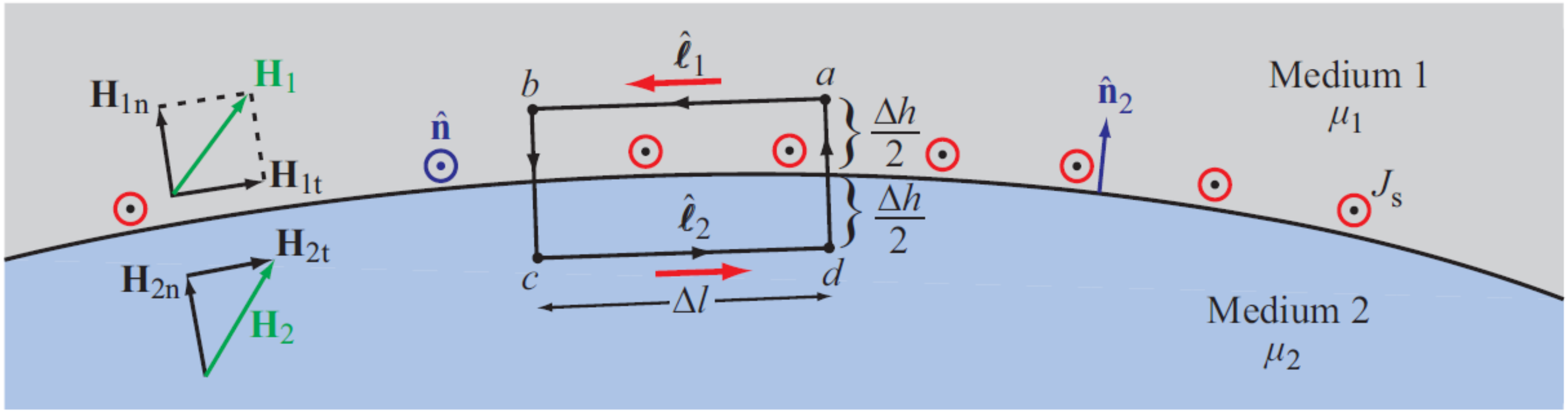


Figure 5-22: Typical hysteresis curve for a ferromagnetic material.

Boundary Conditions



$$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q \rightarrow \boxed{D_{1n} - D_{2n} = \rho_s.} \qquad (5.78)$$

$$\hat{\mathbf{n}}_2 \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s.$$

By analogy, application of Gauss's law for magnetism, as expressed by Eq. (5.44), leads to the conclusion that

$$\oint_S \mathbf{B} \cdot d\mathbf{s} = 0 \rightarrow \boxed{B_{1n} = B_{2n}.} \qquad (5.79)$$

Thus the normal component of **B** *is continuous across the boundary between two adjacent media.*

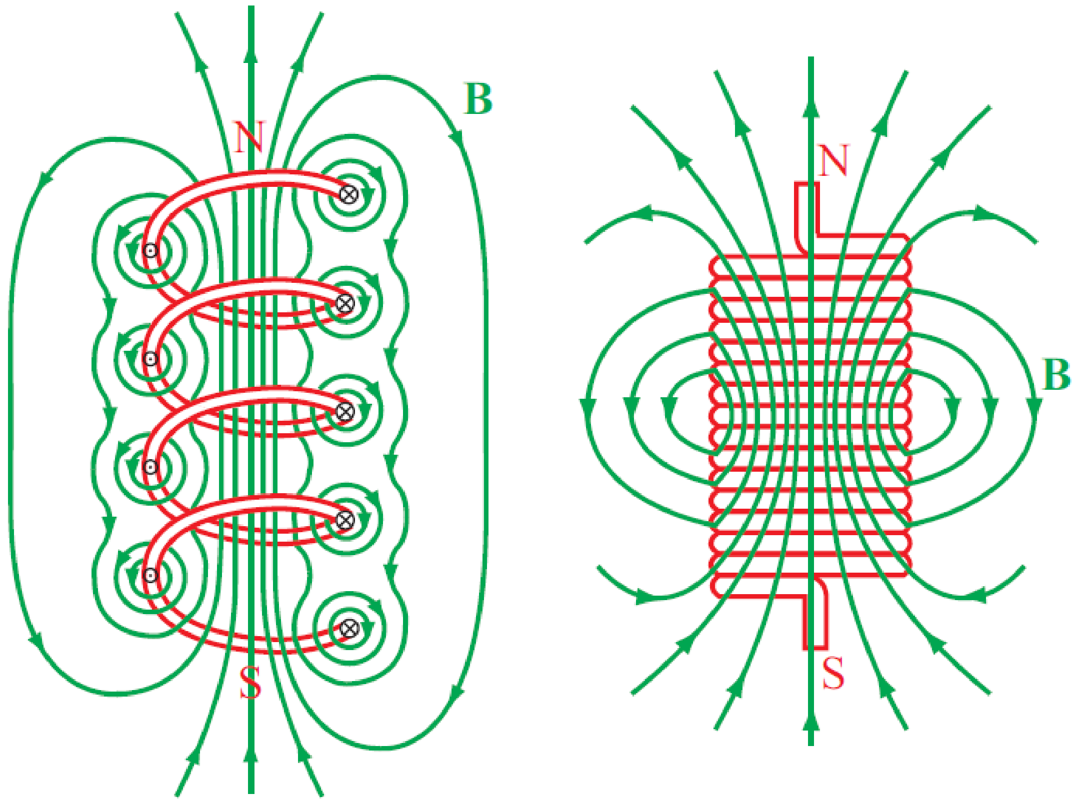
Surface currents can exist only on the surfaces of perfect conductors and superconductors. Hence, ***at the interface between media with finite conductivities,*** $\mathbf{J}_s = 0$ and

$$\boxed{H_{1t} = H_{2t}. \qquad (5.85)}$$

Solenoid

Coil

Inductor



(a) Loosely wound solenoid

(b) Tightly wound solenoid

Inside the solenoid:

$$\mathbf{B} \simeq \hat{\mathbf{z}}\mu n I = \frac{\hat{\mathbf{z}}\mu N I}{l} \qquad (\text{long solenoid with } l/a \gg 1)$$

Inductance

and for two-conductor configurations similar to those of Fig. 5-27,

$$L = \frac{\Lambda}{I} = \frac{\Phi}{I} = \frac{1}{I}\int_S \mathbf{B} \cdot d\mathbf{s}. \qquad (5.96)$$

Magnetic Flux

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{s} \qquad \text{(Wb)}.$$

Flux Linkage

$$\Lambda = N\Phi = \mu \frac{N^2}{l} I S \qquad \text{(Wb)}$$

Inductance

$$L = \frac{\Lambda}{I} \qquad \text{(H)}.$$

Solenoid

$$L = \mu \frac{N^2}{l} S \qquad \text{(solenoid)}, \qquad (5.95)$$

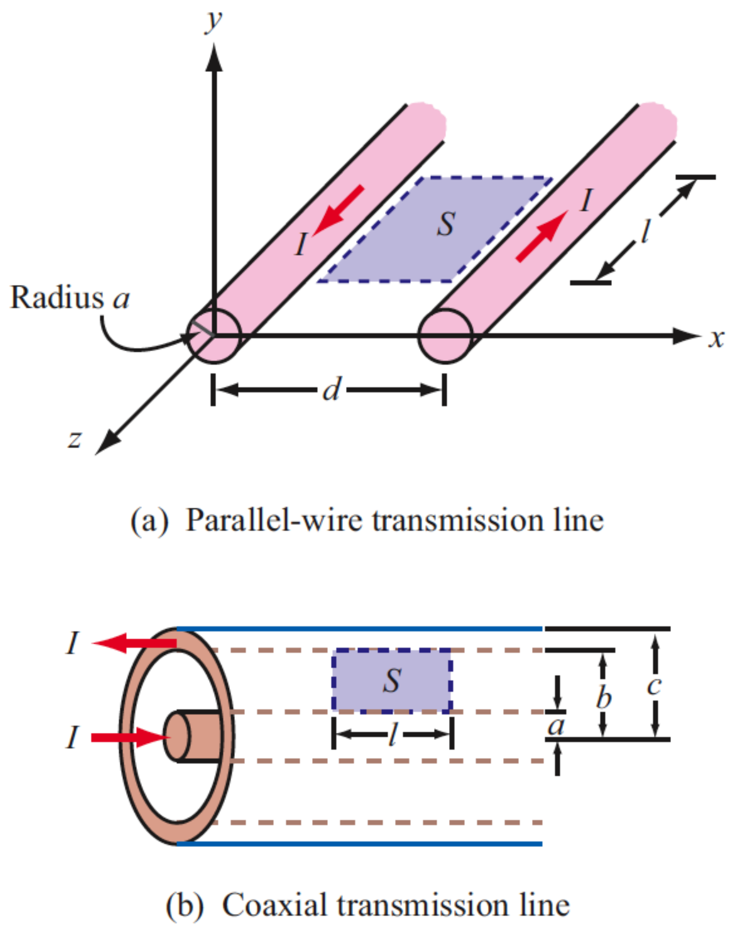


Figure 5-27: To compute the inductance per unit length of a two-conductor transmission line, we need to determine the magnetic flux through the area S between the conductors.

Example 5-7: Inductance of Coaxial Cable

The magnetic field in the region S between the two conductors is approximately

$$\mathbf{B} = \hat{\boldsymbol{\phi}} \frac{\mu I}{2\pi r}$$

Total magnetic flux through S:

$$\Phi = l \int_a^b B \, dr = l \int_a^b \frac{\mu I}{2\pi r} \, dr = \frac{\mu I l}{2\pi} \ln\left(\frac{b}{a}\right)$$

Inductance per unit length:

$$L' = \frac{L}{l} = \frac{\Phi}{l I} = \frac{\mu}{2\pi} \ln\left(\frac{b}{a}\right).$$

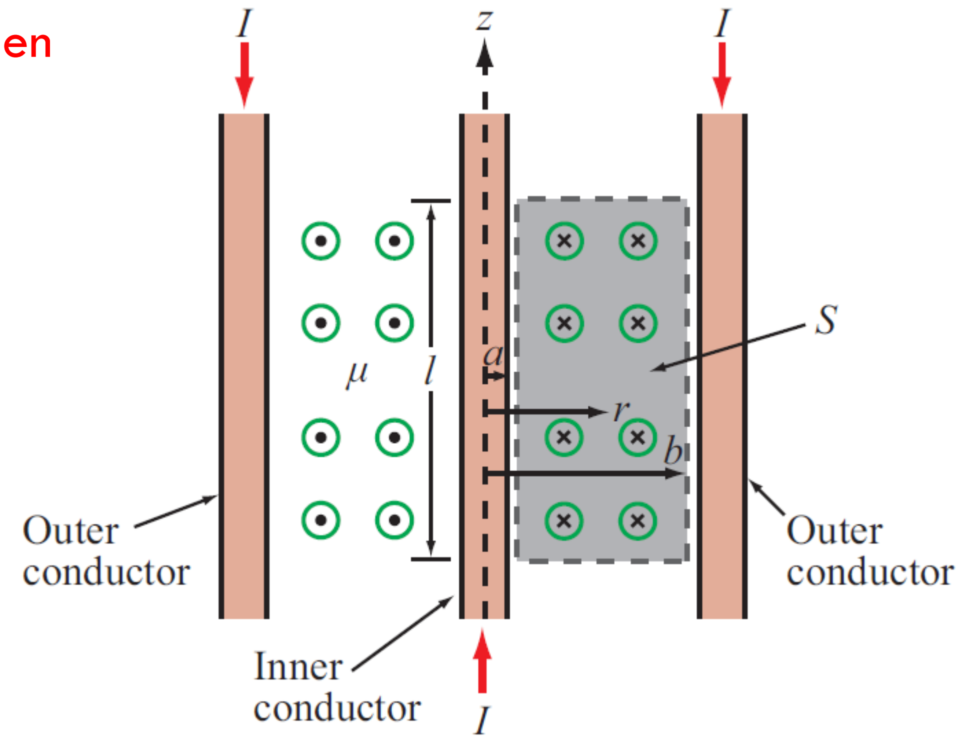


Figure 5-28: Cross-sectional view of coaxial transmission line (Example 5-7).

Tech Brief 11: Inductive Sensors

LVDT can measure displacement with submillimeter precision

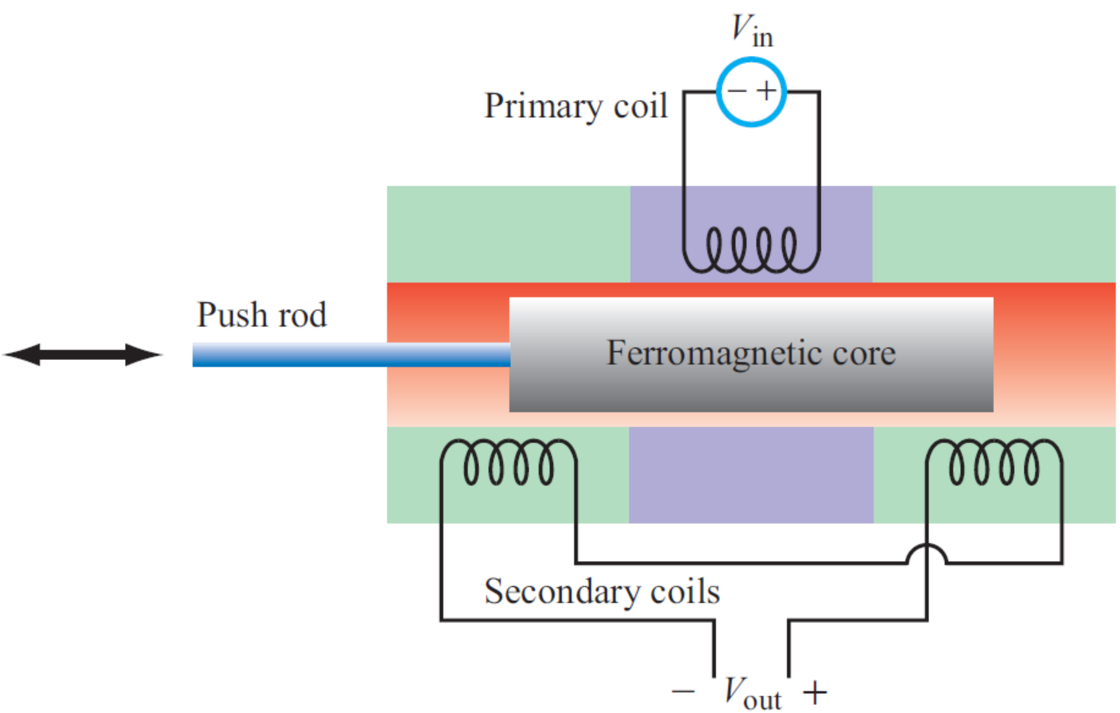


Figure TF11-1: Linear variable differential transformer (LVDT) circuit.

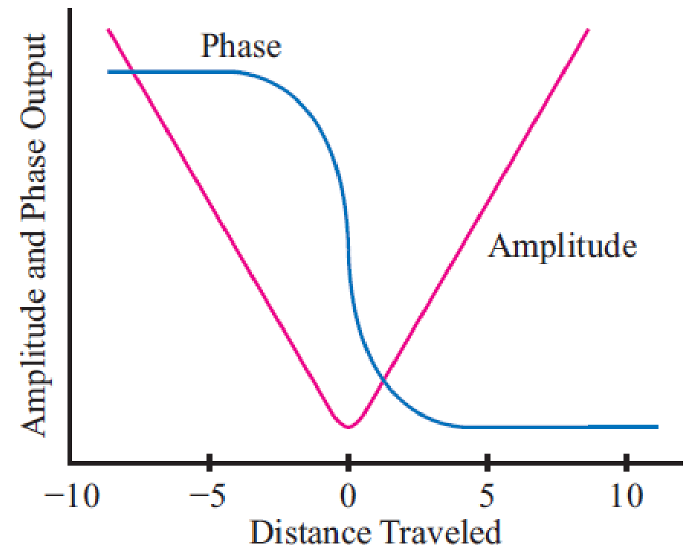


Figure TF11-2: Amplitude and phase responses as a function of the distance by which the magnetic core is moved away from the center position.

Proximity Sensor

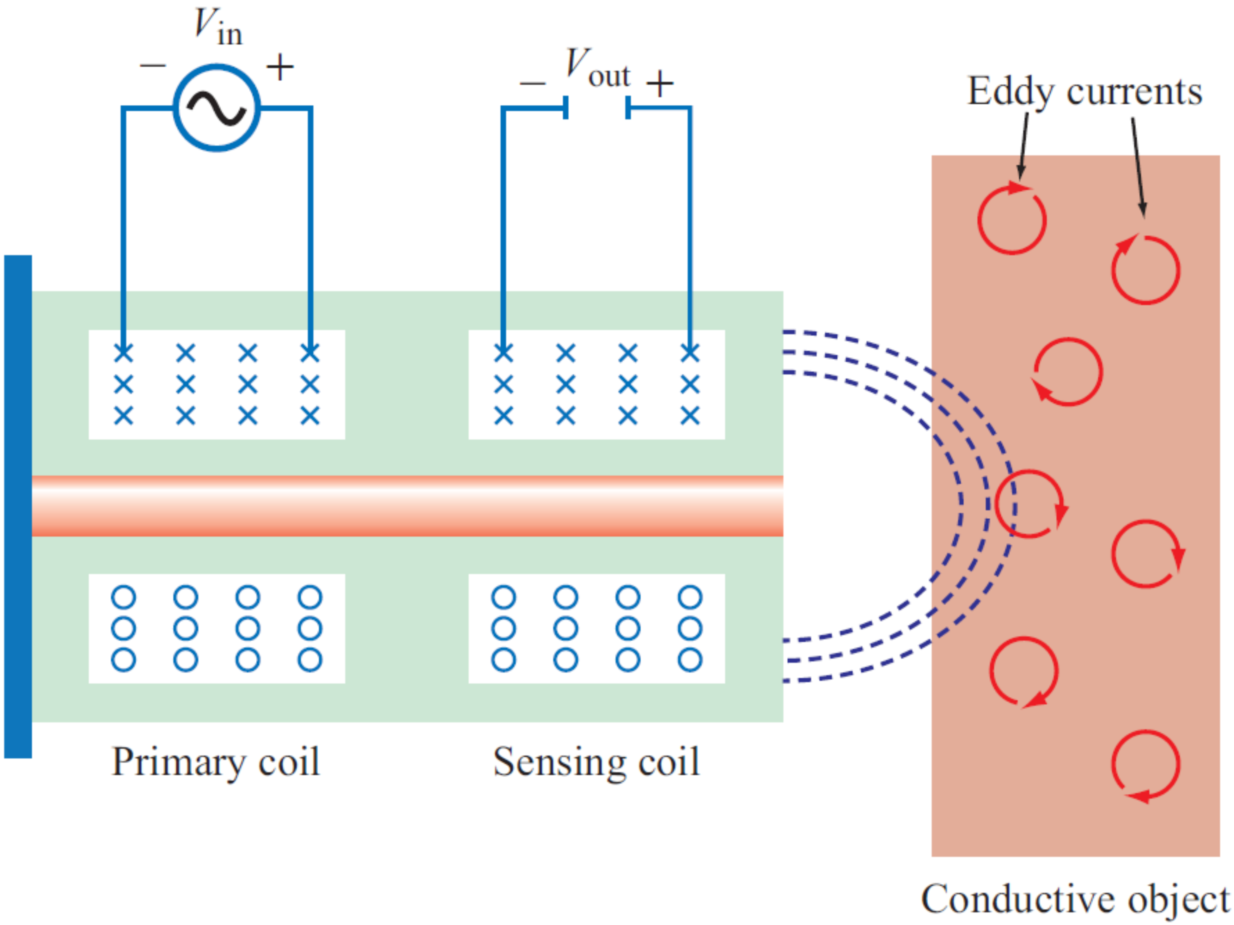
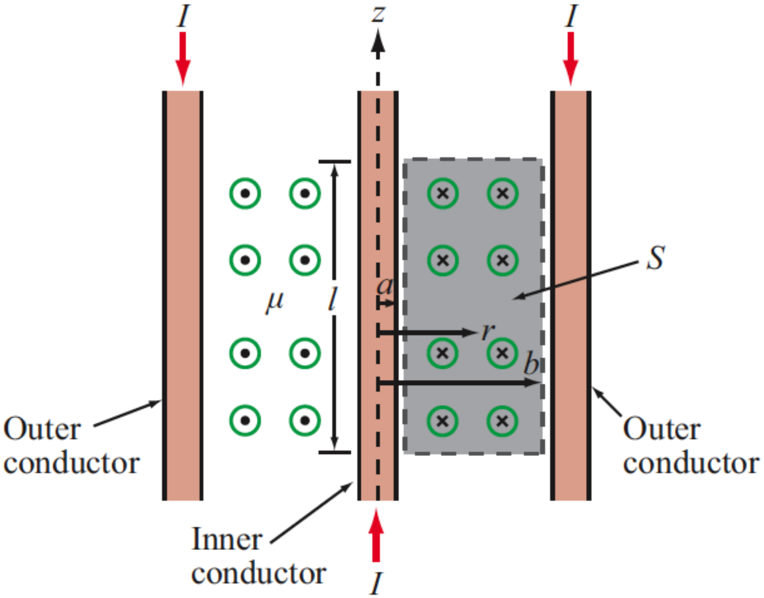


Figure TF11-5: Eddy-current proximity sensor.

Magnetic Energy Density

$$w_m = \frac{W_m}{\mathcal{V}} = \frac{1}{2}\mu H^2 \qquad (\text{J/m}^3).$$

Example 5-8: Magnetic Energy in a Coaxial Cable



Magnetic field in the insulating material is

$$H = \frac{B}{\mu} = \frac{I}{2\pi r}$$

The magnetic energy stored in the coaxial cable is

$$W_m = \frac{1}{2}\int_{\mathcal{V}} \mu H^2 \, d\mathcal{V} = \frac{\mu I^2}{8\pi^2}\int_{\mathcal{V}} \frac{1}{r^2}\, d\mathcal{V}$$

$$W_m = \frac{\mu I^2}{8\pi^2}\int_a^b \frac{1}{r^2}\cdot 2\pi r l \, dr = \frac{\mu I^2 l}{4\pi}\ln\left(\frac{b}{a}\right) = \frac{1}{2}LI^2 \qquad (\text{J}),$$

Summary

Chapter 5 Relationships

Maxwell's Magnetostatics Equations

Gauss's Law for Magnetism

$$\nabla \cdot \mathbf{B} = 0 \quad \leftrightarrow \quad \oint_S \mathbf{B} \cdot d\mathbf{s} = 0$$

Ampère's Law

$$\nabla \times \mathbf{H} = \mathbf{J} \quad \leftrightarrow \quad \oint_C \mathbf{H} \cdot d\boldsymbol{\ell} = I$$

Lorentz Force on Charge q

$$\mathbf{F} = q(\mathbf{E} + \mathbf{u} \times \mathbf{B})$$

Magnetic Force on Wire

$$\mathbf{F}_{\mathrm{m}} = I \oint_C d\mathbf{l} \times \mathbf{B} \qquad \text{(N)}$$

Magnetic Torque on Loop

$$\mathbf{T} = \mathbf{m} \times \mathbf{B} \qquad \text{(N·m)}$$

$$\mathbf{m} = \hat{\mathbf{n}} NIA \qquad \text{(A·m}^2\text{)}$$

Biot–Savart Law

$$\mathbf{H} = \frac{I}{4\pi} \int_l \frac{d\mathbf{l} \times \hat{\mathbf{R}}}{R^2} \qquad \text{(A/m)}$$

Magnetic Field

Infinitely Long Wire $\quad \mathbf{B} = \hat{\boldsymbol{\phi}} \dfrac{\mu_0 I}{2\pi r} \qquad \text{(Wb/m}^2\text{)}$

Circular Loop $\quad \mathbf{H} = \hat{\mathbf{z}} \dfrac{Ia^2}{2(a^2 + z^2)^{3/2}} \qquad \text{(A/m)}$

Solenoid $\quad \mathbf{B} \simeq \hat{\mathbf{z}} \mu n I = \dfrac{\hat{\mathbf{z}} \mu N I}{l} \qquad \text{(Wb/m}^2\text{)}$

Vector Magnetic Potential

$$\mathbf{B} = \nabla \times \mathbf{A} \qquad \text{(Wb/m}^2\text{)}$$

Vector Poisson's Equation

$$\nabla^2 \mathbf{A} = -\mu \mathbf{J}$$

Inductance

$$L = \frac{\Lambda}{I} = \frac{\Phi}{I} = \frac{1}{I} \int_S \mathbf{B} \cdot d\mathbf{s} \qquad \text{(H)}$$

Magnetic Energy Density

$$w_{\mathrm{m}} = \frac{1}{2} \mu H^2 \qquad \text{(J/m}^3\text{)}$$

ELECTROMAGNETICS II COURSE

September 22, 2019
Dr. Omar Saraereh
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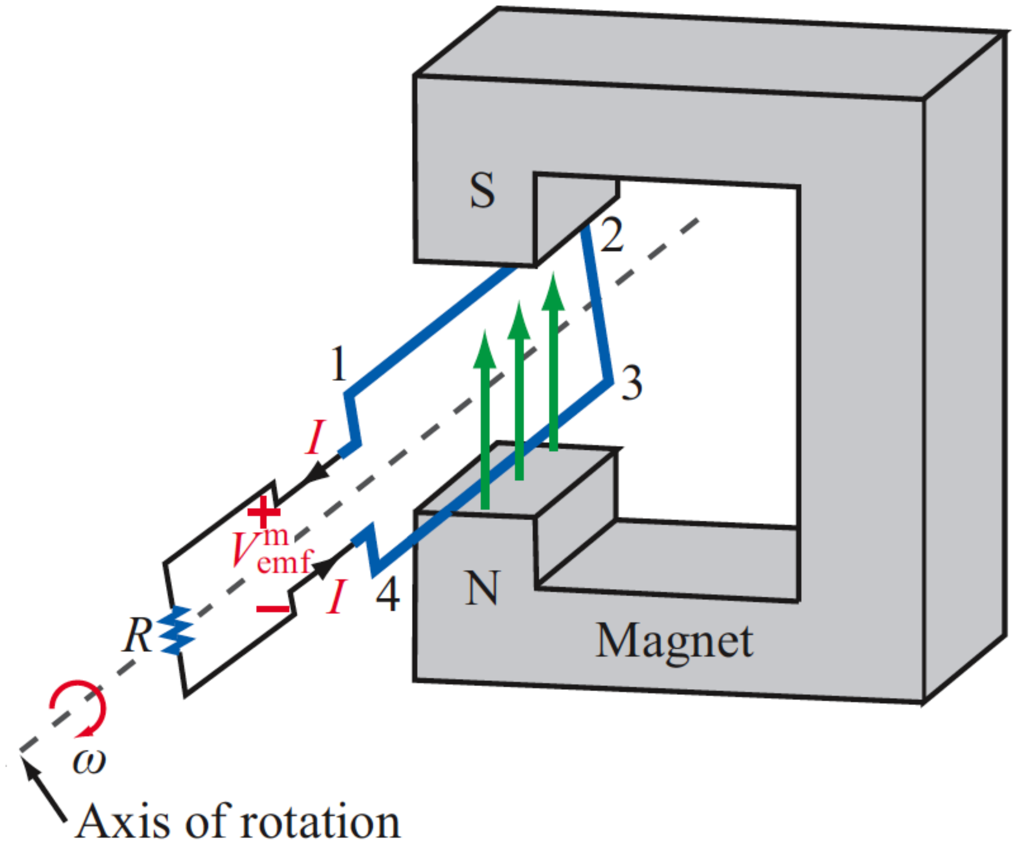


(b) ac generator

6. MAXWELL'S EQUATIONS IN TIME-VARYING FIELDS

7e Applied EM by Ulaby and Ravaioli

Chapter 6 Overview

Chapter Contents

Objectives

Upon learning the material presented in this chapter, you should be able to:

1. Apply Faraday's law to compute the voltage induced by a stationary coil placed in a time-varying magnetic field or moving in a medium containing a magnetic field.
2. Describe the operation of the electromagnetic generator.
3. Calculate the displacement current associated with a time-varying electric field.
4. Calculate the rate at which charge dissipates in a material with known ϵ and σ.

Maxwell's Equations

Table 6-1: Maxwell's equations.

Reference	Differential Form	Integral Form	
Gauss's law	$\nabla \cdot \mathbf{D} = \rho_{\mathrm{v}}$	$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q$	(6.1)
Faraday's law	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}$	(6.2)*
Gauss's law for magnetism	$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{s} = 0$	(6.3)
Ampère's law	$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$	$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right) \cdot d\mathbf{s}$	(6.4)

*For a stationary surface S.

In this chapter, we will examine Faraday's and Ampère's laws

Faraday's Law

Electromotive force (voltage) induced by time-varying magnetic flux:

$$V_{\text{emf}} = -N \frac{d\Phi}{dt} = -N \frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{s} \qquad \text{(V)}$$

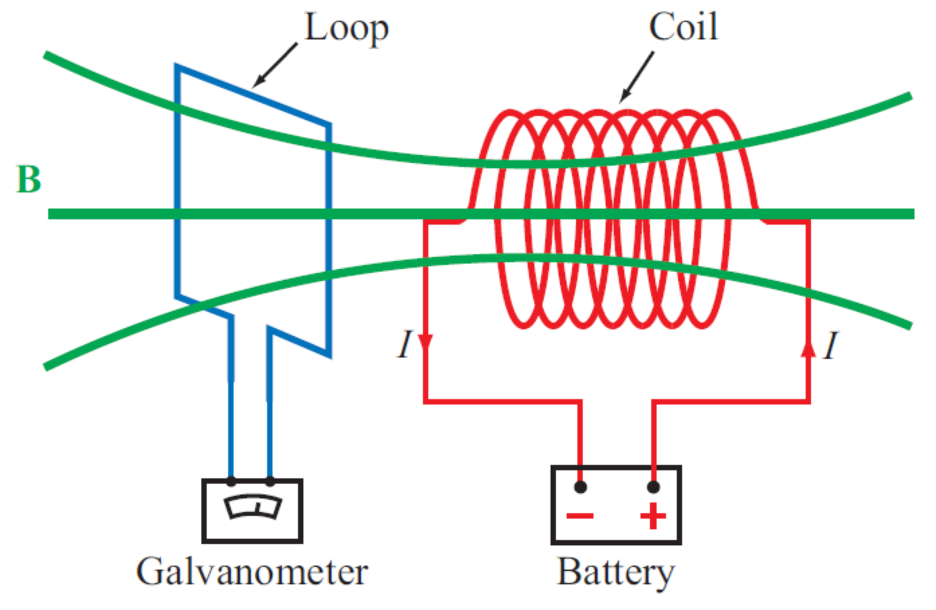


Figure 6-1: The galvanometer (predecessor of the ammeter) shows a deflection whenever the magnetic flux passing through the square loop changes with time.

Magnetic fields can produce an electric current in a closed loop, but only if the magnetic flux linking the surface area of the loop changes with time.The key to the induction process is change.

Three types of EMF

1. A time-varying magnetic field linking a stationary loop; the induced emf is then called the ***transformer emf***, V_{emf}^{tr}.

2. A moving loop with a time-varying surface area (relative to the normal component of **B**) in a static field **B**; the induced emf is then called the ***motional emf***, V_{emf}^{m}.

3. A moving loop in a time-varying field **B**.

The total emf is given by

$$V_{emf} = V_{emf}^{tr} + V_{emf}^{m}, \qquad (6.7)$$

Stationary Loop in Time-Varying B

It is important to remember that $\mathbf{B}_{\text{ind}}$ *serves to oppose the change in* $\mathbf{B}(t)$*, and not necessarily* $\mathbf{B}(t)$ *itself.*

$$V_{\text{emf}}^{\text{tr}} = -N \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s} \quad \text{(transformer emf)},$$

The connection between the direction of $d\mathbf{s}$ *and the polarity of* $V_{\text{emf}}^{\text{tr}}$ *is governed by the following right-hand rule: if* $d\mathbf{s}$ *points along the thumb of the right hand, then the direction of the contour* C *indicated by the four fingers is such that it always passes across the opening from the positive terminal of* $V_{\text{emf}}^{\text{tr}}$ *to the negative terminal.*

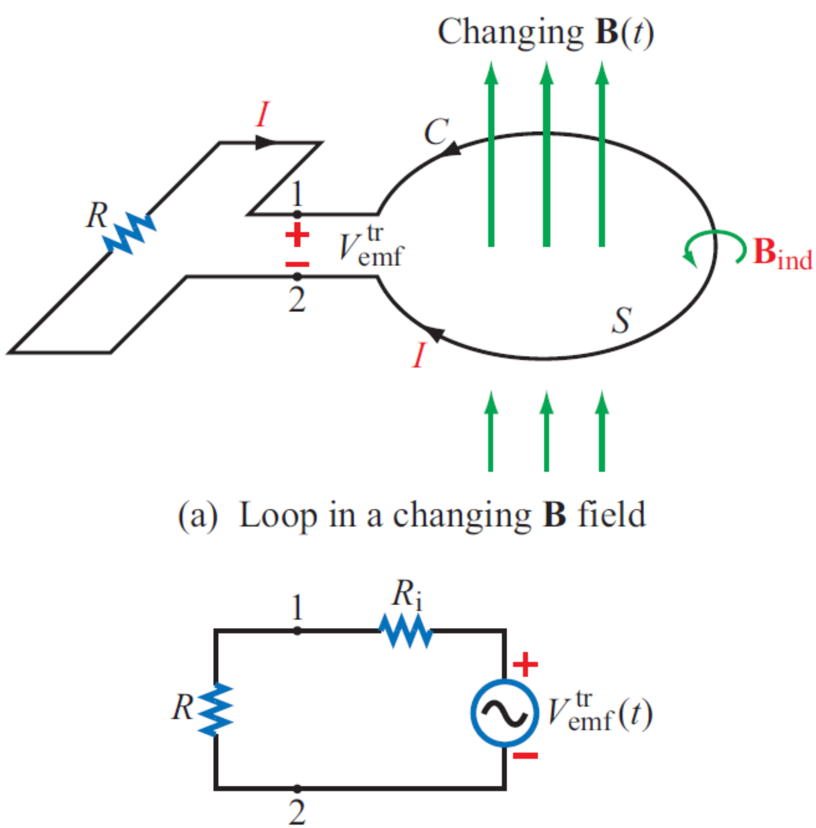


Figure 6-2: (a) Stationary circular loop in a changing magnetic field $\mathbf{B}(t)$, and (b) its equivalent circuit.

$$I = \frac{V_{\text{emf}}^{\text{tr}}}{R + R_{\text{i}}} \,. \qquad (6.9)$$

For good conductors, R_{i} usually is very small, and it may be ignored in comparison with practical values of R.

The polarity of $V_{\text{emf}}^{\text{tr}}$ *and hence the direction of* I *is governed by* ***Lenz's law****, which states that the current in the loop is always in a direction that opposes the change of magnetic flux* $\Phi(t)$ *that produced* I.

Example 6-1: Inductor in a Changing Magnetic Field

An inductor is formed by winding N turns of a thin conducting wire into a circular loop of radius a. The inductor loop is in the x–y plane with its center at the origin, and connected to a resistor R, as shown in Fig. 6-3. In the presence of a magnetic field $\mathbf{B} = B_0(\hat{\mathbf{y}}2 + \hat{\mathbf{z}}3)\sin\omega t$, where ω is the angular frequency, find

(a) the magnetic flux linking a single turn of the inductor,

(b) the transformer emf, given that $N = 10$, $B_0 = 0.2$ T, $a = 10$ cm, and $\omega = 10^3$ rad/s,

(c) the polarity of $V_{\text{emf}}^{\text{tr}}$ at $t = 0$, and

(d) the induced current in the circuit for $R = 1$ kΩ (assume the wire resistance to be much smaller than R).

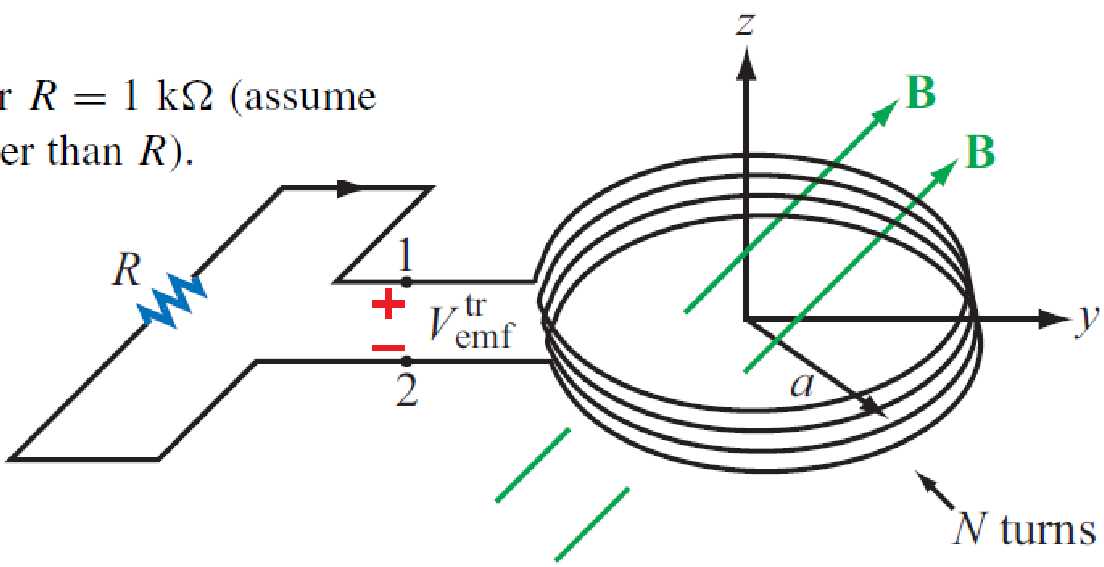


Figure 6-3: Circular loop with N turns in the x–y plane. The magnetic field is $\mathbf{B} = B_0(\hat{\mathbf{y}}2 + \hat{\mathbf{z}}3)\sin\omega t$ (Example 6-1).

cont.

Example 6-1 Solution

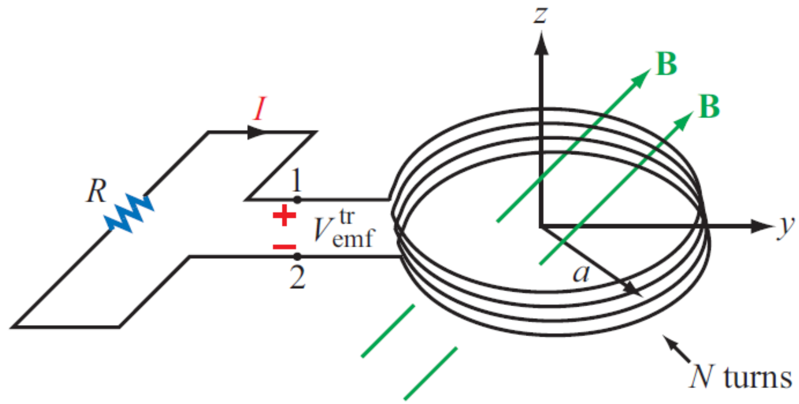


Figure 6-3: Circular loop with N turns in the x–y plane. The magnetic field is $\mathbf{B} = B_0(\hat{\mathbf{y}}2 + \hat{\mathbf{z}}3)\sin\omega t$ (Example 6-1).

Solution: **(a)** The magnetic flux linking each turn of the inductor is

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{s}$$
$$= \int_S [B_0(\hat{\mathbf{y}}\,2 + \hat{\mathbf{z}}\,3)\sin\omega t] \cdot \hat{\mathbf{z}}\,ds$$
$$= 3\pi a^2 B_0 \sin\omega t.$$

(b) To find $V_{\text{emf}}^{\text{tr}}$, we can apply Eq. (6.8) or we can apply the general expression given by Eq. (6.6) directly. The latter approach gives

$$V_{\text{emf}}^{\text{tr}} = -N\,\frac{d\Phi}{dt}$$
$$= -\frac{d}{dt}(3\pi N a^2 B_0 \sin\omega t)$$
$$= -3\pi N\omega a^2 B_0 \cos\omega t.$$

For $N = 10$, $a = 0.1$ m, $\omega = 10^3$ rad/s, and $B_0 = 0.2$ T,

$$V_{\text{emf}}^{\text{tr}} = -188.5 \cos 10^3 t \qquad \text{(V)}.$$

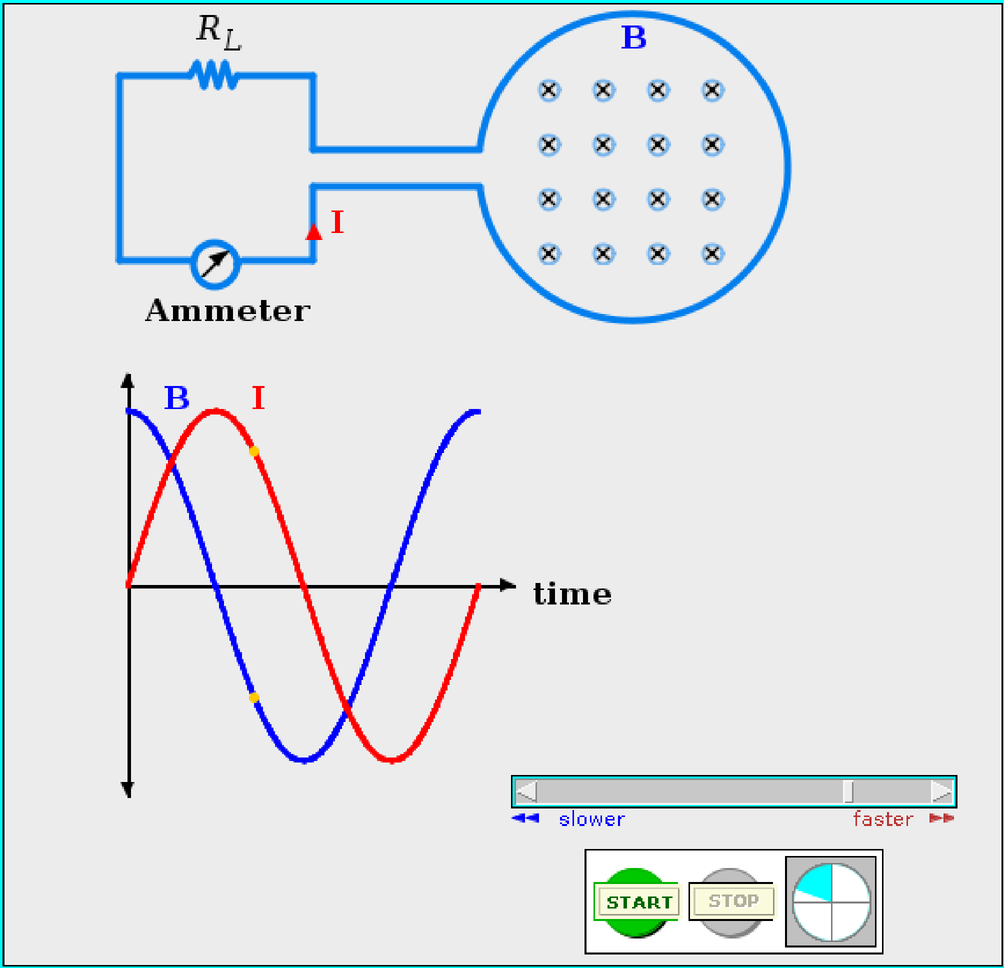
(c) At $t = 0$, $d\Phi/dt > 0$ and $V_{\text{emf}}^{\text{tr}} = -188.5$ V. Since the flux is increasing, the current I must be in the direction shown in Fig. 6-3 in order to satisfy Lenz's law. Consequently, terminal 2 is at a higher potential than terminal 1 and

$$V_{\text{emf}}^{\text{tr}} = V_1 - V_2$$
$$= -188.5 \qquad \text{(V)}.$$

(d) The current I is given by

$$I = \frac{V_2 - V_1}{R}$$
$$= \frac{188.5}{10^3}\cos 10^3 t$$
$$= 0.19 \cos 10^3 t \qquad \text{(A)}.$$

Module 6.1 **Circular Loop in Time-varying Magnetic Field**



Demonstration of Faraday's Law

The circular wire loop shown in the figure is connected to a simple circuit composed of a resistor R_L in series with a current meter. The time-varying magnetic flux linking the surface of the loop induces a V_{emf}, and hence a current through R. The purpose of this demo is to illustrate, in the form of a slow-motion video, how the current I varies with time, in both magnitude and direction, when $B(t) = B_0 \cos \omega t$.

Note that $I(t)$ is a maximum when the slope of $B(t)$ is a maximum, which occurs when B itself is zero. The direction of $I(t)$ is dictated by Lenz's Law.

Applet Design: Janice Richards

Example 6-2: Lenz's Law

Determine voltages V_1 and V_2 across the 2-Ω and 4-Ω resistors shown in Fig. 6-4. The loop is located in the x–y plane, its area is 4 m^2, the magnetic flux density is $\mathbf{B} = -\hat{\mathbf{z}}0.3t$ (T), and the internal resistance of the wire may be ignored.

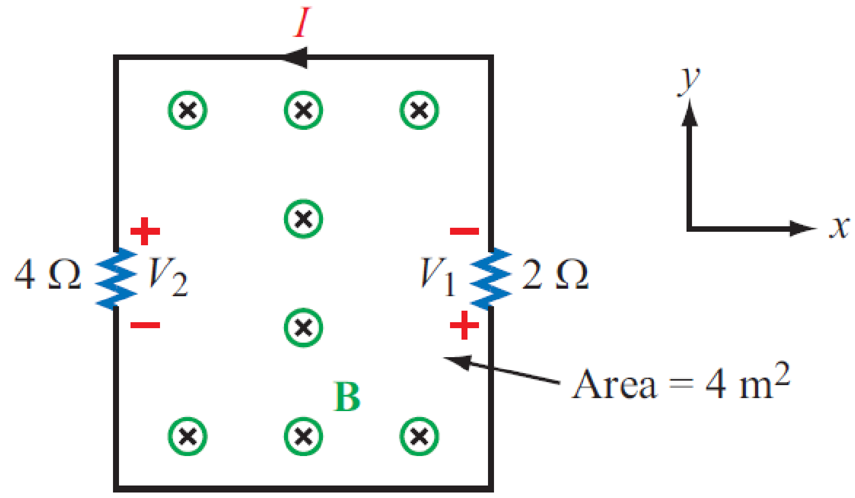


Figure 6-4: Circuit for Example 6-2.

Solution: The flux flowing through the loop is

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{s} = \int_S (-\hat{\mathbf{z}}0.3t) \cdot \hat{\mathbf{z}}\, ds$$

$$= -0.3t \times 4 = -1.2t \qquad \text{(Wb)},$$

and the corresponding transformer emf is

$$V_{\text{emf}}^{\text{tr}} = -\frac{d\Phi}{dt} = 1.2 \qquad \text{(V)}.$$

The total voltage of 1.2 V is distributed across two resistors in series. Consequently,

$$I = \frac{V_{\text{emf}}^{\text{tr}}}{R_1 + R_2}$$

$$= \frac{1.2}{2+4} = 0.2 \text{ A},$$

and

$$V_1 = IR_1 = 0.2 \times 2 = 0.4 \text{ V},$$

$$V_2 = IR_2 = 0.2 \times 4 = 0.8 \text{ V}.$$

Ideal Transformer

$$V_1 = -N_1 \frac{d\Phi}{dt} .$$

A similar relation holds true on the secondary side:

$$V_2 = -N_2 \frac{d\Phi}{dt} .$$

$$\frac{V_1}{V_2} = \frac{N_1}{N_2}$$

$$\frac{I_1}{I_2} = \frac{N_2}{N_1}$$

$$R_{\text{in}} = \frac{V_1}{I_1}$$

$$R_{\text{in}} = \frac{V_2}{I_2}\left(\frac{N_1}{N_2}\right)^2 = \left(\frac{N_1}{N_2}\right)^2 R_{\text{L}} . \quad (6.20)$$

When the load is an impedance Z_L and V_1 is a sinusoidal source, the phasor-domain equivalent of Eq. (6.20) is

$$Z_{\text{in}} = \left(\frac{N_1}{N_2}\right)^2 Z_{\text{L}} . \quad (6.21)$$

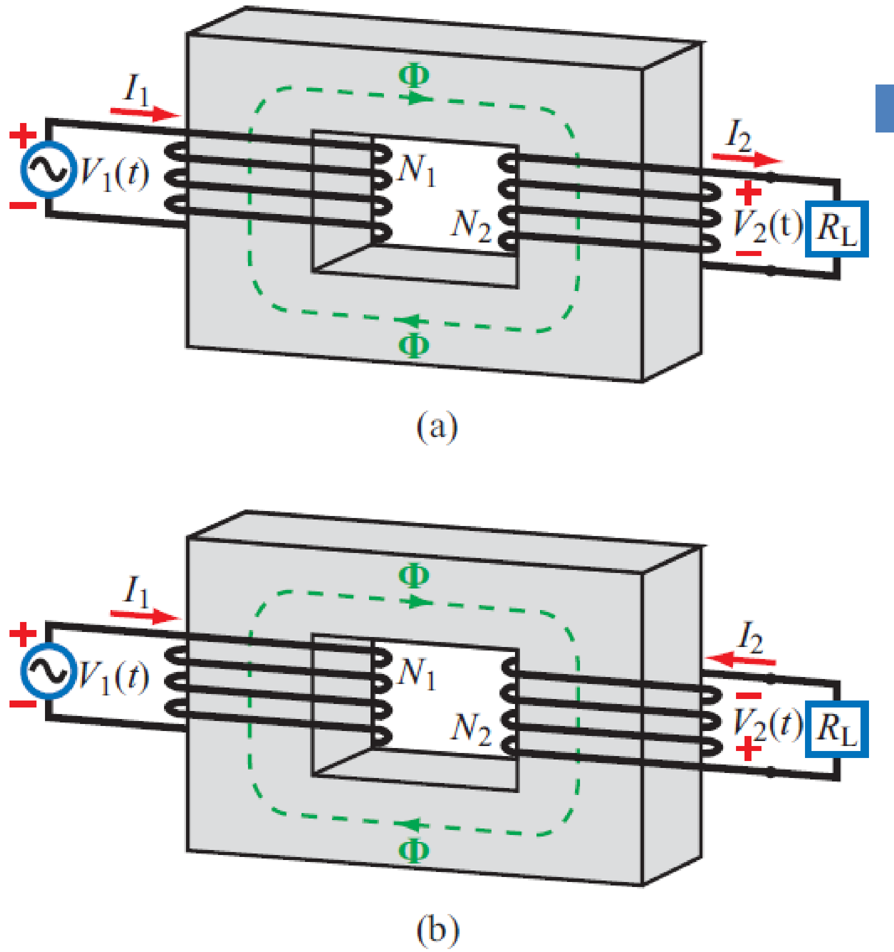


Figure 6-5: In a transformer, the directions of I_1 and I_2 are such that the flux Φ generated by one of them is opposite to that generated by the other. The direction of the secondary winding in (b) is opposite to that in (a), and so are the direction of I_2 and the polarity of V_2.

Motional EMF

Magnetic force on charge q moving with velocity **u** in a magnetic field **B**:

$$\mathbf{F}_m = q(\mathbf{u} \times \mathbf{B}).$$

This magnetic force is equivalent to the electrical force that would be exerted on the particle by the electric field Em given by

$$\mathbf{E}_m = \frac{\mathbf{F}_m}{q} = \mathbf{u} \times \mathbf{B}.$$

This, in turn, induces a voltage difference between ends 1 and 2, with end 2 being at the higher potential. The induced voltage is

$$V^m_{emf} = V_{12} = \int_2^1 \mathbf{E}_m \cdot d\mathbf{l} = \int_2^1 (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l}.$$

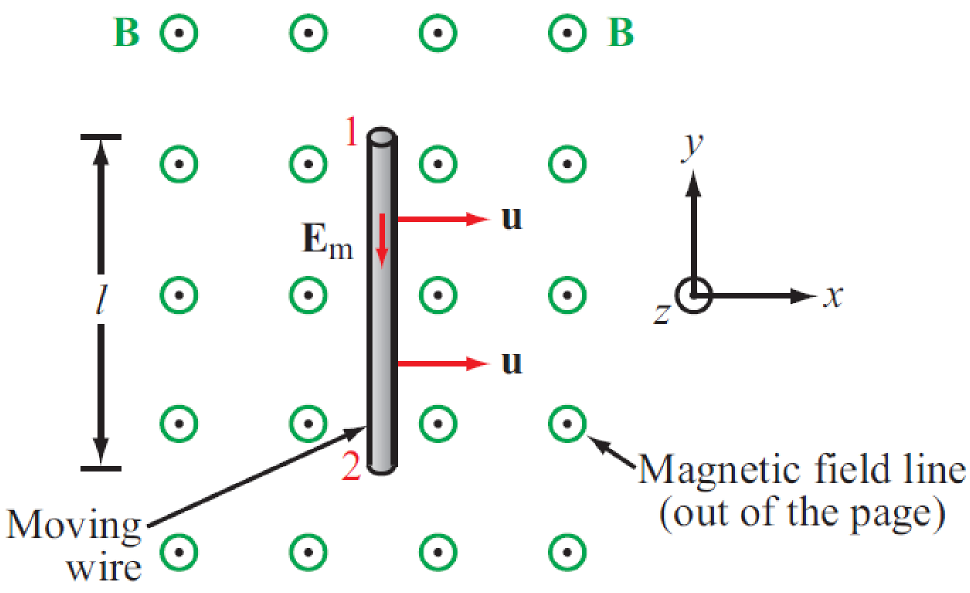


Figure 6-7: Conducting wire moving with velocity **u** in a static magnetic field.

For the conducting wire, $\mathbf{u} \times \mathbf{B} = \hat{\mathbf{x}}u \times \hat{\mathbf{z}}B_0 = -\hat{\mathbf{y}}uB_0$ and $d\mathbf{l} = \hat{\mathbf{y}}\, dl$. Hence,

$$V^m_{emf} = V_{12} = -uB_0 l. \qquad (6.25)$$

Motional EMF

In general, if any segment of a closed circuit with contour C moves with a velocity $\mathbf{u}$ across a static magnetic field $\mathbf{B}$, then the induced motional emf is given by

$$V_{\text{emf}}^{\text{m}} = \oint_C (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l} \qquad \text{(motional emf)}. \qquad (6.26)$$

Only those segments of the circuit that cross magnetic field lines contribute to $V_{\text{emf}}^{\text{m}}$.

Example 6-3: Sliding Bar

$$V_{emf}^{m} = V_{12} = V_{43} = \int_{3}^{4} (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l}$$

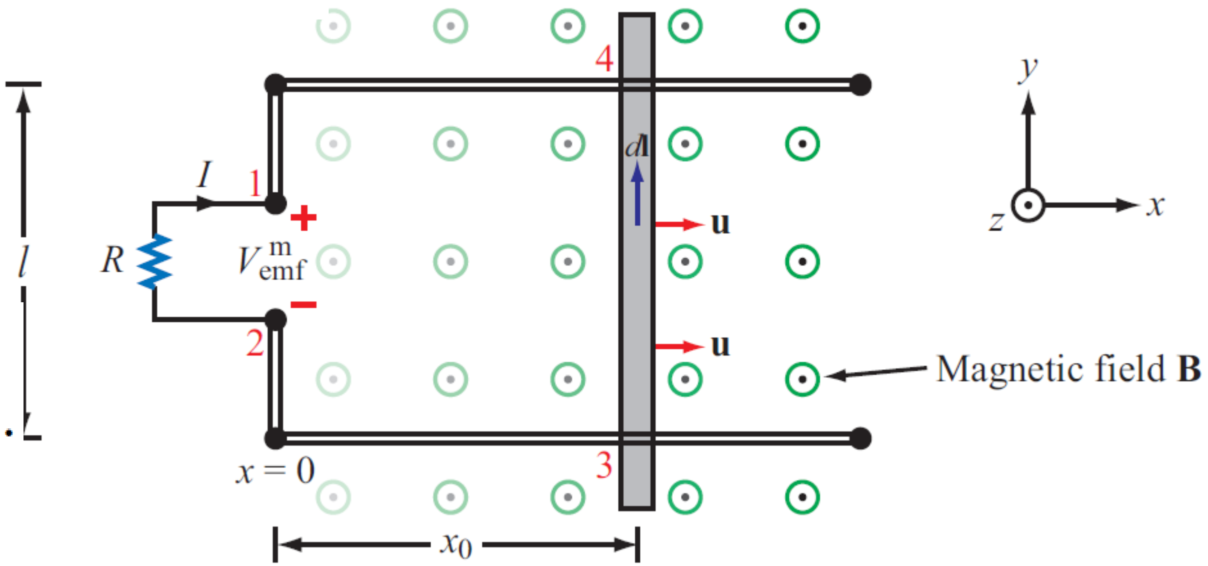
$$= \int_{3}^{4} (\hat{\mathbf{x}}u \times \hat{\mathbf{z}}B_0x_0) \cdot \hat{\mathbf{y}}\, dl = -uB_0x_0l.$$

Note that B increases with x

$$\mathbf{B} = \hat{\mathbf{z}}B_0x$$

The length of the loop is related to u by $x_0 = ut$. Hence

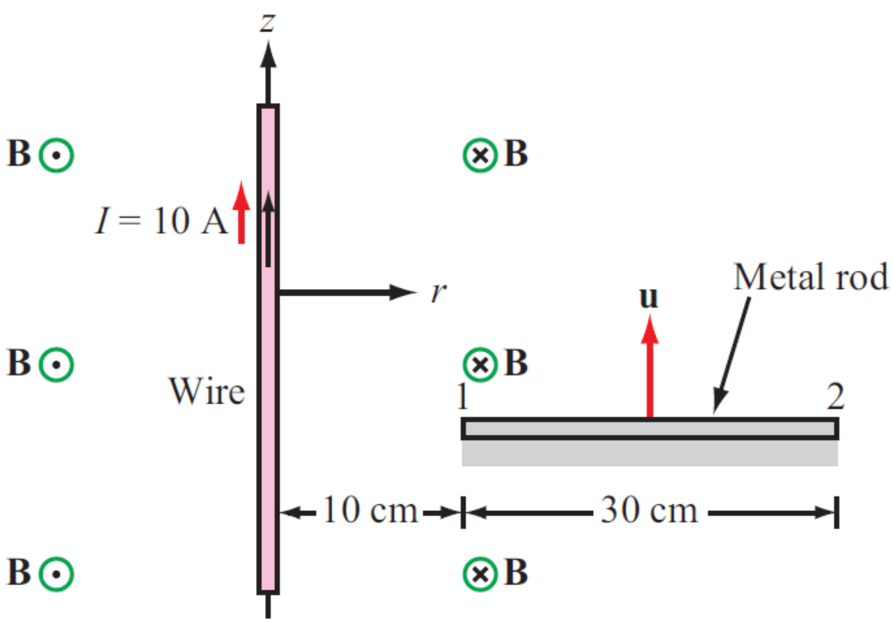
$$V_{emf}^{m} = -B_0u^2lt \qquad (\text{V}).$$



Example 6-5: Moving Rod Next to a Wire

The wire shown in Fig. 6-10 carries a current $I = 10$ A. A 30-cm-long metal rod moves with a constant velocity $\mathbf{u} = \hat{\mathbf{z}}5$ m/s. Find V_{12}.

$$\mathbf{B} = \hat{\boldsymbol{\phi}} \frac{\mu_0 I}{2\pi r}$$



$$V_{12} = \int_{40\text{ cm}}^{10\text{ cm}} (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l}$$

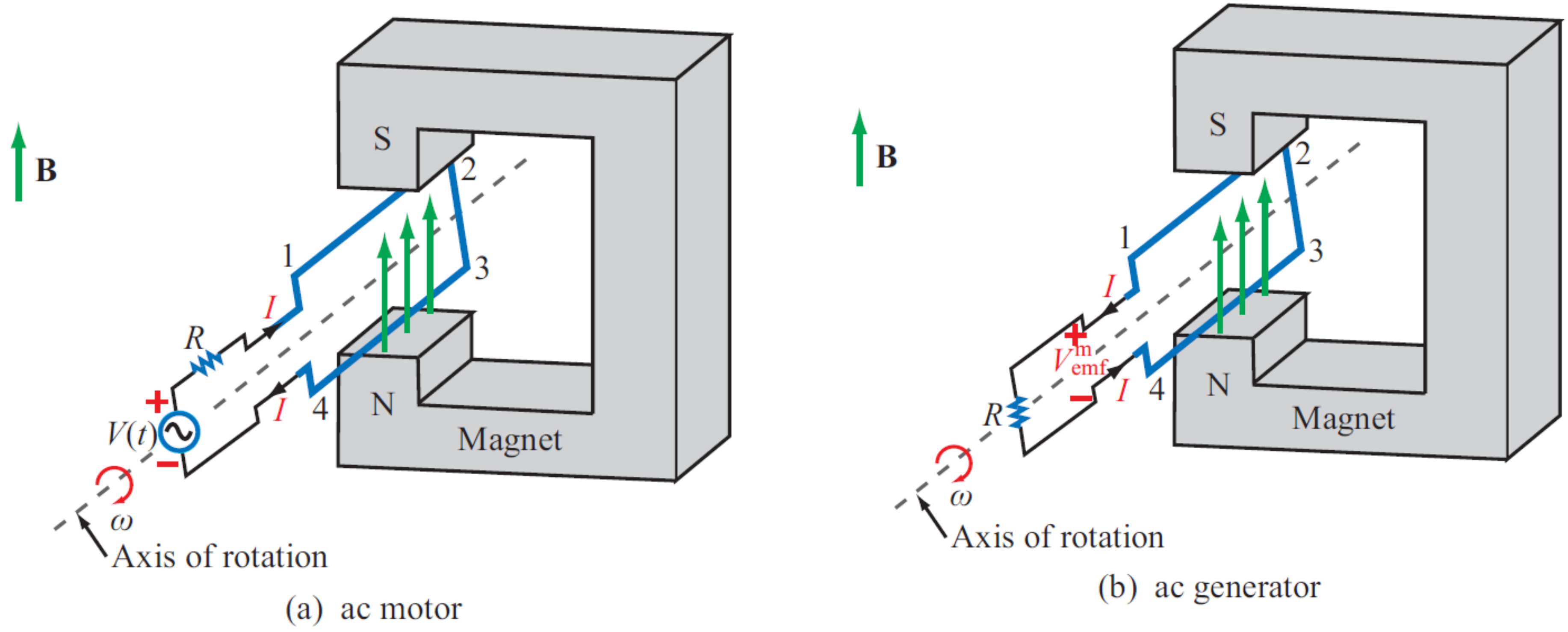
$$= \int_{40\text{ cm}}^{10\text{ cm}} \left(\hat{\mathbf{z}}5 \times \hat{\boldsymbol{\phi}} \frac{\mu_0 I}{2\pi r} \right) \cdot \hat{\mathbf{r}}\, dr$$

$$= -\frac{5\mu_0 I}{2\pi} \int_{40\text{ cm}}^{10\text{ cm}} \frac{dr}{r}$$

$$= -\frac{5 \times 4\pi \times 10^{-7} \times 10}{2\pi} \times \ln\left(\frac{10}{40}\right)$$

$$= 13.9 \quad (\mu\text{V}).$$

EM Motor/ Generator Reciprocity



Motor: Electrical to mechanical energy conversion

Generator: Mechanical to electrical energy conversion

EM Generator EMF

As the loop rotates with an angular velocity *ω about its own* axis, segment 1–2 moves with velocity **u** given by

$$\mathbf{u} = \hat{\mathbf{n}}\omega \frac{w}{2}$$

Also: $\hat{\mathbf{n}} \times \hat{\mathbf{z}} = \hat{\mathbf{x}} \sin\alpha$.

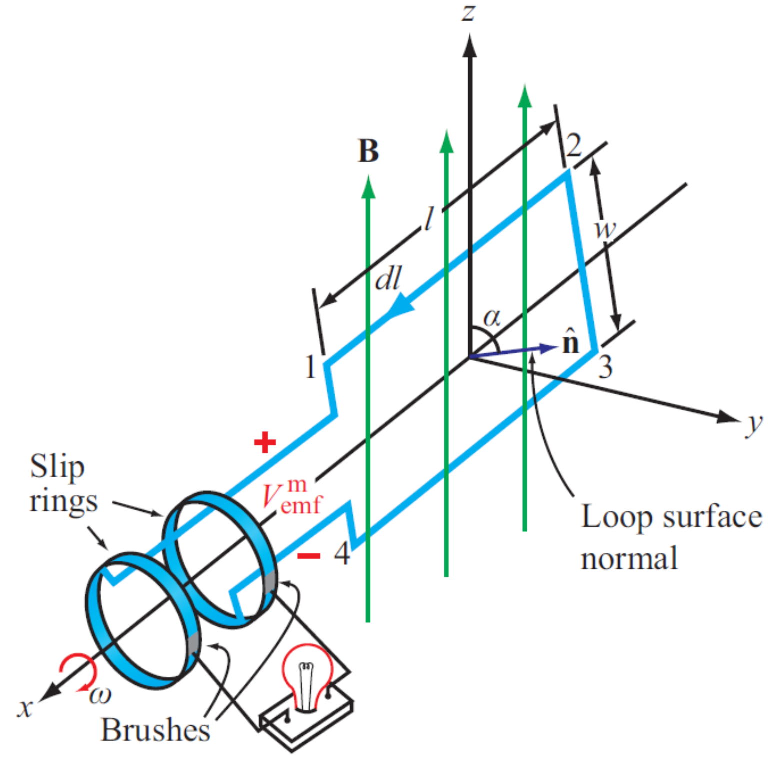


Figure 6-12: A loop rotating in a magnetic field induces an emf.

Segment 3-4 moves with velocity **–u**. Hence:

$$V^{\rm m}_{\rm emf} = V_{14} = \int_2^1 (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l} + \int_4^3 (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l}$$

$$= \int_{-l/2}^{l/2} \left[\left(\hat{\mathbf{n}}\omega \frac{w}{2}\right) \times \hat{\mathbf{z}}B_0\right] \cdot \hat{\mathbf{x}}\, dx$$

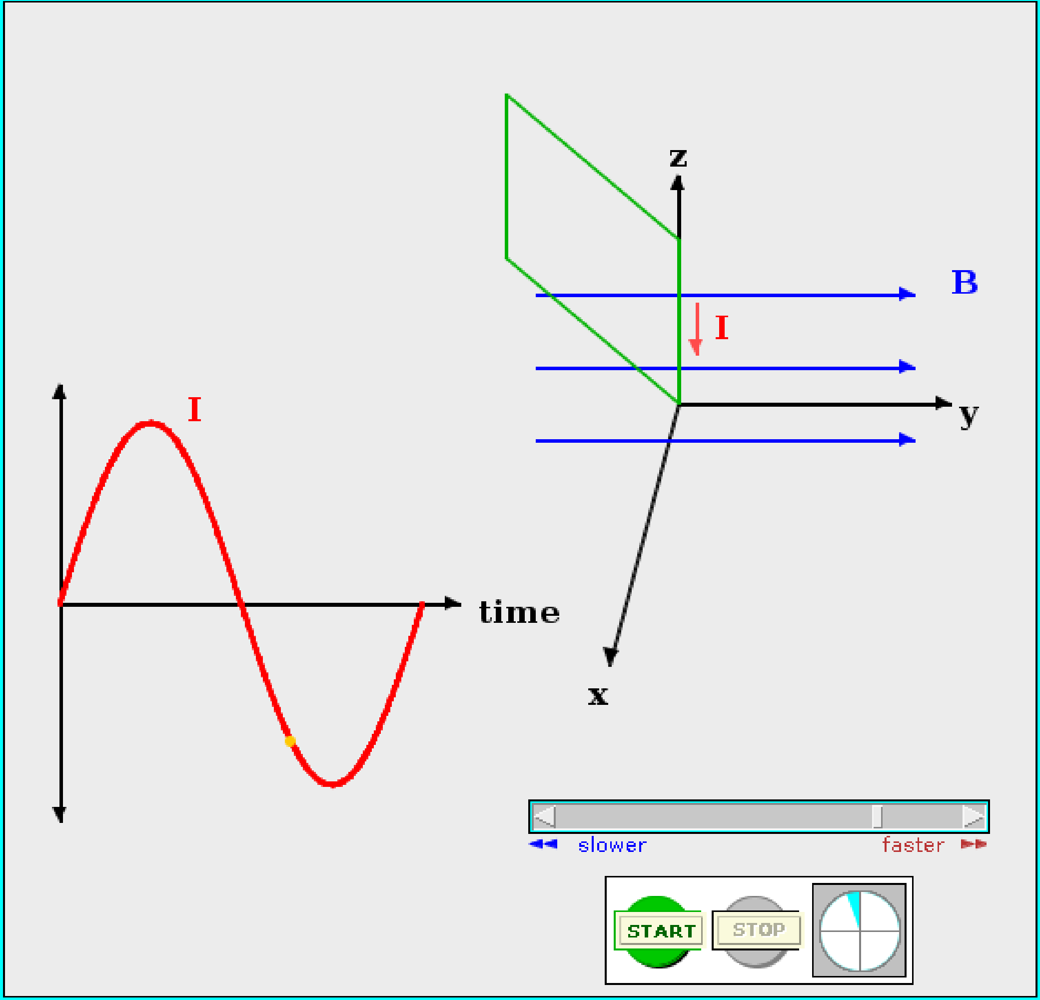
$$+ \int_{l/2}^{-l/2} \left[\left(-\hat{\mathbf{n}}\omega \frac{w}{2}\right) \times \hat{\mathbf{z}}B_0\right] \cdot \hat{\mathbf{x}}\, dx.$$

$$V^{\rm m}_{\rm emf} = wl\omega B_0 \sin\alpha = A\omega B_0 \sin\alpha,$$

$$\alpha = \omega t + C_0,$$

$$V^{\rm m}_{\rm emf} = A\omega B_0 \sin(\omega t + C_0) \qquad \text{(V)}.$$

Module 6.2 Rotating Wire Loop in Constant Magnetic Field



Demonstration of Motional EMF

A rectangular wire loop of area A rotates at an angular frequency ω in a constant magnetic flux density B_0. The purpose of the demo is to illustrate how the current varies in time relative to the loop's position.

Note the direction of the current and its magnitude, as indicated by its brightness.

$$I_{max} = \omega B_0 A$$

Applet Design: Janice Richards

Tech Brief 12: EMF Sensors

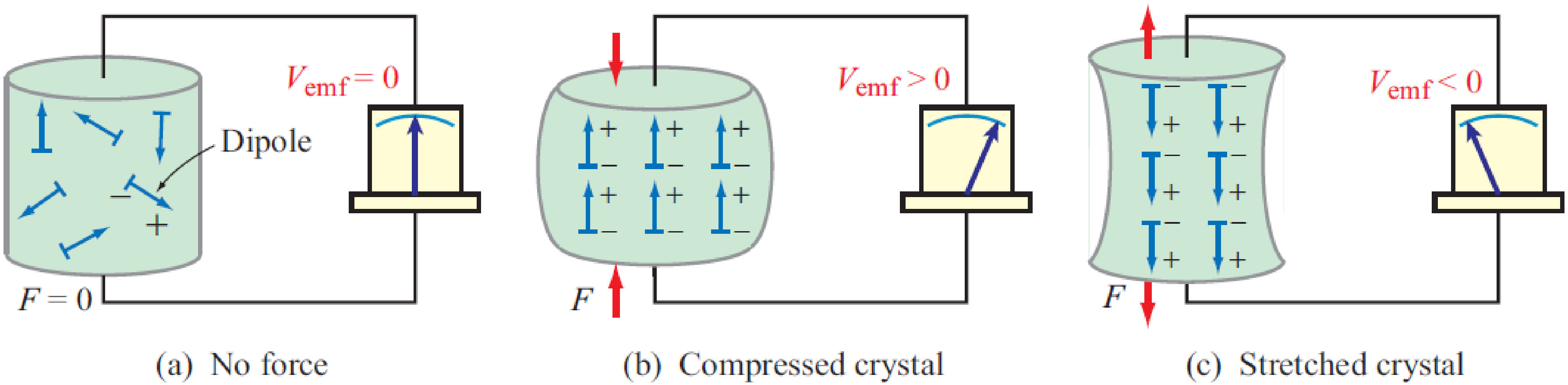


Figure TF12-1: Response of a piezoelectric crystal to an applied force.

- Piezoelectric crystals generate a voltage across them proportional to the compression or tensile (stretching) force applied across them.

- Piezoelectric transducers are used in medical ultrasound, microphones, loudspeakers, accelerometers, etc.

- Piezoelectric crystals are bidirectional: pressure generates emf, and conversely, emf generates pressure (through shape distortion).

Faraday Accelerometer

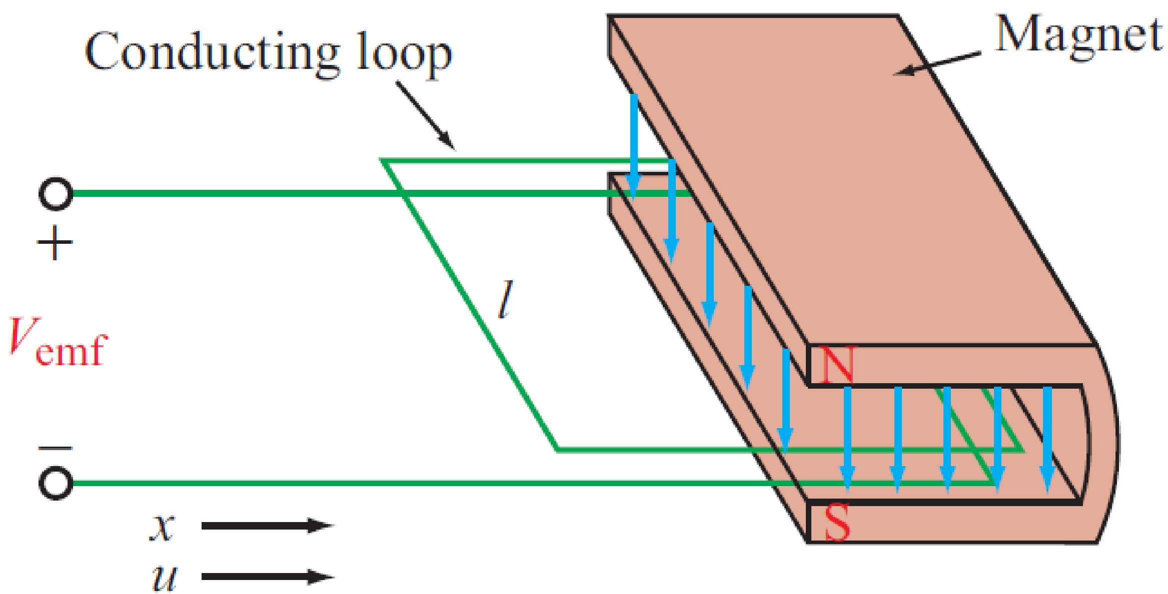


Figure TF12-3: In a Faraday accelerometer, the induced emf is directly proportional to the velocity of the loop (into and out of the magnet's cavity).

The acceleration **a** is determined by differentiating the velocity u with respect to time

The Thermocouple

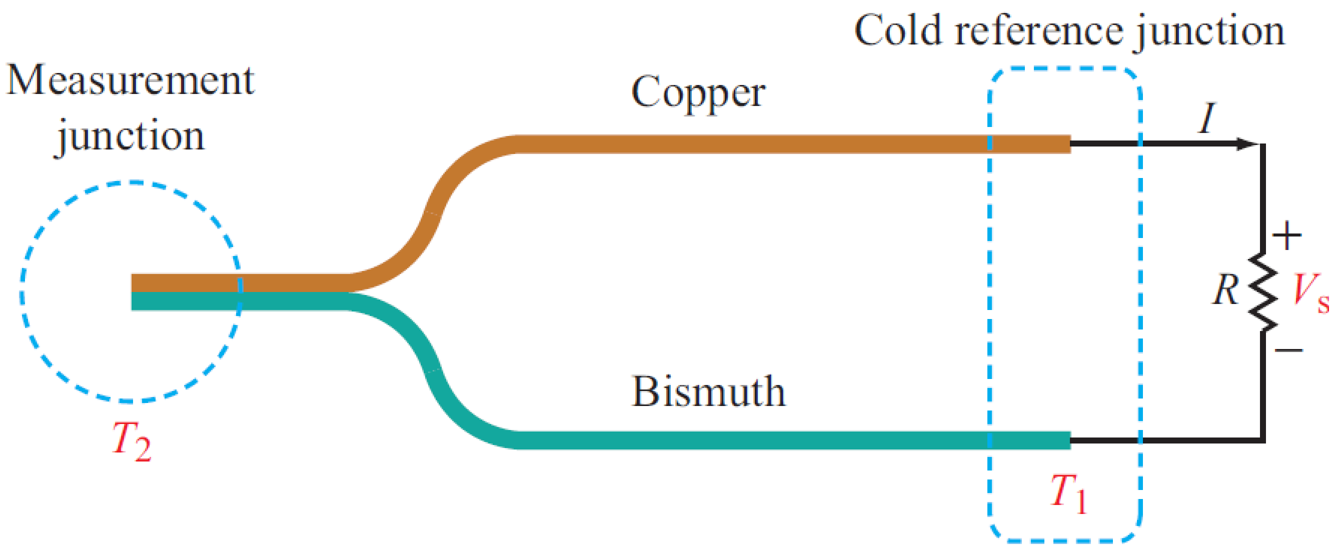


Figure TF12-4: Principle of the thermocouple.

- The thermocouple measures the unknown temperature T_2 at a junction connecting two metals with different thermal conductivities, relative to a reference temperature T_1.
- In today's temperature sensor designs, an artificial cold junction is used instead. The artificial junction is an electric circuit that generates a voltage equal to that expected from a reference junction at temperature T_1.

Displacement Current

Ampère's law in differential form is given by

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \qquad \text{(Ampère's law).} \qquad (6.41)$$

Integrating both sides of Eq. (6.41) over an arbitrary open surface S with contour C, we have

$$\int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{s} = \int_S \mathbf{J} \cdot d\mathbf{s} + \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{s}. \qquad (6.42)$$

This term is conduction current I_C

This term must represent a current

Application of Stokes's theorem gives:

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = I_c + \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{s} \qquad \text{(Ampère's law)}$$

Cont.

Displacement Current

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = I_c + \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{s} \quad \text{(Ampère's law)}$$

Define the displacement current as:

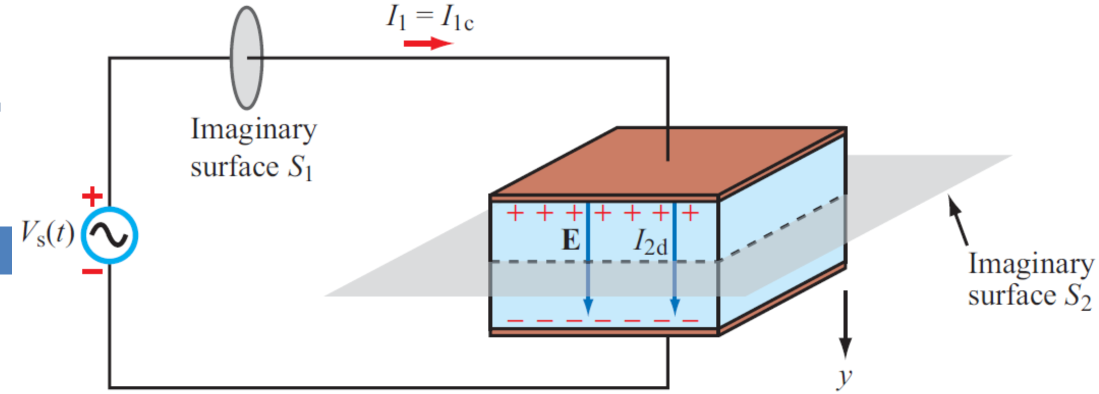
$$I_d = \int_S \mathbf{J}_d \cdot d\mathbf{s} = \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{s}, \qquad (6.44)$$

The displacement current does not involve real charges;
it is an equivalent current that depends on $\partial \mathbf{D}/\partial t$

where $\mathbf{J}_d = \partial \mathbf{D}/\partial t$ represents a ***displacement current density***. In view of Eq. (6.44),

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = I_c + I_d = I, \qquad (6.45)$$

Capacitor Circuit



Given: Wires are perfect conductors and capacitor insulator material is perfect dielectric.

For Surface S_1:

$$I_1 = I_{1c} + I_{1d}$$

$$I_{1c} = C\frac{dV_C}{dt} = C\frac{d}{dt}(V_0 \cos\omega t) = -CV_0\omega \sin\omega t$$

$$I_{1d} = 0$$ (**D** = 0 in perfect conductor)

For Surface S_2:

$$I_2 = I_{2c} + I_{2d}$$

$I_{2c} = 0$ (perfect dielectric)

$$\mathbf{E} = \hat{\mathbf{y}}\frac{V_c}{d} = \hat{\mathbf{y}}\frac{V_0}{d}\cos\omega t$$

$$I_{2d} = \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{s}$$

$$= \int_A \left[\frac{\partial}{\partial t}\left(\hat{\mathbf{y}}\frac{\varepsilon V_0}{d}\cos\omega t\right)\right] \cdot (\hat{\mathbf{y}}\, ds)$$

$$= -\frac{\varepsilon A}{d}V_0\omega \sin\omega t = -CV_0\omega \sin\omega t$$

Conclusion: $I_1 = I_2$

Example 6-7: Displacement Current Density

The conduction current flowing through a wire with conductivity $\sigma = 2 \times 10^7$ S/m and relative permittivity $\varepsilon_r = 1$ is given by $I_c = 2 \sin \omega t$ (mA). If $\omega = 10^9$ rad/s, find the displacement current.

Solution: The conduction current $I_c = JA = \sigma EA$, where A is the cross section of the wire. Hence,

$$E = \frac{I_c}{\sigma A} = \frac{2 \times 10^{-3} \sin \omega t}{2 \times 10^7 A} = \frac{1 \times 10^{-10}}{A} \sin \omega t \qquad \text{(V/m)}.$$

Application of Eq. (6.44), with $D = \varepsilon E$, leads to

$$\begin{aligned} I_d &= J_d A \\ &= \varepsilon A \frac{\partial E}{\partial t} \\ &= \varepsilon A \frac{\partial}{\partial t}\left(\frac{1 \times 10^{-10}}{A} \sin \omega t\right) \\ &= \varepsilon \omega \times 10^{-10} \cos \omega t = 0.885 \times 10^{-12} \cos \omega t \quad \text{(A)}, \end{aligned}$$

where we used $\omega = 10^9$ rad/s and $\varepsilon = \varepsilon_0 = 8.85 \times 10^{-12}$ F/m. Note that I_c and I_d are in phase quadrature (90° phase shift between them). Also, I_d is about nine orders of magnitude smaller than I_c, which is why the displacement current usually is ignored in good conductors.

Boundary Conditions

Table 6-2: Boundary conditions for the electric and magnetic fields.

Field Components	General Form	Medium 1 Dielectric	Medium 2 Dielectric	Medium 1 Dielectric	Medium 2 Conductor
Tangential E	$\hat{\mathbf{n}}_2 \times (\mathbf{E}_1 - \mathbf{E}_2) = 0$	$E_{1t} = E_{2t}$		$E_{1t} = E_{2t} = 0$	
Normal D	$\hat{\mathbf{n}}_2 \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s$	$D_{1n} - D_{2n} = \rho_s$		$D_{1n} = \rho_s$	$D_{2n} = 0$
Tangential H	$\hat{\mathbf{n}}_2 \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s$	$H_{1t} = H_{2t}$		$H_{1t} = J_s$	$H_{2t} = 0$
Normal B	$\hat{\mathbf{n}}_2 \cdot (\mathbf{B}_1 - \mathbf{B}_2) = 0$	$B_{1n} = B_{2n}$		$B_{1n} = B_{2n} = 0$	

Notes: (1) ρ_s is the surface charge density at the boundary; (2) $\mathbf{J}_s$ is the surface current density at the boundary; (3) normal components of all fields are along $\hat{\mathbf{n}}_2$, the outward unit vector of medium 2; (4) $E_{1t} = E_{2t}$ implies that the tangential components are equal in magnitude and parallel in direction; (5) direction of $\mathbf{J}_s$ is orthogonal to $(\mathbf{H}_1 - \mathbf{H}_2)$.

Charge Current Continuity Equation

Current I out of a volume is equal to rate of decrease of charge Q contained in that volume:

$$I = -\frac{dQ}{dt} = -\frac{d}{dt}\int_{\mathcal{V}} \rho_{\mathrm{v}}\, d\mathcal{V}$$

$$\oint_S \mathbf{J}\cdot d\mathbf{s} = -\frac{d}{dt}\int_{\mathcal{V}} \rho_{\mathrm{v}}\, d\mathcal{V}$$

$$\oint_S \mathbf{J}\cdot d\mathbf{s} = \int_{\mathcal{V}} \nabla\cdot\mathbf{J}\, d\mathcal{V} = -\frac{d}{dt}\int_{\mathcal{V}} \rho_{\mathrm{v}}\, d\mathcal{V}$$

Used Divergence Theorem

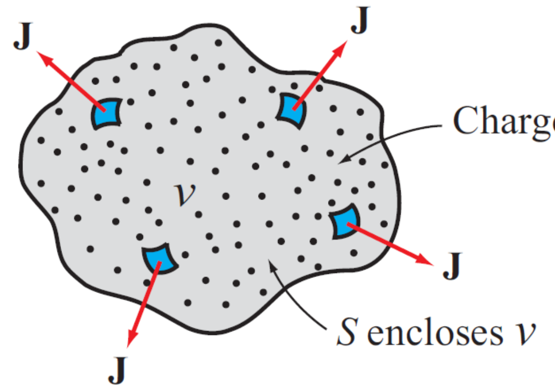


Figure 6-14: The total current flowing out of a volume $\mathcal{V}$ is equal to the flux of the current density **J** through the surface S, which in turn is equal to the rate of decrease of the charge enclosed in $\mathcal{V}$.

$$\nabla\cdot\mathbf{J} = -\frac{\partial\rho_{\mathrm{v}}}{\partial t}, \qquad (6.54)$$

which is known as the ***charge-current continuity relation***, or simply the ***charge continuity equation***.

Charge Dissipation

Question 1: What happens if you place a certain amount of free charge inside of a material?
Answer: The charge will move to the surface of the material, thereby returning its interior to a neutral state.

Question 2: How fast will this happen?
Answer: It depends on the material; in a good conductor, the charge dissipates in less than a femtosecond, whereas in a good dielectric, the process may take several hours.

Derivation of charge density equation:

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t} . \qquad (6.58)$$

In a conductor, the point form of Ohm's law, given by Eq. (4.63), states that $\mathbf{J} = \sigma \mathbf{E}$. Hence,

$$\sigma \nabla \cdot \mathbf{E} = -\frac{\partial \rho_v}{\partial t} . \qquad (6.59)$$

Next, we use Eq. (6.1), $\nabla \cdot \mathbf{E} = \rho_v / \varepsilon$, to obtain the partial differential equation

$$\frac{\partial \rho_v}{\partial t} + \frac{\sigma}{\varepsilon} \rho_v = 0. \qquad (6.60)$$

Cont.

Solution of Charge Dissipation Equation

$$\frac{\partial \rho_v}{\partial t} + \frac{\sigma}{\varepsilon}\rho_v = 0$$

Given that $\rho_v = \rho_{vo}$ at $t = 0$, the solution of Eq. (6.60) is

$$\rho_v(t) = \rho_{vo} e^{-(\sigma/\varepsilon)t} = \rho_{vo} e^{-t/\tau_r} \qquad (\text{C/m}^3),$$

where $\tau_r = \varepsilon/\sigma$ is called the ***relaxation time constant***.

For copper: $\tau_r = 1.53 \times 10^{-19}$ s

For mica: $\tau_r = 5.31 \times 10^4$ s = 15 hours

EM Potentials

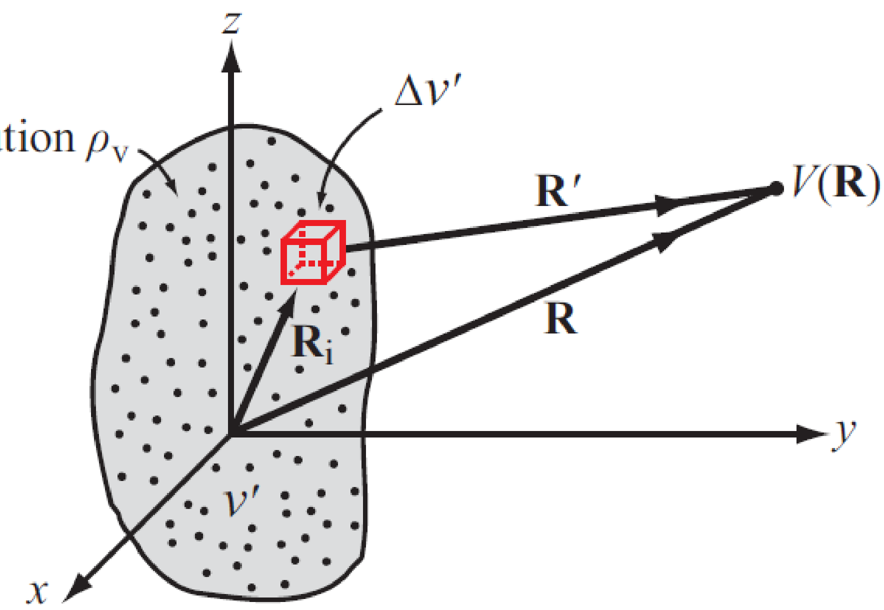


Figure 6-16: Electric potential $V(\mathbf{R})$ due to a charge distribution ρ_v over a volume $\mathcal{V}'$.

Static condition

$$V(\mathbf{R}) = \frac{1}{4\pi\varepsilon} \int_{\mathcal{V}'} \frac{\rho_v(\mathbf{R}_i)}{R'} \, d\mathcal{V}'$$

Dynamic condition

$$V(\mathbf{R}, t) = \frac{1}{4\pi\varepsilon} \int_{\mathcal{V}'} \frac{\rho_v(\mathbf{R}_i, t)}{R'} \, d\mathcal{V}'$$

Dynamic condition with propagation delay:

$$V(\mathbf{R}, t) = \frac{1}{4\pi\varepsilon} \int_{\mathcal{V}'} \frac{\rho_v(\mathbf{R}_i,\ t - R'/u_p)}{R'} \, d\mathcal{V}' \quad \text{(V)},$$

Similarly, for the magnetic vector potential:

$$\mathbf{A}(\mathbf{R}, t) = \frac{\mu}{4\pi} \int_{\mathcal{V}'} \frac{\mathbf{J}(\mathbf{R}_i,\ t - R'/u_p)}{R'} \, d\mathcal{V}' \qquad \text{(Wb/m)}.$$

Time Harmonic Potentials

If charges and currents vary sinusoidally with time:

$$\rho_v(\mathbf{R}_i, t) = \rho_v(\mathbf{R}_i)\cos(\omega t + \phi)$$

we can use phasor notation:

$$\rho_v(\mathbf{R}_i, t) = \mathfrak{Re}\left[\tilde{\rho}_v(\mathbf{R}_i)\, e^{j\omega t}\right],$$

with

$$\tilde{\rho}_v(\mathbf{R}_i) = \rho_v(\mathbf{R}_i)\, e^{j\phi}.$$

Expressions for potentials become:

$$\widetilde{V}(\mathbf{R}) = \frac{1}{4\pi\varepsilon}\int_{v'} \frac{\tilde{\rho}_v(\mathbf{R}_i)\, e^{-jkR'}}{R'}\, dv' \quad \text{(V)}.$$

$$\widetilde{\mathbf{A}}(\mathbf{R}) = \frac{\mu}{4\pi}\int_{v'} \frac{\widetilde{\mathbf{J}}(\mathbf{R}_i)\, e^{-jkR'}}{R'}\, dv',$$

Also:

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \quad \text{(dynamic case).}$$

$$\widetilde{\mathbf{H}} = \frac{1}{\mu}\nabla \times \widetilde{\mathbf{A}}.$$

Maxwell's equations become:

$$\nabla \times \widetilde{\mathbf{E}} = -j\omega\mu\widetilde{\mathbf{H}}$$

$$\text{or} \quad \widetilde{\mathbf{H}} = -\frac{1}{j\omega\mu}\nabla \times \widetilde{\mathbf{E}}.$$

$$\nabla \times \widetilde{\mathbf{H}} = j\omega\varepsilon\widetilde{\mathbf{E}} \quad \text{or} \quad \widetilde{\mathbf{E}} = \frac{1}{j\omega\varepsilon}\nabla \times \widetilde{\mathbf{H}}.$$

$$k = \frac{\omega}{u_p}$$

Example 6-8: Relating E to H

In a nonconducting medium with $\varepsilon = 16\varepsilon_0$ and $\mu = \mu_0$, the electric field intensity of an electromagnetic wave is

$$\mathbf{E}(z,t) = \hat{\mathbf{x}}\, 10 \sin(10^{10} t - kz) \qquad \text{(V/m)}. \tag{6.88}$$

Determine the associated magnetic field intensity **H** and find the value of k.

Solution: We begin by finding the phasor $\widetilde{\mathbf{E}}(z)$ of $\mathbf{E}(z,t)$. Since $\mathbf{E}(z,t)$ is given as a sine function and phasors are defined in this book with reference to the cosine function, we rewrite Eq. (6.88) as

$$\begin{aligned}\mathbf{E}(z,t) &= \hat{\mathbf{x}}\, 10 \cos(10^{10} t - kz - \pi/2) \qquad \text{(V/m)} \\ &= \mathfrak{Re}\left[\widetilde{\mathbf{E}}(z)\, e^{j\omega t}\right],\end{aligned} \tag{6.89}$$

with $\omega = 10^{10}$ (rad/s) and

$$\widetilde{\mathbf{E}}(z) = \hat{\mathbf{x}}\, 10 e^{-jkz} e^{-j\pi/2} = -\hat{\mathbf{x}}\, j 10 e^{-jkz}. \tag{6.90}$$

Cont.

Example 6-8: Relating E to H

To find both $\widetilde{\mathbf{H}}(z)$ and k, we will perform a "circle": we will use the given expression for $\widetilde{\mathbf{E}}(z)$ in Faraday's law to find $\widetilde{\mathbf{H}}(z)$; then we will use $\widetilde{\mathbf{H}}(z)$ in Ampère's law to find $\widetilde{\mathbf{E}}(z)$, which we will then compare with the original expression for $\widetilde{\mathbf{E}}(z)$; and the comparison will yield the value of k. Application of Eq. (6.87) gives

$$\begin{aligned}
\widetilde{\mathbf{H}}(z) &= -\frac{1}{j\omega\mu} \nabla \times \widetilde{\mathbf{E}} \\
&= -\frac{1}{j\omega\mu} \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -j10e^{-jkz} & 0 & 0 \end{vmatrix} \\
&= -\frac{1}{j\omega\mu} \left[\hat{\mathbf{y}} \frac{\partial}{\partial z} (-j10e^{-jkz}) \right] \\
&= -\hat{\mathbf{y}} j \frac{10k}{\omega\mu} e^{-jkz}.
\end{aligned} \tag{6.91}$$

Cont.

Example 6-8 cont.

So far, we have used Eq. (6.90) for $\tilde{\mathbf{E}}(z)$ to find $\tilde{\mathbf{H}}(z)$, but k remains unknown. To find k, we use $\tilde{\mathbf{H}}(z)$ in Eq. (6.86) to find $\tilde{\mathbf{E}}(z)$:

$$\begin{aligned}\tilde{\mathbf{E}}(z) &= \frac{1}{j\omega\varepsilon}\nabla\times\tilde{\mathbf{H}} \\ &= \frac{1}{j\omega\varepsilon}\left[-\hat{\mathbf{x}}\frac{\partial}{\partial z}\left(-j\frac{10k}{\omega\mu}e^{-jkz}\right)\right] \\ &= -\hat{\mathbf{x}}j\frac{10k^2}{\omega^2\mu\varepsilon}e^{-jkz}. \qquad (6.92)\end{aligned}$$

Equating Eqs. (6.90) and (6.92) leads to

$$k^2 = \omega^2\mu\varepsilon,$$

or

$$\begin{aligned}k &= \omega\sqrt{\mu\varepsilon} \\ &= 4\omega\sqrt{\mu_0\varepsilon_0} \\ &= \frac{4\omega}{c} = \frac{4\times10^{10}}{3\times10^8} = 133 \qquad \text{(rad/m)}. \qquad (6.93)\end{aligned}$$

Cont.

Example 6-8 cont.

With k known, the instantaneous magnetic field intensity is then given by

$$\mathbf{H}(z,t) = \mathfrak{Re}\left[\widetilde{\mathbf{H}}(z)\, e^{j\omega t}\right]$$
$$= \mathfrak{Re}\left[-\hat{\mathbf{y}} j\, \frac{10k}{\omega\mu} e^{-jkz} e^{j\omega t}\right]$$
$$= \hat{\mathbf{y}}\, 0.11 \sin(10^{10}t - 133z) \qquad \text{(A/m)}. \qquad (6.94)$$

We note that k has the same expression as the phase constant of a lossless transmission line [Eq. (2.49)].

Summary

Chapter 6 Relationships

Faraday's Law

$$V_{\text{emf}} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{s} = V_{\text{emf}}^{\text{tr}} + V_{\text{emf}}^{\text{m}}$$

Transformer

$$V_{\text{emf}}^{\text{tr}} = -N\int_S \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{s} \qquad (N \text{ loops})$$

Motional

$$V_{\text{emf}}^{\text{m}} = \oint_C (\mathbf{u}\times\mathbf{B})\cdot d\mathbf{l}$$

Charge-Current Continuity

$$\nabla\cdot\mathbf{J} = -\frac{\partial \rho_{\text{v}}}{\partial t}$$

EM Potentials

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$$

$$\mathbf{B} = \nabla\times\mathbf{A}$$

Current Density

Conduction $\mathbf{J}_{\text{c}} = \sigma\mathbf{E}$

Displacement $\mathbf{J}_{\text{d}} = \dfrac{\partial \mathbf{D}}{\partial t}$

Conductor Charge Dissipation

$$\rho_{\text{v}}(t) = \rho_{\text{vo}}e^{-(\sigma/\varepsilon)t} = \rho_{\text{vo}}e^{-t/\tau_r}$$

https://www.youtube.com/watch?v=bxHs9I3lbZc