Solutions Manual for:

Communications Systems, 5th edition

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Published by Wiley, 2009.

Chapter 2

2.1 (a)

$$g(t) = A\cos(2\pi f_c t) \qquad t \in \left[\frac{-T}{2}, \frac{T}{2}\right]$$

$$f_c = \frac{1}{T}$$

We can rewrite the half-cosine as:

$$A\cos(2\pi f_c t) \cdot \mathrm{rect}\left(\frac{t}{T}\right)$$

Using the property of multiplication in the time-domain:

$$G(f) = G_1(f) * G_2(f)$$

$$= \frac{1}{2}\left[\delta(f - f_c) + \delta(f + f_c)\right] * AT\,\frac{\sin(\pi fT)}{\pi fT}$$

Writing out the convolution:

$$G(f) = \int_{-\infty}^{\infty} \frac{AT}{2}\left(\frac{\sin(\pi\lambda T)}{\pi\lambda T}\right)\left[\delta(\lambda - (f + f_c) + \delta(\lambda - (f - f_c)\right] d\lambda$$

$$= \frac{A}{2\pi}\left(\frac{\sin(\pi(f + f_c)T)}{f + f_c} + \frac{\sin(\pi(f - f_c)T)}{f - f_c}\right) \qquad f_c = \frac{1}{2T}$$

$$= \frac{A}{2\pi}\left(\frac{\cos(\pi fT)}{f - \frac{1}{2T}} - \frac{\cos(\pi fT)}{f + \frac{1}{2T}}\right)$$

(b)By using the time-shifting property:

$$g(t - t_0) \rightleftarrows \exp(-j2\pi f t_0) \qquad t_0 = \frac{T}{2}$$

$$G(f) = \frac{A}{2\pi}\left(\frac{\cos(\pi fT)}{f - \frac{1}{2T}} - \frac{\cos(\pi fT)}{f + \frac{1}{2T}}\right) \cdot \exp(-j\pi fT)$$

(c)The half-sine pulse is identical to the half-cosine pulse except for the centre frequency and time-shift.

$$f_c = \frac{1}{2Ta}$$

$$\begin{aligned}
G(f) &= \frac{A}{2\pi}\left[\frac{\cos(\pi fTa)}{f-f_c} - \frac{\cos(\pi fTa)}{f+f_c}\right]\cdot(\cos(\pi fTa) - j\sin(\pi fTa)) \\
&= \frac{A}{4\pi}\left[\frac{\cos(2\pi fTa)}{f-f_c} - \frac{\cos(2\pi fTa)}{f+f_c} + j\frac{\sin(2\pi fTa)}{f-f_c} - j\frac{\sin(2\pi fTa)}{f+f_c}\right] \\
&= \frac{A}{4\pi}\left[\frac{\exp(-j2\pi fTa)}{f-f_c} - \frac{\exp(-j2\pi fTa)}{f+f_c}\right]
\end{aligned}$$

(d) The spectrum is the same as for (b) except shifted backwards in time and multiplied by -1.

$$\begin{aligned}
G(f) &= \frac{A}{2\pi}\left(\frac{\cos(\pi fT)}{f-\frac{1}{2T}} - \frac{\cos(\pi fT)}{f+\frac{1}{2T}}\right)\cdot\exp(j\pi fT) \\
&= \frac{A}{4\pi}\left[\frac{\exp(j2\pi fT)}{f-\frac{1}{2T}} - \frac{\exp(j2\pi fT)}{f+\frac{1}{2T}}\right]
\end{aligned}$$

(e) Because the Fourier transform is a linear operation, this is simply the summation of the results from (b) and (d)

$$\begin{aligned}
G(f) &= \frac{A}{4\pi}\left[\frac{\exp(j2\pi fT)+\exp(-j2\pi fT)}{f-\frac{1}{2T}} - \frac{\exp(j2\pi fT)+(-j2\pi fT)}{f+\frac{1}{2T}}\right] \\
&= \frac{A}{2\pi}\left[\frac{\cos(2\pi fT)}{f-\frac{1}{2T}} - \frac{\cos(2\pi fT)}{f+\frac{1}{2T}}\right]
\end{aligned}$$

2.2

$$
\begin{aligned}
g(t) &= \exp(-t)\sin(2\pi f_c t)\mathrm{u}(t) \\
&= \left(\exp(-t)\mathrm{u}(t)\right)\left(\sin(2\pi f_c t)\right)
\end{aligned}
$$

$$
\begin{aligned}
\therefore G(f) &= \frac{1}{1+j2\pi f} * \left[\frac{1}{2j}\left(\delta(f-f_c)-\delta(f+f_c)\right)\right] \\
&= \frac{1}{2j}\left[\frac{1}{1+j2\pi(f-f_c)} - \frac{1}{1+j2\pi(f+f_c)}\right]
\end{aligned}
$$

2.3 (a)

$$g(t) = g_e(t) + g_o(t)$$

$$g_e(t) = \frac{1}{2}\left[g(t)+g(-t)\right]$$

$$g_e(t) = A\mathrm{rect}\left(\frac{t}{2T}\right)$$

$$g_o(t) = \frac{1}{2}\left[g(t)-g(-t)\right]$$

$$g_o(t) = A\left(\mathrm{rect}\left(\frac{t-\frac{1}{2}T}{T}\right) - \mathrm{rect}\left(\frac{t+\frac{1}{2}T}{T}\right)\right)$$

(b)

By the time-scaling property $g(-t) \rightleftarrows G(-f)$

$$\begin{aligned}
G_e(f) &= \frac{1}{2}\left[G(f) + G(-f)\right] \\
&= \frac{1}{2}\left[\mathrm{sinc}(fT)\exp(-j2\pi fT) + \mathrm{sinc}(fT)\exp(j2\pi fT)\right] \\
&= \mathrm{sinc}(fT)\cos(\pi fT)
\end{aligned}$$

$$\begin{aligned}
G_o(f) &= \frac{1}{2}\left[G(f) - G(-f)\right] \\
&= \frac{1}{2}\left[\mathrm{sinc}(fT)\exp(-j2\pi fT) - \mathrm{sinc}(fT)\exp(j2\pi fT)\right] \\
&= -j\,\mathrm{sinc}(fT)\sin(\pi fT)
\end{aligned}$$

2.4. We need to find a function with the stated properties.

We can verify that:
$G(f) = -j\,\mathrm{sgn}(f) + j\mathrm{u}(f-W) - j\mathrm{u}(-f-W)$
meets the stated criteria.
By duality *g(f)* $\rightleftarrows$ *G(-t)*

$$g(t) = \frac{1}{\pi t} + j\left(\frac{1}{2}\delta(t) - \frac{1}{j2\pi t}\right)\exp(-j2\pi Wt) - j\left(\frac{1}{2}\delta(t) - \frac{1}{j2\pi t}\right)\exp(j2\pi Wt)$$
$$= \frac{1}{\pi t} + j\frac{\sin(2\pi Wt)}{2\pi t}$$

2.5
$$g(t) = \frac{1}{\tau}\int_{t-T}^{t+T}\exp\left(-\frac{\pi u^2}{\tau^2}\right)du$$
$$= \frac{1}{\tau}\int_{t-T}^{0} h(\tau)d\tau + \frac{1}{\tau}\int_{0}^{t+T} h(\tau)d\tau$$
$$\frac{dg(t)}{dt} = -\frac{1}{\tau}h(t-T) + \frac{1}{\tau}h(t+T)$$

By the differentiation property:

$$F\left(\frac{dg(t)}{dt}\right) = j2\pi fG(f)$$
$$= \frac{1}{\tau}\left[H(f)\exp(j2\pi f\tau) - H(f)\exp(-j2\pi f\tau)\right]$$
$$= \frac{2j}{\tau}H(f)\sin(2\pi f\tau)$$

But $H(f) = \tau\exp(-\pi f^2\tau^2)$

$$\therefore G(f) = \frac{1}{\pi f}\exp(-\pi f^2\tau^2)\sin(2\pi fT)$$
$$= \exp(-\pi f^2\tau^2)\frac{\sin(2\pi fT)}{\pi f}$$
$$= 2T\exp(-\pi f^2\tau^2)\mathrm{sinc}(2\pi fT)$$

$$\lim_{\tau\to 0} G(f) = 2T\mathrm{sinc}(2\pi fT)$$

2.6 (a)

If g(t) is even and real then $g(t)=\frac{1}{2}\left[g(t)+g(-t)\right]$

and $g(t)=g^{*}(t) \Rightarrow G(f)=G^{*}(-f)$

$$G^{*}(f)=\frac{1}{2}[G^{*}(f)+G^{*}(-f)]$$

$$\frac{1}{2}G^{*}(f)=\frac{1}{2}G^{*}(-f)$$

$$G^{*}(f)=G(f)$$

$\therefore G(f)$ is all real

If g(t) is odd and real then $g(t)=\frac{1}{2}\left[g(t)-g(-t)\right]$

and $g(t)=g^{*}(t) \Rightarrow G(f)=G^{*}(-f)$

$$G(f)=\frac{1}{2}[G(f)-G(-f)]$$

$$G^{*}(f)=\frac{1}{2}G^{*}(f)-\frac{1}{2}G^{*}(-f)$$

$$G^{*}(f)=-G^{*}(-f)$$

$$G^{*}(f)=-G(f)$$

$\therefore G(f)$ must be all imaginary

(b)

$$(-j2\pi t)G(t) \rightleftarrows \frac{d}{df}g(-f) \quad \text{by duality}$$

$$t \cdot G(t) \rightleftarrows \frac{j}{2\pi}\frac{d}{df}g(-f)$$

The previous step can be repeated *n* times so:

$$(-j2\pi ft)^{n}G(t) \rightleftarrows \frac{d^{n}}{df^{n}}g(-f)$$

But each factor $(-j2\pi ft)$ represents another differentiation.

$$t^{n} \cdot G(t) \rightleftarrows \left(\frac{j}{2\pi}\right)^{n} g^{(n)}(-f)$$

Replacing g with h

$$t^{n}h(t) \rightleftarrows \left(\frac{j}{2\pi}\right)^{n} H^{(n)}(f)$$

(c)

Let $h(t) = t^n g(t)$ and $H(f) = \left(\frac{j}{2\pi}\right)^n G^{(n)}(f)$

$$\int_{-\infty}^{\infty} h(t)dt = H(0) = \left(\frac{j}{2\pi}\right)^n G^{(n)}(0)$$

(d)

$$g_1(t) \rightleftarrows G_1(f)$$
$$g_2^*(t) \rightleftarrows G_2(-f)$$

$$g_1(t)g_2(t) \rightleftarrows \int_{-\infty}^{\infty} G_1(\lambda)G_2(f-\lambda)d\lambda$$

$$g_1(t)g_2^*(t) \rightleftarrows \int_{-\infty}^{\infty} G_1(\lambda)G_2(-(f-\lambda))d\lambda$$
$$= \int_{-\infty}^{\infty} G_1(\lambda)G_2(\lambda - f)d\lambda$$

(e)

$$g_1(t)g_2^*(t) \rightleftarrows \int_{-\infty}^{\infty} G_1(\lambda)G_2(\lambda - f)d\lambda$$

$$\int_{-\infty}^{\infty} g_1(t)g_2^*(t)dt \rightleftarrows G(0)$$

$$\int_{-\infty}^{\infty} g_1(t)g_2^*(t)dt \rightleftarrows \int_{-\infty}^{\infty} G_1(\lambda)G_2(\lambda - 0)d\lambda$$

$$\int_{-\infty}^{\infty} g_1(t)g_2^*(t)dt \rightleftarrows \int_{-\infty}^{\infty} G_1(\lambda)G_2(\lambda)d\lambda$$

2.7 (a)

$$g(t) \rightleftharpoons AT\,\mathrm{sinc}^2(fT)$$

$$\int_{-\infty}^{\infty} |g(t)|\,dt = AT$$

$$\begin{aligned}\max G(f) &= G(0)\\ &= AT\,\mathrm{sinc}^2(0)\\ &= AT\end{aligned}$$

$\therefore$ The first bound holds true.

(b)

$$\int_{-\infty}^{\infty} \left|\frac{dg(t)}{dt}\right| dt = 2A$$

$$\begin{aligned}|j2\pi fG(f)| &= |2\pi fAT\,\mathrm{sinc}^2(fT)|\\ &= \left|2\pi fAT\frac{\sin(\pi fT)}{\pi fT}\cdot\frac{\sin(\pi fT)}{\pi fT}\right|\\ &= \left|2A\frac{\sin(\pi fT)}{\pi fT}\cdot\sin(\pi fT)\right|\end{aligned}$$

But, $|\sin(\pi fT)| \le 1\ \forall f$ and $|\mathrm{sinc}(\pi fT)| \le 1\ \forall f$

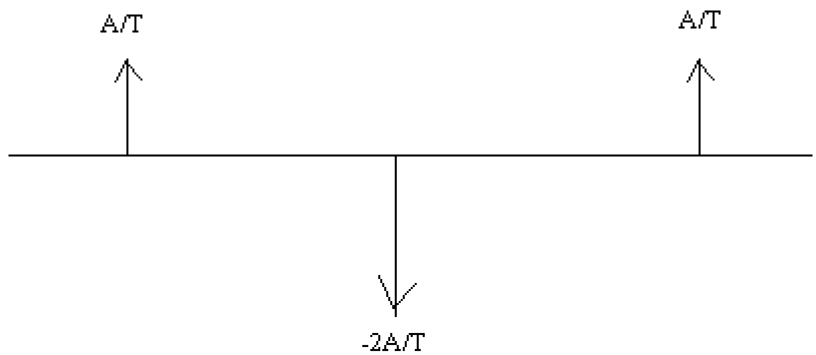
$$\therefore \left|2A\frac{\sin(\pi fT)}{\pi fT}\cdot\sin(\pi fT)\right| \le 2A$$

$$\therefore |j2\pi fG(f)| \le 2A$$

2.7 c)

$$\left|(j2\pi f)^2 G(f)\right| = \left|4\pi^2 f^2 G(f)\right|$$
$$= \left|4\pi^2 f^2 AT \frac{\sin^2(\pi fT)}{(\pi fT)^2}\right|$$
$$= \left|\frac{4A}{T}\sin^2(\pi fT)\right|$$
$$\leq \frac{4A}{T}$$

The second derivative of the triangular pulse is plotted as:



Integrating the absolute value of the delta functions gives:

$$\int_{-\infty}^{\infty}\left|\frac{d^2 g(t)}{dt^2}\right| dt = \frac{4A}{T}$$

$$\therefore \left|(j2\pi f)^2 G(f)\right| \leq \int_{-\infty}^{\infty}\left|\frac{d^2 g(t)}{dt^2}\right| dt$$

2.8. (a)

$$g_1(t) * g_2(t) \rightleftarrows G_1(f)G_2(f)$$
$$= G_2(f)G_1(f) \text{ by the commutative property of multiplication}$$

b)

$$g_1(f) * \left[g_2(f) * g_3(f)\right] \rightleftarrows G_1(f)\left[G_2(f)G_3(f)\right]$$

Because multiplication is commutative, the order of the multiplication doesn't matter.

$$\therefore G_1(f)\left[G_2(f)G_3(f)\right] = \left[G_1(f)G_2(f)\right]G_3(f)$$
$$\therefore G_1(f)\left[G_2(f)G_3(f)\right] \rightleftarrows \left[g_1(f) * g_2(f)\right] * g_3(f)$$

c)

Taking the Fourier transform gives:

$$G_1(f)\left[G_2(f) + G_3(f)\right]$$

Multiplication is distributive so:

$$G_1(f)G_2(f) + G_2(f)G_3(f) \rightleftarrows g_1(t)g_2(t) + g_1(t)g_2(t)$$

2.9 a)

Let $h(t) = g_1(t) * g_2(t)$

$$\frac{dh(t)}{dt} \rightleftarrows j2\pi f H(f)$$

$$= j2\pi f G_1(f) G_2(f)$$

$$= \left(j2\pi f G_1(f) \right) G_2(f)$$

$$\left(j2\pi f G_1(f) \right) G_2(f) \rightleftarrows \left[\frac{dg_1(t)}{dt} \right] * g_2(t)$$

$$\therefore \frac{d}{dt}\left[g_1(t) * g_2(t) \right] = \left[\frac{dg_1(t)}{dt} \right] * g_2(t)$$

b)

$$\int_{-\infty}^{t} g_1(t) * g_2(t) dt \rightleftarrows \frac{1}{j2\pi f} G_1(f) G_2(f) + \frac{G_1(0) G_2(0)}{2} \delta(f)$$

$$= \left[\frac{1}{j2\pi f} G_1(f) \right] G_2(f) + \left[\frac{G_1(0)}{2} \delta(f) \right] G_2(f)$$

$$= \left[\frac{1}{j2\pi f} G_1(f) + \frac{G_1(0)}{2} \delta(f) \right] G_2(f)$$

$$\therefore \int_{-\infty}^{t} g_1(t) * g_2(t) dt = \left[\int_{-\infty}^{t} g_1(t) \right] * g_2(t)$$

2.10. $Y(f) = \int_{-\infty}^{t} X(\nu) X(f-\nu) d\nu$

$|X(\nu)| \neq 0$ if $|\nu| \leq W$

$|X(f-\nu)| \neq 0$ if $|f-\nu| \leq W$

$(f-\nu) \leq W$ for $f \leq W + \nu$ when $\nu \geq 0$ and $\nu \leq W$

$(f-\nu) \geq -W$ for $f \leq -W + \nu$ when $\nu \leq 0$ and $\nu \geq -W$

$\therefore (f-\nu) \leq W$ for $0 \leq \nu \leq W$ when $f \leq 2W$

$(f-\nu) \geq -W$ for $-W \leq \nu \leq 0$ when $f \geq -2W$

$\therefore$ Over the range of integration $[-W, W]$, the integral is non-zero if $|f| \leq 2W$

2.11 a) Given a rectangular function: $g(t) = \frac{1}{T}\text{rect}\left(\frac{t}{T}\right)$, for which the area under *g(t)* is always equal to 1, and the height is 1/T.

$$\frac{1}{T}\text{rect}\left(\frac{t}{T}\right) \rightleftarrows \text{sinc}(fT)$$

Taking the limits:

$$\lim_{T\to 0}\frac{1}{T}\text{rect}\left(\frac{t}{T}\right) = \delta(t)$$

$$\lim_{T\to 0}\frac{1}{T}\text{sinc}(fT) = 1$$

b)

$$g(t) = 2W\text{sinc}(2Wt)$$

$$2W\text{sinc}(2Wt) \rightleftarrows \text{rect}\left(\frac{f}{2W}\right)$$

$$\lim_{W\to\infty} 2W\text{sinc}(2Wt) = \delta(t)$$

$$\lim_{W\to\infty} \text{rect}\left(\frac{2}{2W}\right) = 1$$

2.12.

$$G(f) = \frac{1}{2} + \frac{1}{2}\text{sgn}(f)$$

By duality:

$$G(f) \rightleftarrows \frac{1}{2}\delta(-t) - \frac{1}{j2\pi t}$$

$$\therefore g(t) = \frac{1}{2}\delta(t) + \frac{j}{2\pi t}$$

2.13. a) By the differentiation property:

$$\left(j2\pi f\right)^2 G(f) = \sum_i k_i \exp(-j2\pi f t_i)$$

$$\therefore G(f) = -\frac{1}{4\pi^2 f^2}\sum_i k_i \exp(-j2\pi f t_i)$$

b)the slope of each non-flat segment is: $\pm\dfrac{A}{t_b - t_a}$

$$G(f) = -\left(\frac{1}{4\pi^2 f^2}\right)\left(\frac{A}{t_b - t_a}\right)\left[\exp(j2\pi f t_b) - \exp(j2\pi f t_a) - \exp(j2\pi f t_a) + \exp(j2\pi f t_b)\right]$$

$$= -\frac{A}{2\pi^2 f^2\left(t_b - t_a\right)}\left[\cos(2\pi f t_b) - \cos(2\pi f t_a)\right]$$

But: $\sin(\pi f(t_b - t_a))\sin(\pi f(t_b + t_a)) = \dfrac{1}{2}\left[\cos(2\pi f t_a) - \cos(2\pi f t_b)\right]$ by a trig identity.

$$\therefore G(f) = \frac{A}{\pi^2 f^2 (t_b - t_a)}\left[\sin(\pi f(t_b - t_a))\sin(\pi f(t_b + t_a))\right]$$

2.14 a) let *g(t)* be the half cosine pulse of Fig. P2.1a, and let *g(t-t*$_0$*)* be its time-shifted counterpart in Fig.2.1b

$$\varepsilon = G(f)G^*(f)$$
$$= \left\|G(f)\right\|^2$$

$$\left(G(f)\exp(-j2\pi f t_0)\right)\left(G^*(f)\exp(j2\pi f t_0)\right) = \left\|G(f)\right\|^2 \exp(-j2\pi f t_0)\exp(j2\pi f t_0)$$

$$\left(G(f)\exp(-j2\pi f t_0)\right)\left(G^*(f)\exp(j2\pi f t_0)\right) = \left\|G(f)\right\|^2$$

2.14 b)Given that the two energy densities are equal, we only need to prove the result for one. From before, it was shown that the Fourier transform of the half-cosine pulse was:

$$\frac{AT}{2}\left[\operatorname{sinc}((f+f_c)T)+\operatorname{sinc}((f-f_c)T)\right] \quad \text{for } f_c=\frac{1}{2T}$$

After squaring, this becomes:

$$\frac{A^2T^2}{4}\left[\frac{\sin^2(\pi(f+f_c)T)}{(\pi(f+f_c)T)^2}+\frac{\sin^2(\pi(f-f_c)T)}{(\pi(f-f_c)T)^2}+2\frac{\sin(\pi(f+f_c)T)\sin(\pi(f-f_c)T)}{\pi^2T^2(f+f_c)(f-f_c)}\right]$$

The first term reduces to:

$$\frac{\sin^2\left(\pi fT+\frac{\pi}{2}\right)}{\left(\pi fT+\frac{\pi}{2}\right)^2}=\frac{\cos^2(\pi fT)}{\left(\pi fT+\frac{\pi}{2}\right)^2}=\frac{\cos^2(\pi fT)}{\pi^2T^2(f+f_c)^2}$$

The second term reduces to:

$$\frac{\sin^2\left(\pi fT-\frac{\pi}{2}\right)}{\left(\pi fT-\frac{\pi}{2}\right)^2}=\frac{\cos^2(\pi fT)}{\pi^2T^2(f-f_c)^2}$$

The third term reduces to:

$$\begin{aligned}2\frac{\sin(\pi(f+f_c)T)\sin(\pi(f-f_c)T)}{\pi^2T^2(f+f_c)(f-f_c)}&=\frac{\cos(\pi)-\cos^2(2\pi fT)}{\pi^2T^2\left(f^2-\frac{1}{4T^2}\right)}\\&=\frac{-1-\cos(2\pi fT)}{\pi^2T^2\left(f^2-\frac{1}{4T^2}\right)}\\&=-\frac{2\cos^2(\pi fT)}{\pi^2T^2\left(f^2-\frac{1}{4T^2}\right)}\end{aligned}$$

Summing these terms gives:

$$\frac{A^2T^2}{4\pi^2T^2}\left[\frac{\cos^2(\pi fT)}{\left(f+\frac{1}{2T}\right)^2}+\frac{\cos^2(\pi fT)}{\left(f-\frac{1}{2T}\right)^2}-2\frac{\cos^2(\pi fT)}{\left(f+\frac{1}{2T}\right)\left(f-\frac{1}{2T}\right)}\right]$$

2.14 b)Cont'd

By rearranging the previous expression, and summing over a common denominator, we get:

$$\frac{A^2T^2}{4\pi^2T^2}\left[\frac{\cos^2(\pi fT)}{\left(f^2-\frac{1}{4T^2}\right)^2}\right]$$

$$=\frac{A^2T^2}{4\pi^2T^4}\left[\frac{\cos^2(\pi fT)}{\frac{1}{16}\frac{1}{T^4}\left(4T^2f^2-1\right)^2}\right]$$

$$=\frac{A^2T^2}{\pi^2}\left[\frac{\cos^2(\pi fT)}{\left(4T^2f^2-1\right)^2}\right]$$

2.15 a)The Fourier transform of $\frac{dg(t)}{dt} \rightleftarrows j2\pi fG(f)$

Let $g'(t) = \frac{dg(t)}{dt}$

By Rayleigh's theorem: $\int_{-\infty}^{\infty} |g(t)|^2\, dt = \int_{-\infty}^{\infty} |G(f)|^2\, df$

$$\therefore W^2T^2 = \frac{\int t^2 |g(t)|^2\, dt \cdot \int f^2 |G(f)|^2\, df}{\left(\int |g(t)|^2\, dt\right)^2}$$

$$= \frac{\int t^2 |g(t)|^2\, dt \cdot \int g'(t) g'^*(t) dt}{4\pi^2 \left(\int |g(t)|^2\, dt\right)^2}$$

$$\geq \frac{\left[\int t^2 g^*(t) g'(t) - tg(t) g'^*(t) dt\right]^2}{16\pi^2 \left(\int |g(t)|^2\, dt\right)^2}$$

$$= \frac{\left[\int t \cdot \frac{d}{dt}\left(g(t) g^*(t)\right) dt\right]^2}{16\pi^2 \left(\int g(t) g^*(t) dt\right)^2}$$

Using integration by parts, we can show that:

$$\int_{-\infty}^{\infty} t \cdot \frac{d}{dt} |g(t)|^2\, dt = \int_{-\infty}^{\infty} |g(t)|^2$$

$$\therefore W^2T^2 \geq \frac{1}{16\pi^2}$$

$$\therefore WT \geq \frac{1}{4\pi}$$

2.15 b) For $g(t) = \exp(-\pi t^2)$

$g(t) \rightleftarrows \exp(-\pi f^2)$

$$\therefore W^2T^2 = \frac{\int_{-\infty}^{\infty} t^2 \exp(-2\pi t^2)dt \cdot \int_{-\infty}^{\infty} f^2 \exp(-2\pi f^2)df}{\int_{-\infty}^{\infty} \exp(-2\pi t^2)dt}$$

Using a table of integrals: $\int_0^{\infty} x^2 \exp(-ax^2)dx = \frac{1}{4a}\sqrt{\frac{\pi}{a}}$ for $a > 0$

$$\therefore \int_{-\infty}^{\infty} t^2 \exp(-2\pi t^2)dt = \frac{1}{4\pi}\sqrt{\frac{1}{2}}$$

$$\int_{-\infty}^{\infty} f^2 \exp(-2\pi t^2)df = \frac{1}{4\pi}\sqrt{\frac{1}{2}}$$

$$\int_{-\infty}^{\infty} \exp(-2\pi t^2) = \sqrt{\frac{1}{2}}$$

$$\therefore T^2W^2 = \frac{\left(\frac{1}{4\pi}\sqrt{\frac{1}{2}}\right)^2}{\frac{1}{2}}$$

$$= \left(\frac{1}{4\pi}\right)^2$$

$$\therefore TW = \frac{1}{4\pi}$$

2.16.

Given: $\int_{-\infty}^{\infty} |x(t)|^2 \, dt < \infty$ and $\int_{-\infty}^{\infty} |h(t)| \, dt < \infty$, which implies that $\int_{-\infty}^{\infty} |h(t)| dt < \infty$.

However, if $\int_{-\infty}^{\infty} |x(t)|^2 \, dt < \infty$ then $\int_{-\infty}^{\infty} |X(f)|^2 \, df < \infty$ and $\int_{-\infty}^{\infty} |X(f)|^4 \, df < \infty$. This result also applies to *h(t)*.

$$Y(f) = H(f)X(f)$$

$$\int_{-\infty}^{\infty} |Y(f)|^2 \, df = \int_{-\infty}^{\infty} X(f)H(f) \cdot X^*(f)H^*(f)df$$

$$= \int_{-\infty}^{\infty} |X(f)|^2 |H(f)|^2 \, df$$

$$\left| \int_{-\infty}^{\infty} |Y(f)|^2 \, df \right|^2 \leq \int_{-\infty}^{\infty} |X(f)|^4 \, df \int_{-\infty}^{\infty} |H(f)|^4 \, df$$

$$< \infty$$

$$\therefore \int_{-\infty}^{\infty} |Y(f)|^2 \, df < \infty$$

By Rayleigh's theorem: $\int_{-\infty}^{\infty} |Y(f)|^2 \, df = \int_{-\infty}^{\infty} |y(t)|^2 \, dt$

$$\therefore \int_{-\infty}^{\infty} |y(t)|^2 \, dt < \infty$$

2.17.

The transfer function of the summing block is: $H_1(f) = \left[1 - \exp(-j2\pi fT)\right]$.

The transfer function of the integrator is: $H_2(f) = \dfrac{1}{j2\pi f}$

These elements are cascaded :

$$\begin{aligned}H(f) &= \left(H_1(f)H_2(f)\right)\cdot\left(H_1(f)H_2(f)\right)\\ &= -\frac{1}{\left(2\pi f\right)^2}\left[1-\exp(-j2\pi fT)\right]^2\\ &= -\frac{1}{\left(2\pi f\right)^2}\left[1-2\exp(-j2\pi fT)+\exp(-j4\pi fT)\right]\end{aligned}$$

2.18.a) Using the Laplace transform representation of a single stage, the transfer function is:

$$H_0(s) = \frac{1}{1+RCs}$$
$$= \frac{1}{1+\tau_0 s}$$
$$H_0(f) = \frac{1}{1+j2\pi f\tau_0}$$

These units are cascaded, so the transfer function for *N* stages is:

$$H(f) = \left(H(f)\right)^N = \left(\frac{1}{1+j2\pi f\tau_0}\right)^N$$

b) For $N\to\infty$, and $\tau_0^2 = \dfrac{T^2}{4\pi^2 N}$

$$\ln H(f) = N\ln\left(\frac{1}{1+j2\pi f\tau_0}\right)$$
$$= -N\ln\left(1+j2\pi f\tau_0\right)$$
$$= -N\ln\left(1+\frac{jfT}{\sqrt{N}}\right)$$

let $z = \dfrac{jfT}{\sqrt{N}}$, then for very large N, $|z|<1$

$\therefore$ We can use the Taylor series expansion of $\ln(1+z)$

$$-N\ln(1+z) = -N\left[\sum_{m=1}^{\infty}\frac{1}{m}(-1)^{m+1}z^m\right]$$
$$= -N\left[\sum_{m=1}^{\infty}\frac{1}{m}(-1)^{m+1}\left(j\frac{fT}{\sqrt{N}}\right)^m\right]$$

(next page)

2.18 (b) Cont'd

Taking the limit as $N \rightarrow \infty$:

$$\lim_{N\to\infty}\left(-N\left[\sum_{m=1}^{\infty}\frac{1}{m}(-1)^{m+1}\left(j\frac{fT}{\sqrt{N}}\right)^{m}\right]\right)=-N\left(j\frac{fT}{\sqrt{N}}+\frac{f^2T^2}{2N}\right)$$

$$=-\frac{1}{2}f^2T^2-j\sqrt{N}\,fT$$

$$\therefore H(f)=\exp(-\frac{1}{2}f^2T^2)\exp(-j\sqrt{N}\,ft)$$

$$\therefore |H(f)|=\exp(-\frac{1}{2}f^2T^2)$$

2.19.a) $y(t) = \int_{t-T}^{T} x(\tau) d\tau$

This is the convolution of a rectangular function with *x(τ)*. The interval of the rectangular function is [(t-T),T], and the midpoint is T/2.

$rect\left(\frac{t}{T}\right) \rightleftarrows T\text{sinc}(fT)$, but the function is shifted by $\frac{\mathrm{T}}{2}$.

$\therefore H(f) = T\text{sinc}(fT)\exp(-j\pi fT)$

b)$BW = \frac{1}{RC} = \frac{1}{T}$

$$H(f) = \frac{T}{1 + j2RC\pi f}\exp(-j2\pi f\frac{T}{2})$$

$$= \frac{T}{RC}\left(\frac{1}{\frac{1}{RC} + j2\pi f}\right)\exp(-j\pi fT)$$

$$\therefore h(t) = \frac{T}{RC}\exp\left(-\frac{1}{RC}(t - \frac{T}{2})\right)u(t - \frac{T}{2})$$

$$= \exp\left(-\frac{1}{T}(t - \frac{T}{2})\right)u(t - \frac{T}{2})$$

2.20. a) For the sake of convenience, let h(t) be the filter time-shifted so that it is symmetric about the origin (t = 0).

$$H(f) = \sum_{k=1}^{\frac{N-1}{2}} w_k \exp(-j2\pi fk) + \sum_{k=-1}^{-\frac{N-1}{2}} w_k \exp(-j2\pi fk) + w_0$$
$$= 2\sum_{k=1}^{\frac{N-1}{2}} w_k \cos(2\pi fk)$$

Let *G(f)* be the filter returned to its correct position. Then

$G(f) = H(f)\exp(-j2\pi f \left(\frac{N-1}{2}\right))$, which is a time-shift of $\left(\frac{N-1}{2}\right)$ samples.

$$\therefore G(f) = \exp\left(-j\pi f\left(N-1\right)\right) 2\sum_{k=1}^{\frac{N-1}{2}} w_k \cos(2\pi fk)$$

b)By inspection, it is apparent that:

$\measuredangle G(f) = \measuredangle \exp(-j\pi f(N-1))$

This meets the definition of linear phase.

2.21 Given an ideal bandpass filter of the type shown in Fig P2.7, we need to find the response of the filter for $x(t) = A\cos(2\pi f_0 t)$

$$|H(f)| = \frac{1}{2B}\mathrm{rect}\left(\frac{f - f_c}{2B}\right) + \frac{1}{2B}\mathrm{rect}\left(\frac{f + f_c}{2B}\right)$$

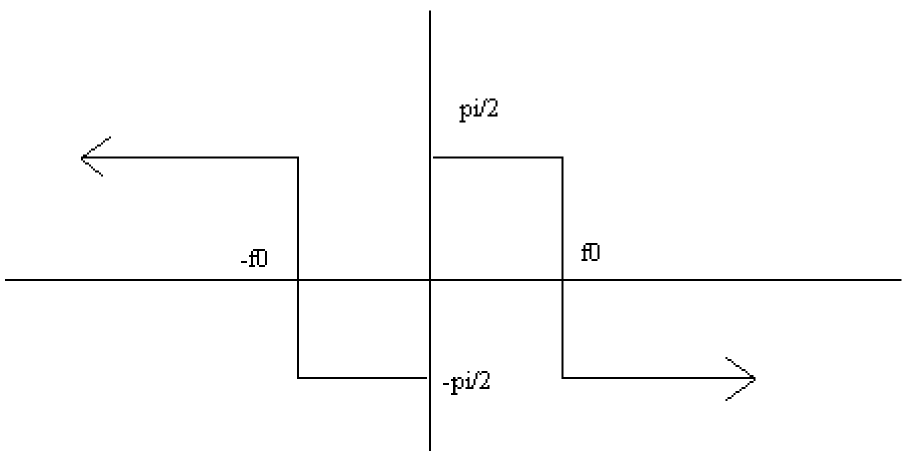
$$|X(f)| = \frac{1}{2}[\delta(f - f_0) + \delta(f - f_0)]$$

If $|f_c - f_0|$ is large compared to 2B, then the response is zero in the steady state.
However:

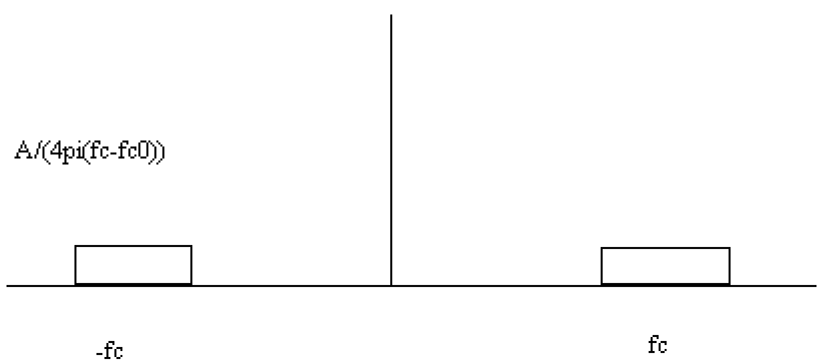
$$x(t)u(t) \rightleftarrows \left(\frac{A}{j2\pi(f - f_0)} + \frac{A}{2}\delta(f - f_0) + \frac{A}{j2\pi(f + f_0)} + \frac{A}{2}\delta(f + f_0)\right)$$

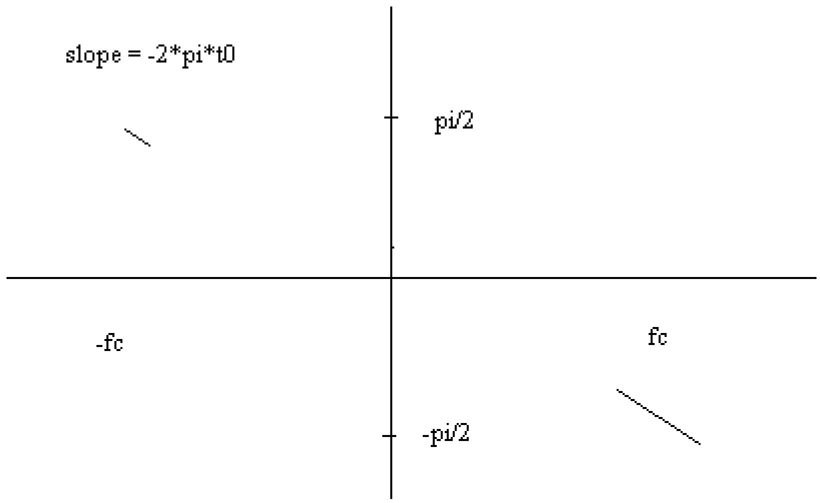
Since $|f_c - f_0|$ is large, assume that the portion of the amplitude spectrum lying inside the passband is approximately uniform with a magnitude of $\dfrac{A}{4\pi(f_c - f_0)}$.

The phase spectum of the input is plotted as:



The approximate magnitude and phase spectra of the output:





Taking the envelope by retaining the positive frequency components, shifting them to the origin, and scaling by 2:

$$\tilde{Y}(f) \simeq \begin{cases} \dfrac{A\exp\left(-j\left(\dfrac{\pi}{2}\right)-j2\pi f t_0\right)}{2\pi(f_c - f_0)} & \text{if } -B < f < B \\ 0 & \text{otherwise} \end{cases}$$

$$\tilde{y}(t) = \frac{AB}{j\pi(f_c - f_0)}\operatorname{sinc}\left[2B(t-t_0)\right]$$

$$\therefore\ y(t) \simeq \frac{AB}{\pi(f_c - f_0)}\operatorname{sinc}\left[2B(t-t_0)\right]\sin(2\pi f_c t)$$

2.22

$$H(f) = X(-f)\exp(j2\pi fT)$$

$$X(f) = \frac{A}{2}\left[\delta(f - f_c) + \delta(f + f_c)\right] * T\operatorname{sinc}(fT)\exp(-j2\pi f\frac{T}{2})$$

$$= \frac{AT}{2}\left[\operatorname{sinc}(T(f - f_c)) + \operatorname{sinc}(T(f + f_c))\right]\exp(-j\pi fT)$$

Let $f_c = \dfrac{N}{T}$ for N large

$$Y(f) = H(f)X(f)$$

$$= X(-f)\exp(j2\pi fT)\exp(-j\pi fT)\frac{AT}{2}\left[\operatorname{sinc}\left(T(f - f_c)\right) + \operatorname{sinc}\left(T(f + f_c)\right)\right]$$

$$= \exp(j2\pi fT)\frac{A^2T^2}{4}\left[\operatorname{sinc}\left(T(f - f_c)\right) + \operatorname{sinc}\left(T(f + f_c)\right)\right]\left[\operatorname{sinc}\left(T(-f - f_c)\right) + \operatorname{sinc}\left(T(-f + f_c)\right)\right]$$

$$= \exp(j2\pi fT)\frac{A^2T^2}{4}\left[\operatorname{sinc}(-fT - N) + \operatorname{sinc}(-fT + N)\right]\left[\operatorname{sinc}(fT - N) + \operatorname{sinc}(fT + N)\right]$$

But sinc(x)=sinc(-x)

$$\therefore Y(f) = \exp(j2\pi fT)\frac{A^2T^2}{2}\left[\operatorname{sinc}(fT - N) + \operatorname{sinc}(fT + N)\right]$$

2.23 *G(k)=G*

$$g_n = \frac{1}{N}\sum_{k=0}^{N-1} G(k)\exp(j\frac{2\pi}{N}k\cdot n)$$
$$= \frac{G}{N}\sum_{k=0}^{N-1} \exp(j\frac{2\pi}{N}k\cdot n)$$
$$= \frac{G}{N}\sum_{k=0}^{N-1} \cos(j\frac{2\pi}{N}k\cdot n) + j\sin(j\frac{2\pi}{N}k\cdot n)$$

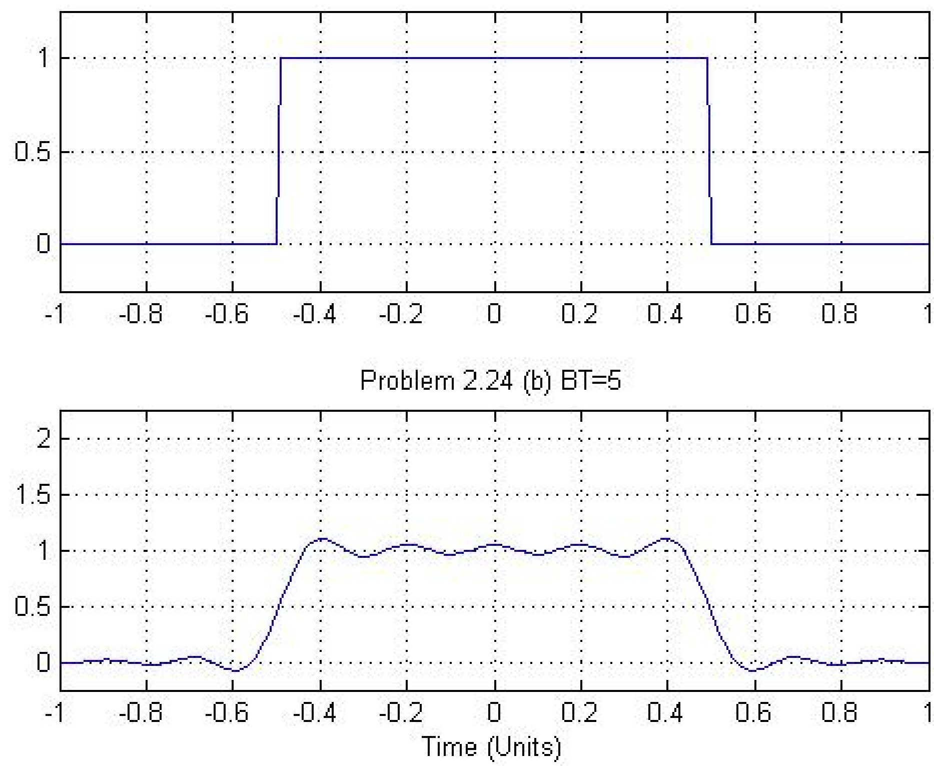
If $n = 0$, $g(n) = \frac{G}{N}\sum_{k=0}^{N-1} 1 = G$

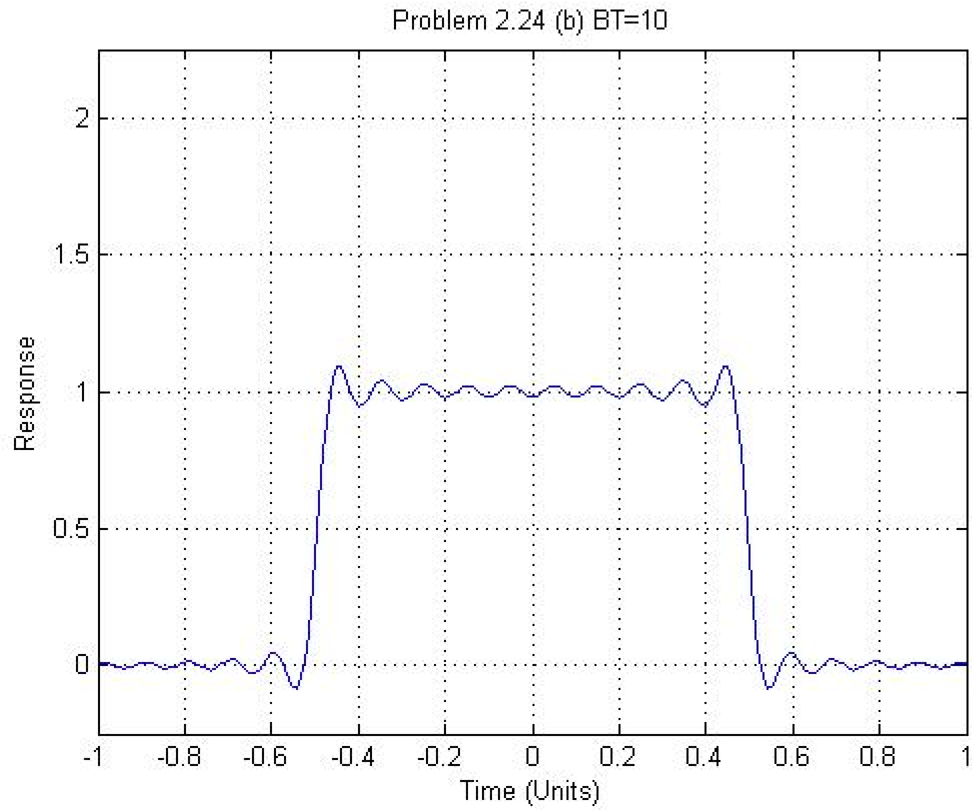
For $n \neq 0$, we are averaging over one full wavelength of a sine or cosine, with regularly sampled points. These sums must always be zero.

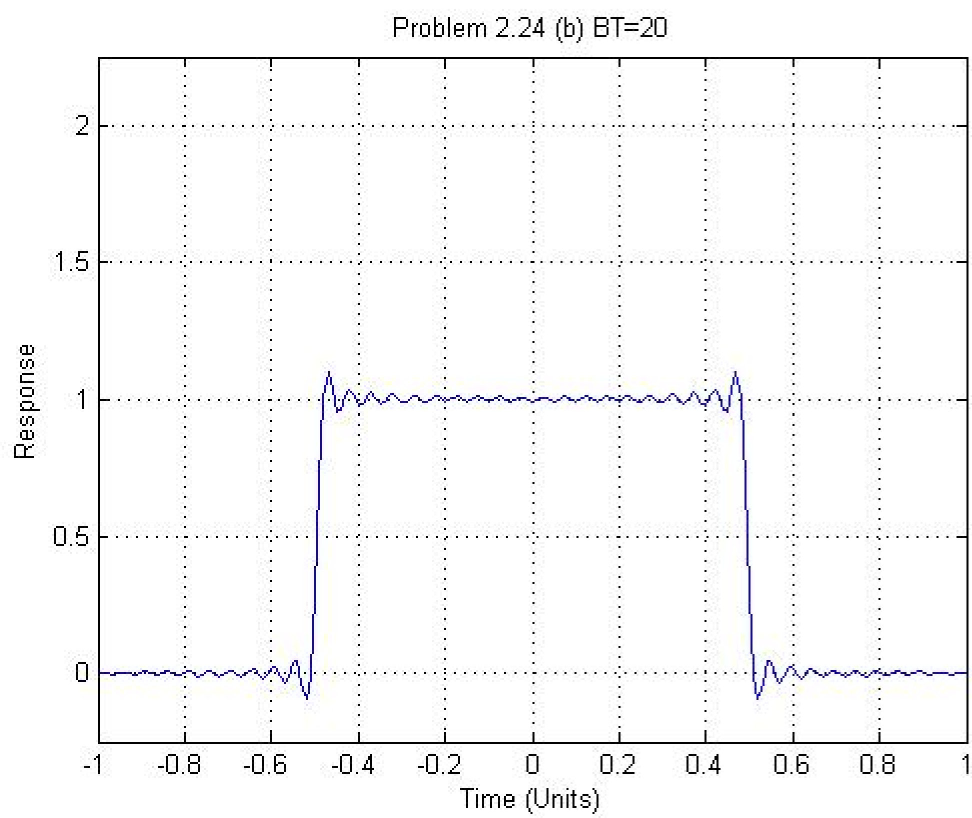
2.24. a) By the duality and frequency-shifting properties, the impulse response of an ideal low-pass filter is a phase-shifted sinc pulse. The resulting filter is non-causal and therefore not realizable in practice.

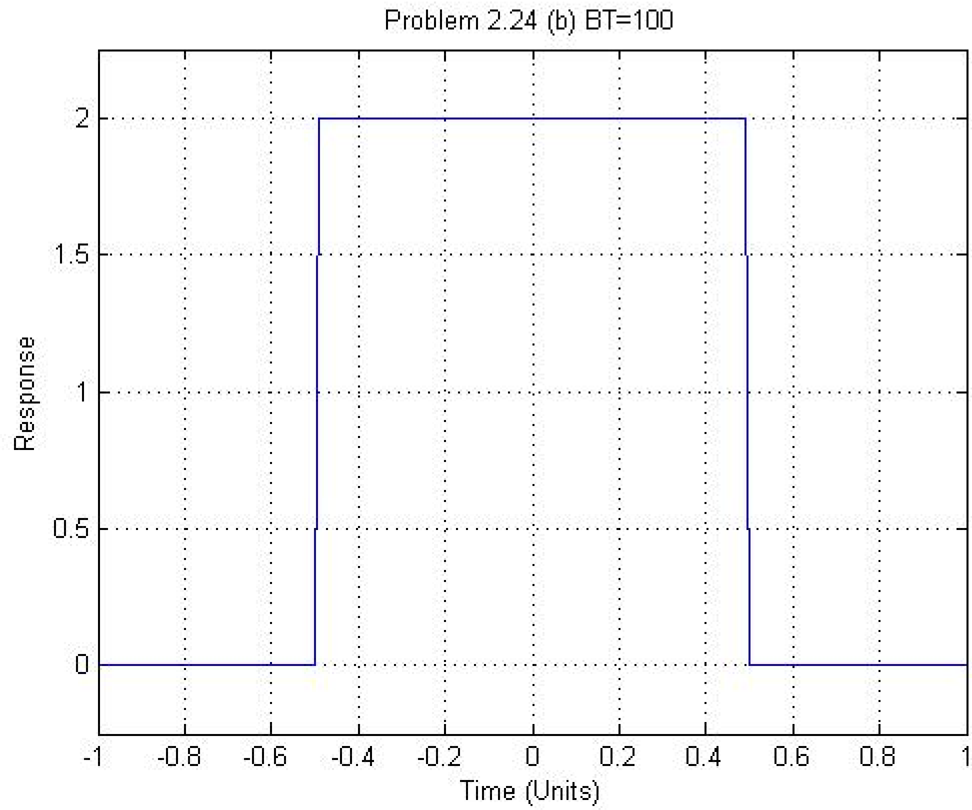
c)Refer to the appropriate graphs for a pictorial representation.
i)$\Delta t = T/100$

BT	**Overshoot (%)**	**Ripple Period**
5	9,98	1/5
10	9.13	1/10
20	9.71	1/20
100	100	No visible ripple




Problem 2.24 (b) BT=10
Response
2
1.5
1
0.5
0
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
Time (Units)


Problem 2.24 (b) BT=20
Response
2
1.5
1
0.5
0
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
Time (Units)



2.24 (d)

Δ*t*	**Overshoot (%)**	**Ripple Period**
T/100	100	No visible ripple.
T/150	16.54	1/100
T/200	~0	No visible ripple.

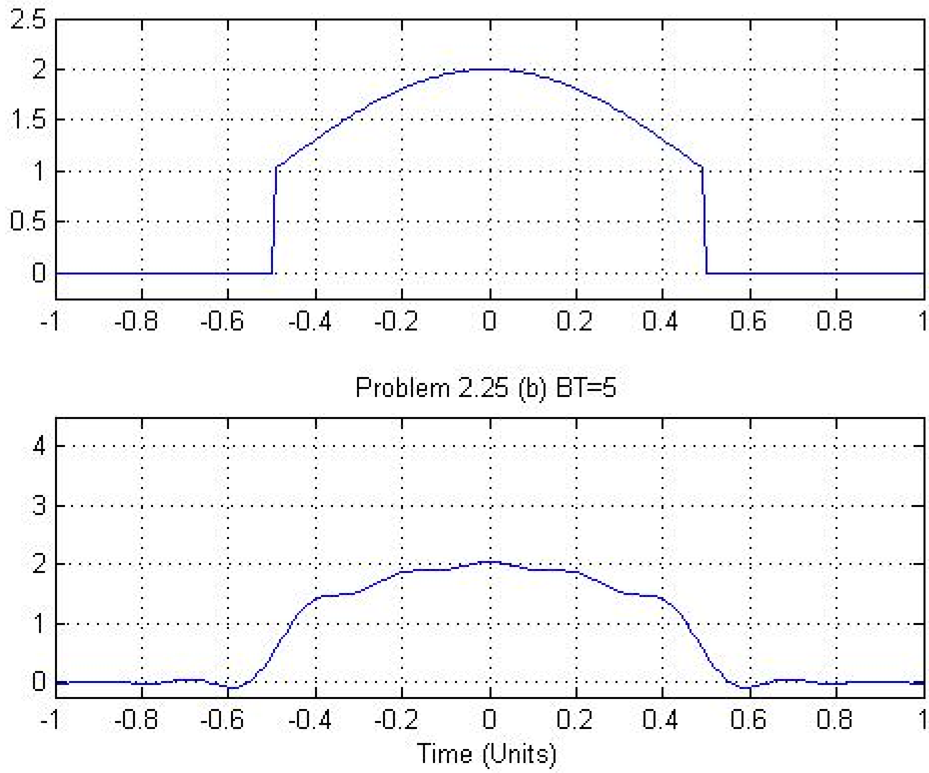
Discussion

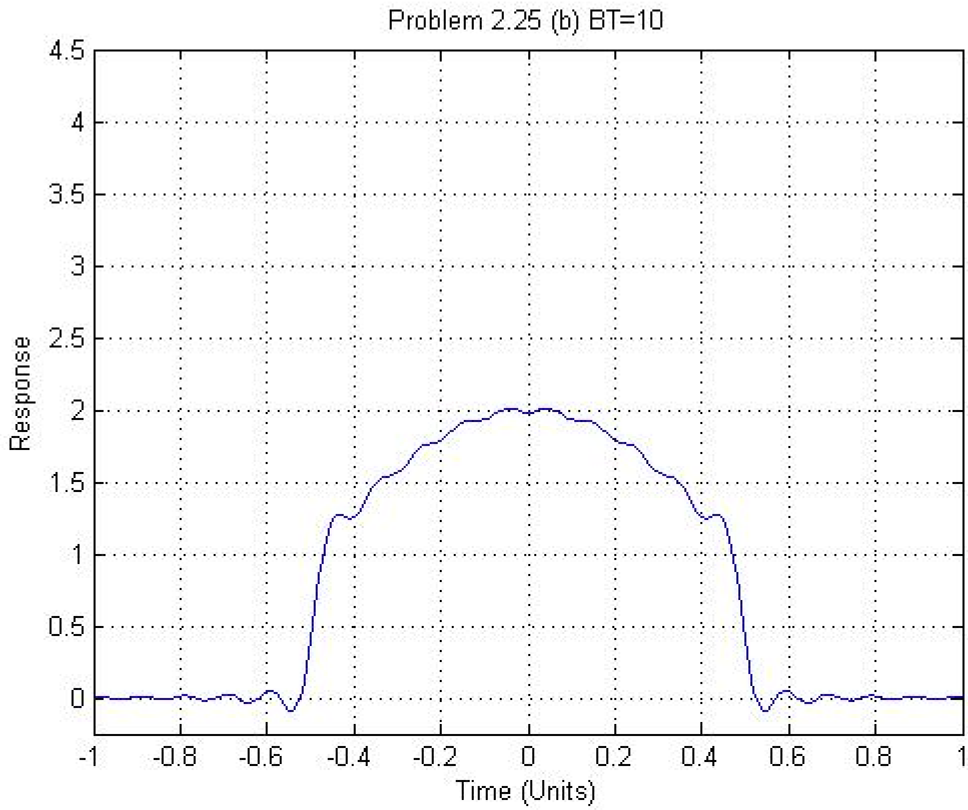
Increasing B, which also increases the filter's bandwidth, allows for more of the high-frequency components to be accounted for. These high-frequency components are responsible for producing the sharper edges. However, this accuracy also depends on the sampling rate being high enough to include the higher frequencies.

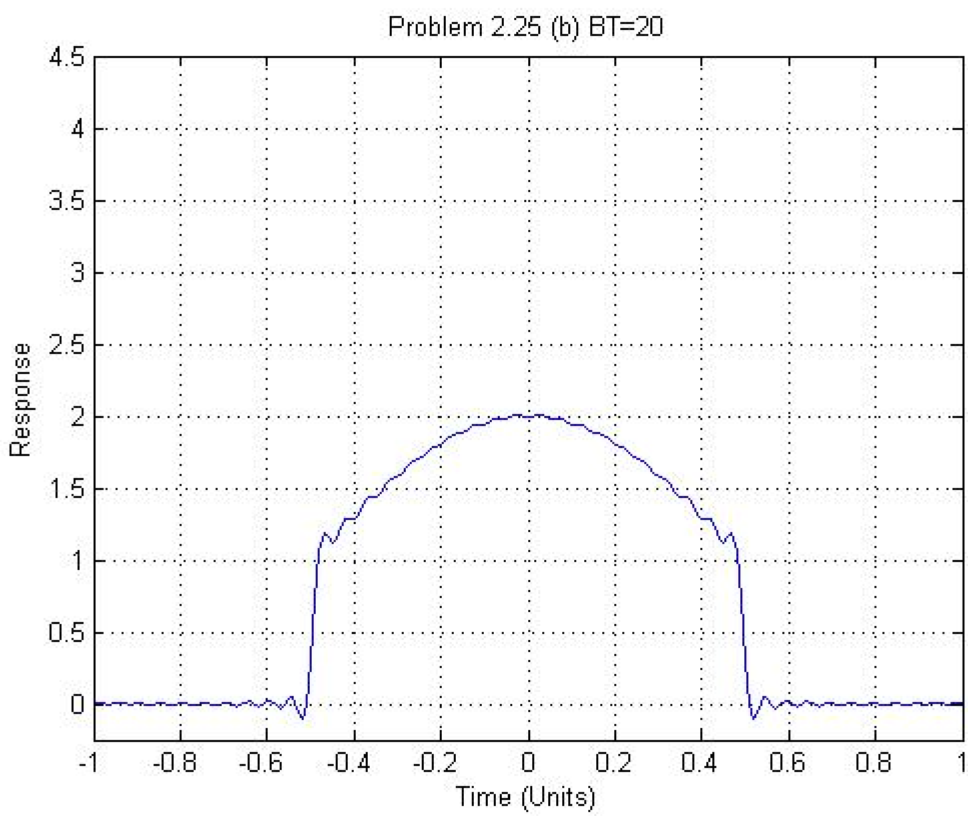
2.25

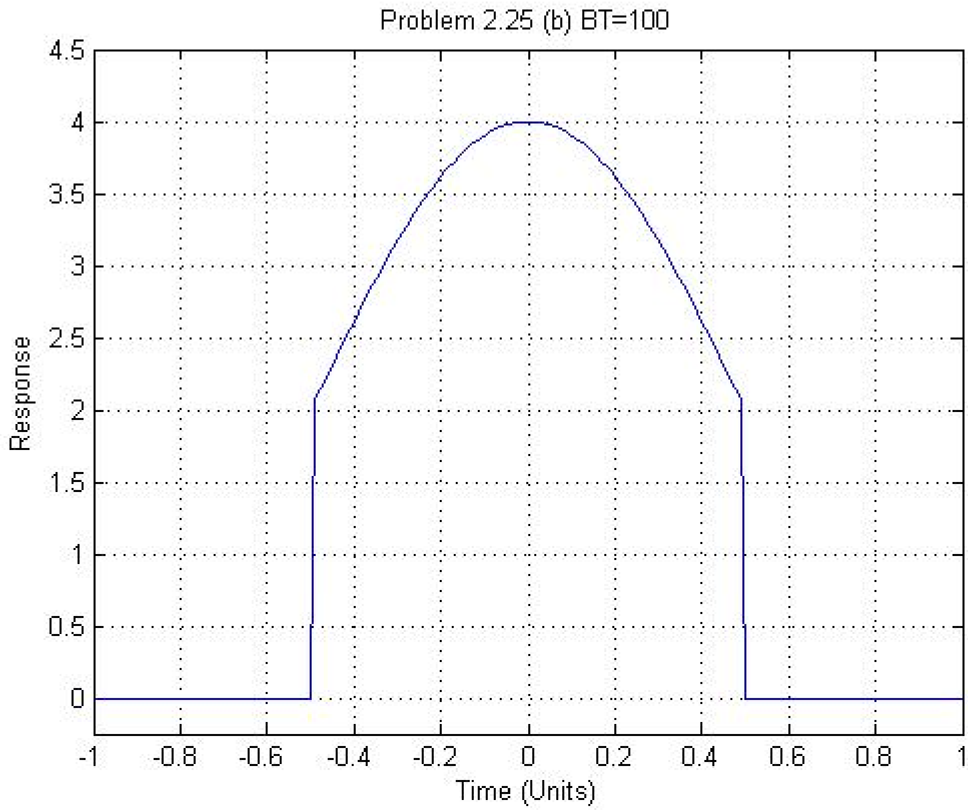
BT	**Overshoot (%)**	**Ripple Period**
5	8.73	1/5
10	8.8	1/10
20	9.8	1/20
100	100	-

The overshoot figures better for the raised cosine pulse that for the square pulse. This is likely because a somewhat greater percentage of the pulse's energy is concentrated at lower frequencies, and so a greater percentage is within the bandwidth of the filter.

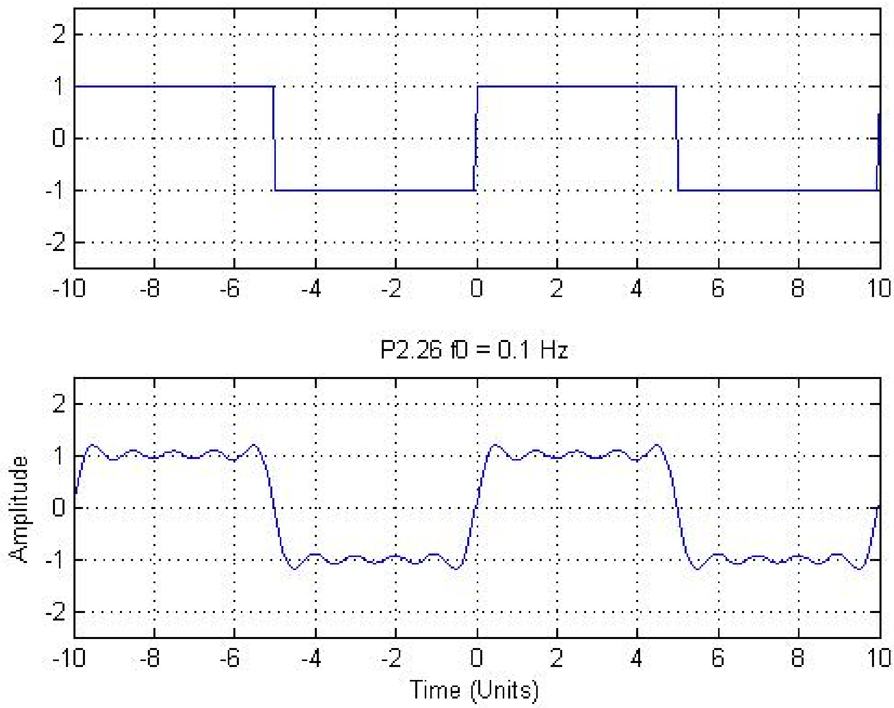


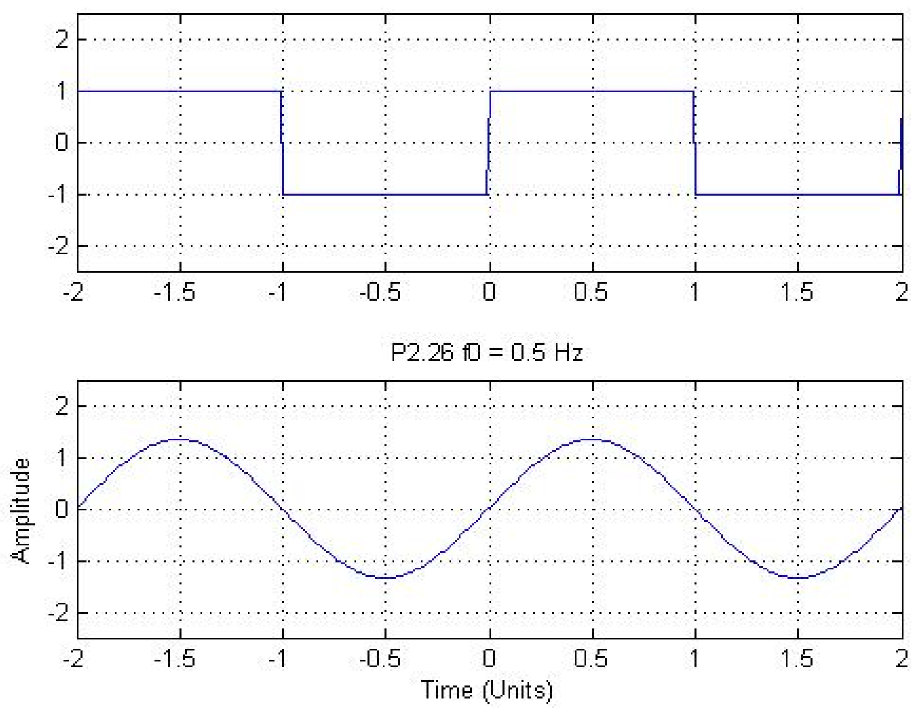

Problem 2.25 (b) BT=10
Response
Time (Units)
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1

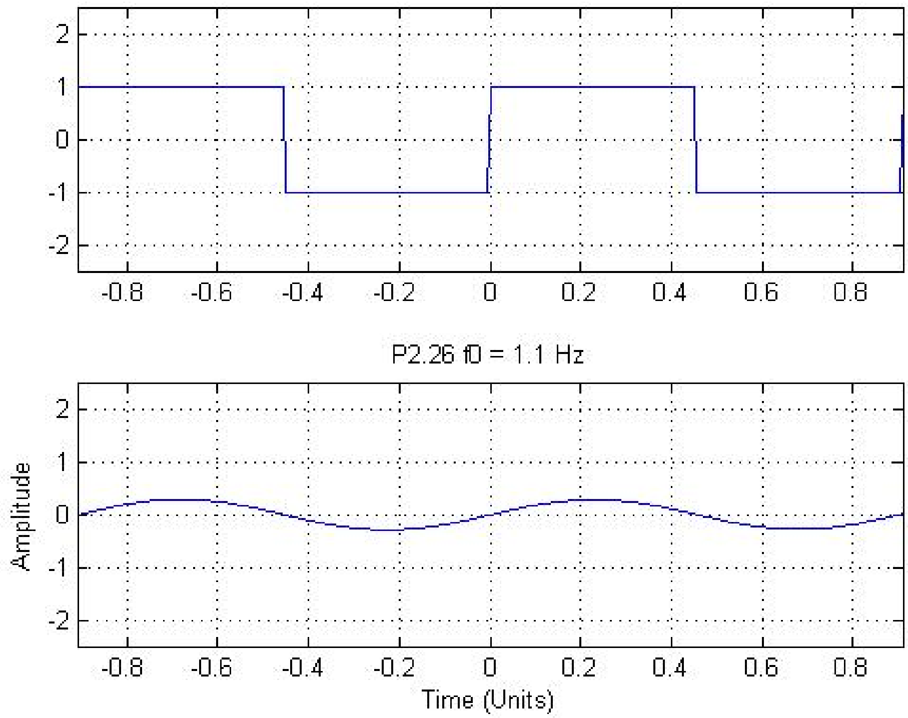

Problem 2.25 (b) BT=20
Response
Time (Units)
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1


Problem 2.25 (b) BT=100
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
Response
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
Time (Units)

2.26.b)


P2.26 f0 = 0.1 Hz
Amplitude
Time (Units)


P2.26 f0 = 0.5 Hz
Amplitude
Time (Units)



2.26 b)
If B is left fixed, at B=1, and only T is varied, the results are as follows

BT	**Max. Amplitude**
5	1.194
2	1.23
1	1.34
0.5	0.612
0.45	0.286

As the centre frequency of the square wave increases, so does the bandwidth of the signal (and its own bandwidth shifts its centre as well). This means that the filter passes less of the signal's energy, since more of it will lie outside of the pass band. This results in greater overshoot.
However, as the frequency of the pulse train continues to increase, the centre frequency is no longer in the pass band, and the resulting output will also be attenuated.

c)

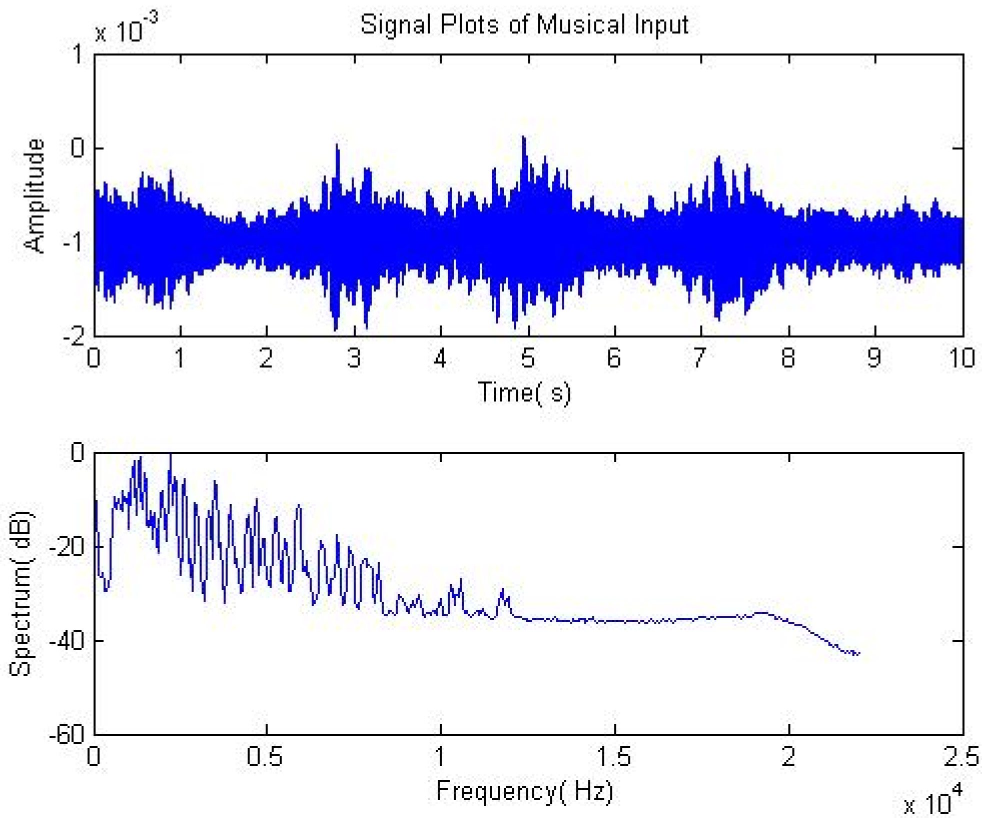
BT	**Max. Amplitude**
5	1.18
2	1.20
1	1.27
0.5	0.62
0.45	0.042

Extending the length of the filter's impulse response has allowed it to better approximate the ideal filter in that there is less ripple. However, this does not extend the bandwidth of the filter, so the reduction in overshoot is minimal. The dramatic change in the last entry (BT=0.45) can be accounted for by the reduction in ripple.

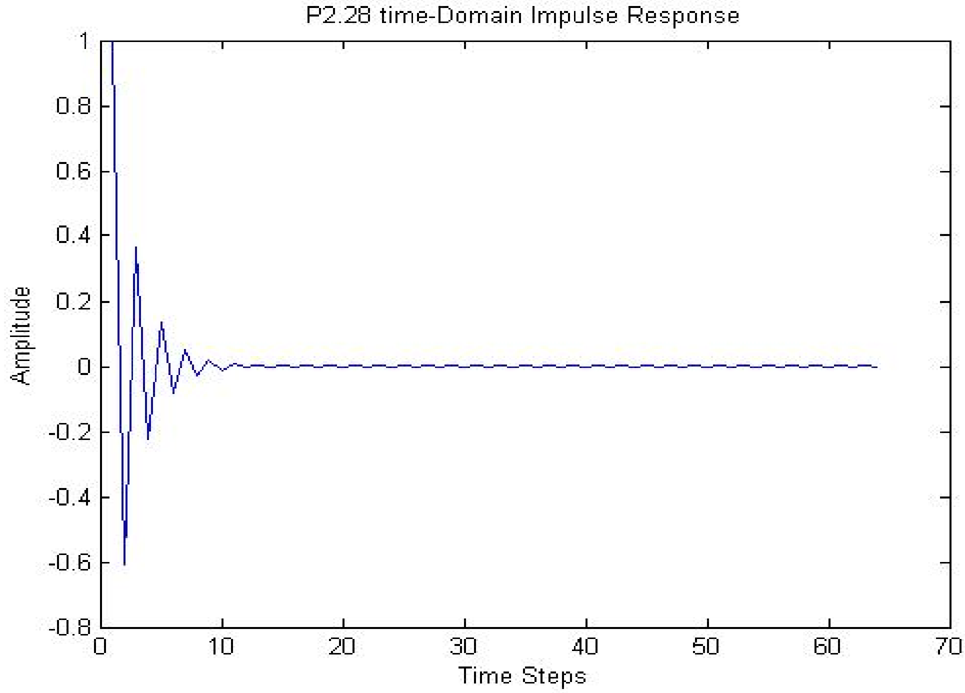
2.27

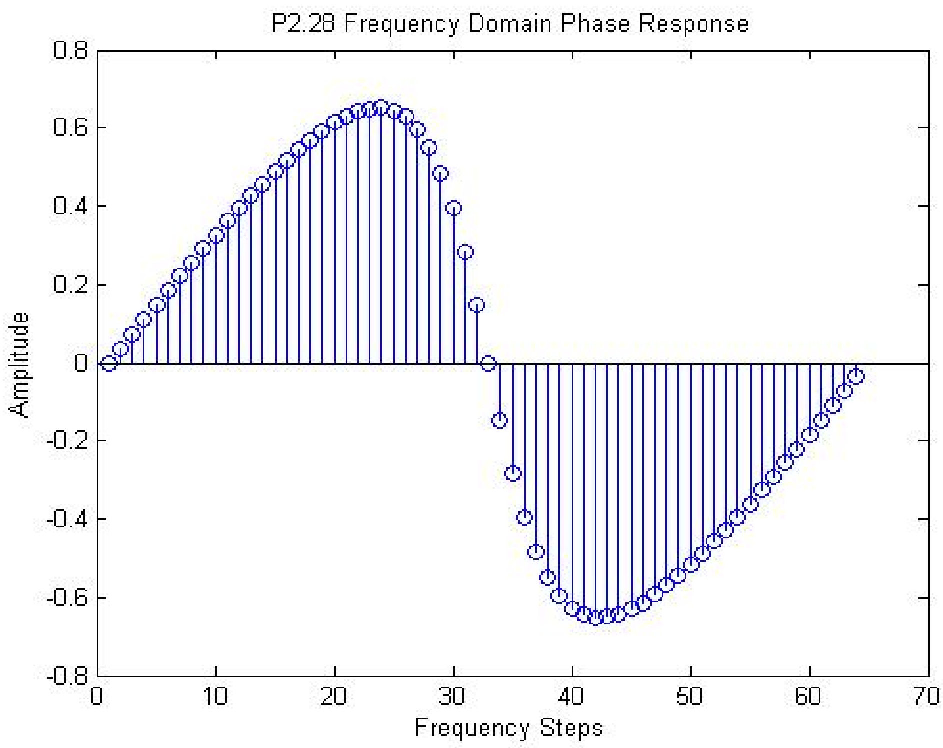
a)At fs = 4000 and fs = 8000, there is a muffled quality to the signals. This improves with higher sampling rates. Lower sampling rates throw away more of the signal's high frequencies, which results in a lower quality approximation.

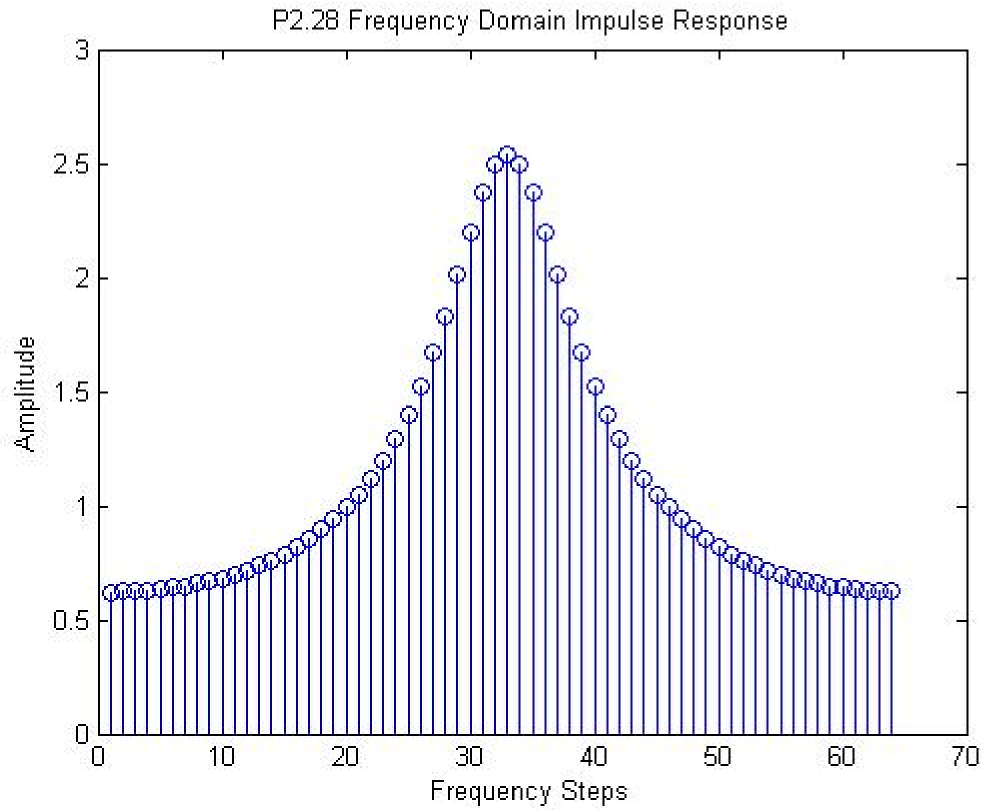
b)Speech suffers from less "muffling" than do other forms of music. This is because a greater percentage of the signal energy is concentrated at low frequencies. Musical instruments create notes that have significant energy in frequencies beyond the human vocal range. This is particularly true of instruments whose notes have sharp attack times.



2.28


P2.28 time-Domain Impulse Response
Amplitude
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
0
10
20
30
40
50
60
70
Time Steps


P2.28 Frequency Domain Phase Response
Amplitude
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
0
10
20
30
40
50
60
70
Frequency Steps


P2.28 Frequency Domain Impulse Response
Amplitude
Frequency Steps
0
0.5
1
1.5
2
2.5
3
0
10
20
30
40
50
60
70

Chapter 3

3.1

$s(t) = A_c[1+k_a m(t)]\cos(2\pi f_c t)$

where $m(t) = \sin(2\pi f_s t)$ and f_s=5 kHz and f_c = 1 MHz.

$$\therefore s(t) = A_c[\cos(2\pi f_c t) + \frac{k_a}{2}(\sin(2\pi(f_c + fs)t) + \sin(2\pi(f_c - f_s)t)]$$

s(t) is the signal before transmission.

The filter bandwidth is: $BW = \frac{f_c}{Q} = \frac{10^6}{175} = 5714$ Hz

m(t) lies close to the 3dB bandwidth of the filter, *m(t)* is therefore attenuated by a factor of a half.

$\therefore m^{'}(t) = 0.5m(t)$ or $k_a^{'} = 0.5k_a$

$\therefore k_a^{'} = 0.25$

The modulation depth is 0.25

3.2 (a)

$$i = I_0[\exp(-\frac{v}{V_T}) - 1]$$

Using the Taylor series expansion of exp(x) up to the third order terms, we get:

$$i = I_0[-\frac{v}{V_T} + \frac{1}{2}\left(\frac{v}{V_T}\right)^2 - \frac{1}{6}\left(\frac{v}{V_T}\right)^3]$$

(b) $v(t) = 0.01[\cos(2\pi f_m t) + \cos(2\pi f_c t)]$

Let $\theta = 2\pi t\frac{f_c + f_m}{2}, \quad \phi = 2\pi t\frac{f_c - f_m}{2}$

then $v(t) = 0.02[\cos\theta\cos\phi]$

$$\therefore v^2(t) = 0.02^2[1+\cos(2\theta)][1+\cos(2\phi)]$$
$$= 0.02^2[1+\cos(2\theta)+\cos(2\phi)+\frac{1}{2}(\cos(2\theta+2\phi)+\cos(2\theta-2\phi))]$$
$$= 0.02^2[1+\cos(2\pi(f_c+f_m)t)+\cos(2\pi(f_c-f_m)t)+\frac{1}{2}(\cos(4\pi f_c t)+\cos(4\pi f_m t))]$$

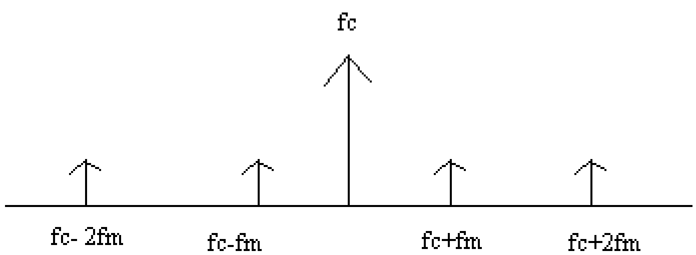
$$v^3(t) = 0.02^3\left[\frac{3\cos\theta+\cos 3\theta}{4}\right]\left[\frac{3\cos\phi+\cos 3\phi}{4}\right]$$
$$= \frac{0.02^3}{16}[\frac{9}{2}(\cos(\theta+\phi)+\cos(\theta-\phi))+\frac{3}{2}(\cos(\theta+3\phi)+\cos(\theta-3\phi)$$
$$+\frac{3}{2}(\cos(3\theta+\phi)+\cos(3\theta-\phi))+\frac{1}{2}(\cos(3\theta+3\phi)+\cos(3\theta-3\phi))]$$

$$\therefore v^3(t) = \frac{0.02^2}{16}[\frac{9}{2}(\cos(2\pi f_c t)+\cos(2\pi f_m t))+\frac{3}{2}(\cos(2\pi(2f_c-f_m)t)+\cos(2\pi(2f_m-f_t)t)$$
$$+\frac{3}{2}(\cos(2\pi(2f_c+f_m)t)+\cos(2\pi(2f_m+f_t)t))+\frac{1}{2}(\cos(6\pi f_c t)+\cos(6\pi f_m t))]$$

The output will have spectral components at:

f_m
f_c
$f_c + f_m$
$f_c - f_m$
$2f_c$
$2f_m$
$2f_c - f_m$
$2f_c + f_m$
$f_c - 2f_m$
$f_c + 2f_m$
$3f_c$
$3f_m$

(c)



The bandpass filter must be symmetric and centred around f_c . It must pass components at $f_c + f_m$, but reject those at $f_c + 2f_m$ and higher.

(d)

Term #	Carrier	Message	Taylor Coef.
1	0.01		-38.46
2		0.0001	739.6
3	2.25×10^{-6}		-9.48×10^{3}

After filtering and assuming a filter gain of 1, we get:

$$i(t) = 0.41\cos(2\pi f_c t) + 0.074[\cos(2\pi(f_c - f_m)t) + \cos(2\pi(f_c + f_m)t)]$$
$$= 0.41\cos(2\pi f_c t) + .148[\cos(2\pi f_c t)\cos(2\pi f_m t)]$$
$$= [0.41 + 0.148\cos(2\pi f_m t)]\cos(2\pi f_c t)$$
$$= [1 + 0.36\cos(2\pi f_m t)]\cos(2\pi f_c t)$$

$\therefore$ The modulation percentage is ~36%

Problem 3.3

(a) Let the input voltage v_i consist of a sinusoidal wave of frequency $\frac{1}{2}f_c$ (i.e., half the desired carrier frequency) and the message signal $m(t)$:

$$v_i = A_c \cos(\pi f_c t) + m(t)$$

Then, the output current i_o is

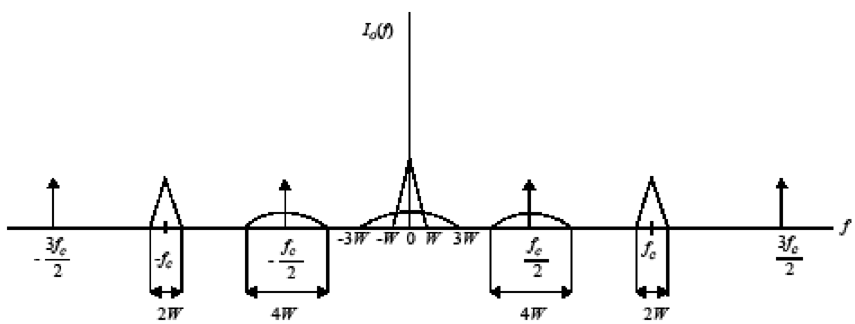
$$i_o = a_1 v_i + a_3 v_i^3$$

$$= a_1[A_c \cos(\pi f_c t) + m(t)] + a_3[A_c \cos(\pi f_c t) + m(t)]^3$$

$$= a_1[A_c \cos(\pi f_c t) + m(t)] + \frac{1}{4}a_3 A_c^3[\cos 3(\pi f_c t) + 3\cos(\pi f_c t)]$$

$$+ \frac{3}{2}a_3 A_c^2 m(t)[1 + \cos 2(\pi f_c t)] + 3a_3 A_c \cos(\pi f_c t) m^2(t) + a_3 m^3(t)$$

Assume that $m(t)$ occupies the frequency interval $-W \le f \le W$. Then, the amplitude spectrum of the output current i_o is as shown below:

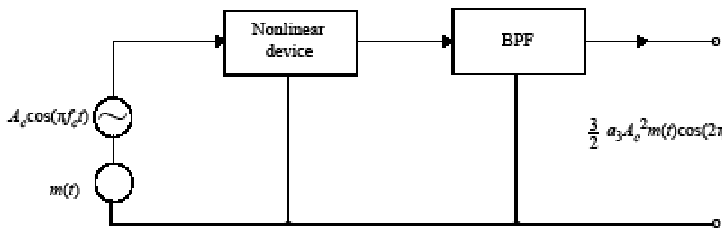


From this diagram we see that in order to extract a DSBSC wave, with carrier frequency f_c from i_o, we need a bandpass filter with mid-band frequency f_c and bandwidth $2W$, which satisfy the requirement:

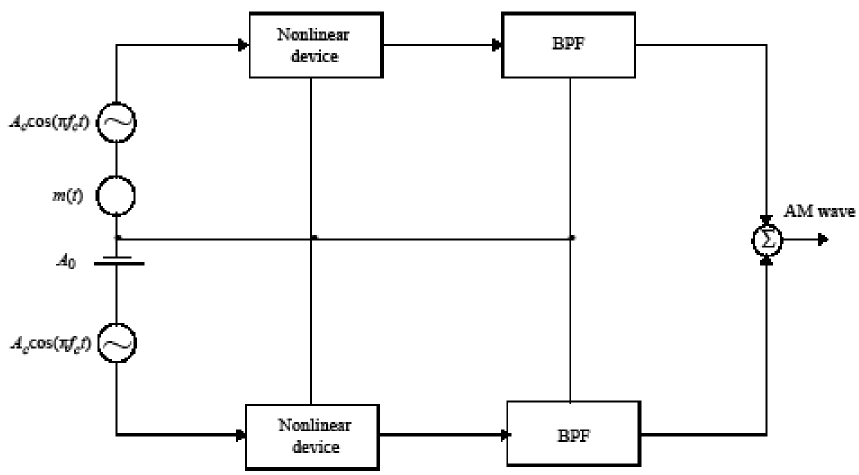
$$f_c - W > \frac{f_c}{2} + 2W$$

that is, $f_c > 6W$

Therefore, to use the given nonlinear device as a product modulator, we may use the following configuration:



(b) To generate an AM wave with carrier frequency f_c we require a sinusoidal component of frequency f_c to be added to the DSBSC generated in the manner described above. To achieve this requirement, we may use the following configuration involving a pair of the nonlinear devices and a pair of identical bandpass filters.



The resulting AM wave is therefore $\frac{3}{2} a_3 A_c^2 [A_0 + m(t)] \cos(2\pi f_c t)$. Thus, the choice of the dc level A_0 at the input of the lower branch controls the percentage modulation of the AM wave.

Problem 3.4

Consider the square-law characteristic:

$$v_2(t) = a_1 v_1(t) + a_2 v_1^2(t) \tag{1}$$

where a_1 and a_2 are constants. Let

$$v_1(t) = A_c \cos(2\pi f_c t) + m(t) \tag{2}$$

Therefore substituting Eq. (2) into (1), and expanding terms:

$$v_2(t) = a_1 A_c \left[1 + \frac{2a_2}{A_1} m(t)\right] \cos(2\pi f_c t) \tag{3}$$

$$+ a_1 m(t) + a_2 m^2(t) + a_2 A_c^2 \cos^2(2\pi f_c t)$$

The first term in Eq. (3) is the desired AM signal with $k_a = 2a_2/a_1$. The remaining three terms are unwanted terms that are removed by filtering.

Let the modulating wave $m(t)$ be limited to the band $-W \le f \le W$, as in Fig. 1(a). Then, from Eq. (3) we find that the amplitude spectrum $|V_2(f)|$ is as shown in Fig. 1(b). It follows therefore that the unwanted terms may be removed from $v_2(t)$ by designing the tuned filter at the modulator output of Fig. P2.2 to have a mid-band frequency fc and bandwidth $2W$, which satisfy the requirement that $f_c > 3W$.

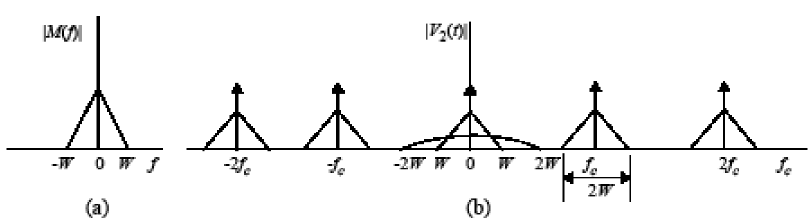


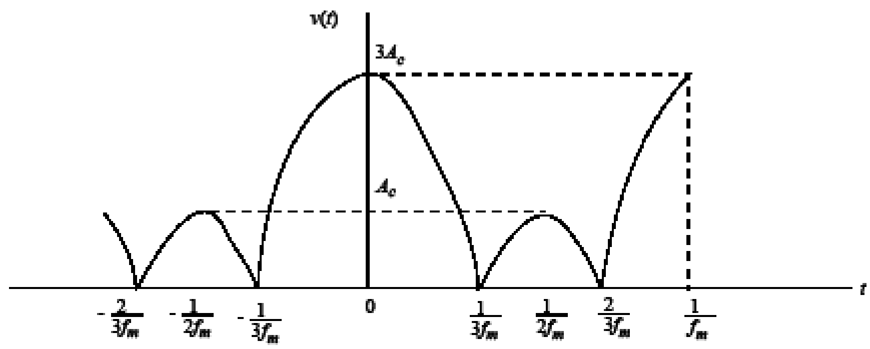
Figure 1

Problem 3.5

(a) The envelope detector output is

$$v(t) = A_c\left|1 + \mu\cos(2\pi f_m t)\right|$$

which is illustrated below for the case when $\mu = 2$.



We see that $v(t)$ is periodic with a period equal to f_m, and an even function of t, and so we may express $v(t)$ in the form:

$$v(t) = a_0 + 2\sum_{n=1}^{\infty} a_n \cos(2n\pi f_m t)$$

where

$$a_0 = 2f_m\int_0^{1/2f_m} v(t)dt$$

$$= 2A_c f_m\int_0^{1/3f_m}[1 + 2\cos(2n\pi f_m t)]dt + 2A_c f_m\int_{1/3f_m}^{1/2f_m}[-1-2\cos(2n\pi f_m t)]dt$$

$$= \frac{A_c}{3} + \frac{4A_c}{\pi}\sin\left(\frac{2\pi}{3}\right) \tag{1}$$

$$a_n = 2f_m\int_0^{1/2f_m} v(t)\cos(2n\pi f_m t)dt$$

$$= 2A_c f_m \int_0^{1/3f_m} [1 + 2\cos(2\pi f_m t)] \cos(2n\pi f_m t) dt$$

$$+ 2A_c f_m \int_{1/3f_m}^{1/2f_m} [-1 - 2\cos(2\pi f_m t)] \cos(2n\pi f_m t) dt$$

$$= \frac{A_c}{n\pi}\left[2\sin\left(\frac{2n\pi}{3}\right) - \sin(n\pi)\right] + \frac{A_c}{(n+1)\pi}\left\{2\sin\left[\frac{2\pi}{3}(n+1)\right] - \sin[\pi(n+1)]\right\}$$

$$+ \frac{A_c}{(n-1)\pi}\left\{2\sin\left[\frac{2\pi}{3}(n-1)\right] - \sin[\pi(n-1)]\right\} \qquad (2)$$

For $n = 0$, Eq. (2) reduces to that shown in Eq. (1).

(b) For $n = 1$, Eq. (2) yields

$$a_1 = A_c\left(\frac{\sqrt{3}}{2\pi} + \frac{1}{3}\right)$$

For $n = 2$, it yields

$$a_2 = \frac{A_c\sqrt{3}}{2\pi}$$

Therefore, the ratio of second-harmonic amplitude to fundamental amplitude in $v(t)$ is

$$\frac{a_2}{a_1} = \frac{3\sqrt{3}}{2\pi + 3\sqrt{3}} = 0.452$$

Problem 3.6

Let

$$v_1(t) = A_c[1 + k_a m(t)]\cos(2\pi f_c t)$$

(a) Then the output of the square-law device is

$$\begin{aligned} v_2(t) &= a_1 v_1 + a_2 v_1^2(t) \\ &= a_1 A_c[1 + k_a m(t)]\cos(2\pi f_c t) \\ &+ \frac{1}{2} a_2 A_c^2 [1 + k_a m(t) + k_a^2 m^2(t)][1 + \cos(4\pi f_c t)] \end{aligned}$$

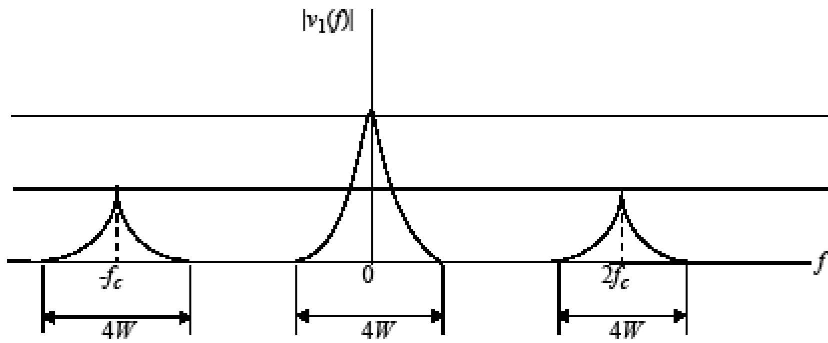
(b) The desired signal, namely $a_2 A_c^2 k_a m(t)$, is due to the $a_2 v_1^2(t)$ - hence, the name "square-law detection". This component can be extracted by means of a low-pass filter. This is not the only contribution within the baseband spectrum, because the term $1/2 a_2 A_c^2 k_a^2 m^2(t)$ will give rise to a plurality of similar frequency components. The ratio of wanted signal to distortion is $2/k_a m(t)$. To make this ratio large, the percentage modulation, that is, $|k_a m(t)|$ should be kept small compared with unity.

Problem 3.7

The squarer output is

$$v_1(t) = A_c^2[1 + k_a m(t)]^2 \cos^2(2\pi f_c t)$$
$$= \frac{A_c^2}{2}[1 + 2k_a m^2(t)][1 + \cos(4\pi f_c t)]$$

The amplitude spectrum of $v_1(t)$ is therefore as follows, assuming that $m(t)$ is limited to the interval $-W \le f \le W$:



Since $f_c > 2W$, we find that $2f_c - 2W > 2W$. Therefore, by choosing the cutoff frequency of the low-pass filter greater than $2W$, but less than $2f_c - 2W$, we obtain the output

$$v_2(t) = \frac{A_c^2}{2}[1 + k_a m(t)]^2$$

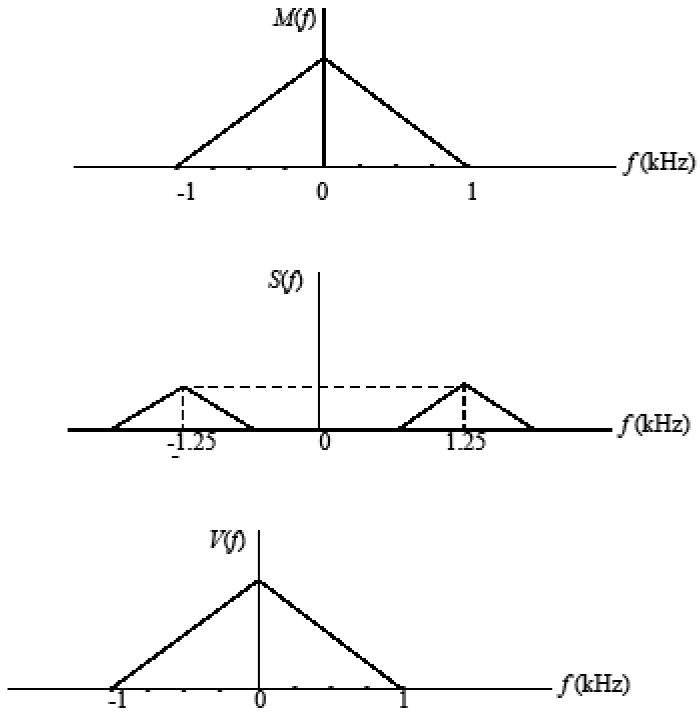
Hence, the square-rooter output is

$$v_3(t) = \frac{A_c}{\sqrt{2}}[1 + k_a m(t)]$$

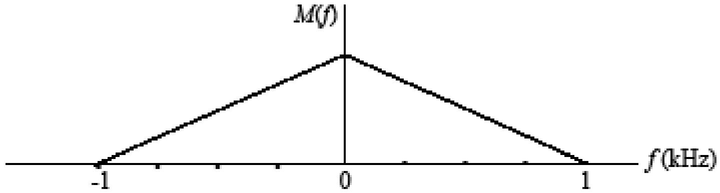
which, except for the dc component $\frac{A_c}{\sqrt{2}}$, is proportional to the message signal $m(t)$.

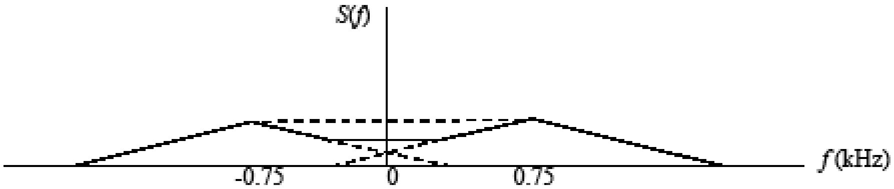
Problem 3.8

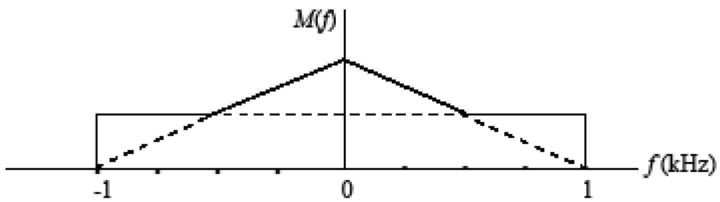
(a) For f_c = 1.25 kHz, the spectra of the message signal $m(t)$, the product modulator output $s(t)$, and the coherent detector output $v(t)$ are as follows, respectively:



(b) For the case when $f_c = 0.75$, the respective spectra are as follows:







To avoid sideband-overlap, the carrier frequency f_c must be greater than or equal to 1 kHz. The lowest carrier frequency is therefore 1 kHz for each sideband of the modulated wave $s(t)$ to be uniquely determined by $m(t)$.

Problem 3.9

The two AM modulator outputs are

$$s_1(t) = A_c[1 + k_a m(t)]\cos(2\pi f_c t)$$

$$s_2(t) = A_c[1 + k_a m(t)]\cos(2\pi f_c t)$$

Subtracting $s_2(t)$ from $s_1(t)$:

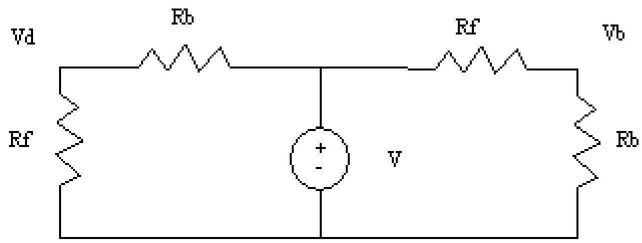
$$s(t) = s_2(t) - s_1(t)$$

$$= 2k_a m(t)\cos(2\pi f_c t)$$

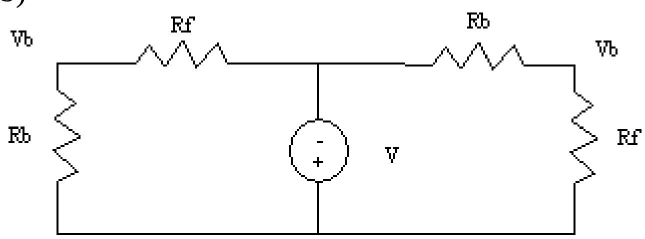
which represents a DSB-SC modulated wave.

3.10. The circuit can be rearranged as follows:

(a)



(b)



Let the voltage V_b-V_d be the voltage across the output resistor, with V_b and V_d being the voltages at each node.

Using the voltage divider rule for condition (a):

$$V_b = V\frac{R_b}{R_f + R_b}\ , \quad V_d = V\frac{R_f}{R_f + R_b}\ , \quad V_b - V_d = V\frac{R_b - R_f}{R_f + R_b}$$

and for (b):

$$V_b = -V\frac{R_f}{R_f + R_b}\ , \quad V_d = -V\frac{R_b}{R_f + R_b}\ , \quad V_b - V_d = V\frac{-R_b + R_f}{R_f + R_b}$$

$\therefore$ The two voltages are of the same magnitude, but are of the opposite sign.

Problem 3.11

(a) Multiplying the signal by the local oscillator gives:

$$s_1(t) = A_c m(t)\cos(2\pi f_c t)\cos[2\pi(f_c + \Delta f)t]$$

$$= \frac{A_c}{2} m(t)\{\cos(2\pi\Delta f t) + \cos[2\pi 2(f_c + \Delta f)t]\}$$

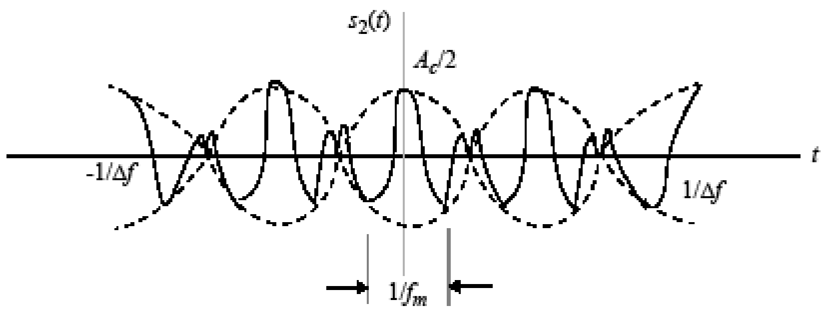
Low pass filtering leaves:

$$s_2(t) = \frac{A_c}{2} m(t)\cos(2\pi\Delta f t)$$

Thus the output signal is the message signal modulated by a sinusoid of frequency Δf.

(b) If $m(t) = \cos(2\pi f_m t)$,

then $s_2(t) = \frac{A_c}{2}\cos(2\pi f_m t)\cos(2\pi\Delta f t)$



Problem 3.12

(a) $y(t) = s^2(t)$

$$= A_c^2 \cos^2(2\pi f_c t) m^2(t)$$

$$= \frac{A_c^2}{2}[1 + \cos(4\pi f_c t)]m^2(t)$$

Therefore, the spectrum of the multiplier output is

$$Y(f) = \frac{A_c^2}{2}\int_{-\infty}^{\infty} M(\lambda)M(f-\lambda)d\lambda + \frac{A_c^2}{4}\left[\int_{-\infty}^{\infty} M(\lambda)M(f-2f_c-\lambda)d\lambda + \int_{-\infty}^{\infty} M(\lambda)M(f+2f_c-\lambda)d\lambda\right]$$

where $M(f) = F[m(t)]$.

(b) At $f = 2f_c$, we have

$$Y(2f_c) = \frac{A_c^2}{2}\int_{-\infty}^{\infty} M(\lambda)M(2f_c\text{-}\lambda)d\lambda$$

$$+\frac{A_c^2}{4}\left[\int_{-\infty}^{\infty} M(\lambda)M(\text{-}\lambda)d\lambda + \int_{-\infty}^{\infty} M(\lambda)M(4f_c\text{-}\lambda)d\lambda\right]$$

Since $M(\text{-}\lambda) = M^*(\lambda)$, we may write

$$Y(2f_c) = \frac{A_c^2}{2}\int_{-\infty}^{\infty} M(\lambda)M(2f_c\text{-}\lambda)d\lambda$$

$$+\frac{A_c^2}{4}\left[\int_{-\infty}^{\infty} |M(\lambda)|^2 d\lambda + \int_{-\infty}^{\infty} M(\lambda)M(4f_c-\lambda)d\lambda\right] \qquad (1)$$

With $m(t)$ limited to $-W \le f \le W$ and $f_c > W$, we find that the first and third integrals reduce to zero, and so we may simplify Eq. (1) as follows

$$Y(2f_c) = \frac{A_c^2}{4}\int_{-\infty}^{\infty} |M(\lambda)|^2 d\lambda$$

$$= \frac{A_c^2 E}{4}$$

where E is the signal energy (by Rayleigh's energy theorem). Similarly, we find that

$$Y(-2f_c) = \frac{A_c^2}{4}E$$

The band-pass filter output, in the frequency domain, is therefore defined by

$$V(f) \approx \frac{A_c^2}{4}E\Delta f[(\delta f - 2f_c) + \delta(f + 2f_c)]$$

Hence,

$$v(t) \approx \frac{A_c^2}{4}E\Delta f\cos(4\pi f_c t)$$

Problem 3.13

The multiplexed signal is

$$s(t) = A_c m_1(t)\cos(2\pi f_c t) + A_c m_2(t)\sin(2\pi f_c t)$$

Therefore,

$$S(f) = \frac{A_c}{2}[M_1(f-f_c) + M_1(f+f_c)] + \frac{A_c}{2j}[M_2(f-f_c) - M_2(f+f_c)]$$

where $M_1(f) = F(m_1(t)]$ and $M_2(f) = F(m_2(t)]$. The spectrum of the received signal is therefore

$$R(f) = H(f)S(f)$$

$$= \frac{A_c}{2}H(f)\left[M_1(f-f_c) + M_1(f+f_c) + \frac{1}{j}M_2(f-f_c) - \frac{1}{j}M_2(f+f_c)\right]$$

To recover $m_1(t)$, we multiply $r(t)$, the inverse Fourier transform of $R(f)$, by $\cos(2\pi f_c t)$ and then pass the resulting output through a low-pass filter, producing a signal with the following spectrum

$$F[r(t)\cos(2\pi f_c t)] = \frac{1}{2}[R(f-f_c) + R(f+f_c)]$$

$$= \frac{A_c}{4}H(f-f_c)[M_1(f-f_c) + M_1(f) + \frac{1}{j}M_2(f-f_c) - \frac{1}{j}M_2(f)]$$

$$+ \frac{A_c}{4}H(f+f_c)\left[M_1(f) + M_1(f+2f_c) + \frac{1}{j}M_2 - \frac{1}{j}M_2(f+f_c)\right] \qquad (1)$$

The condition $H(f_c+f) = H^*(f_c - f)$ is equivalent to $H(f+f_c) = H(f-f_c)$; this follows from the fact that for a real-valued impulse response $h(t)$, we have $H(-f) = H^*(f)$. Hence, substituting this condition in Eq. (1), we get

$$F[r(t)\cos(2\pi f_c t)] = \frac{A_c}{2}H(f-f_c)M_1(f)$$

$$+ \frac{A_c}{4}H(f-f_c)\left[M_1(f-2f_c) + \frac{1}{j}M_2(f-2f_c) + M_1(f+2f_c) - \frac{1}{j}M_2(f+2f_c)\right]$$

The low-pass filter output, therefore, has a spectrum equal to $(A_c/2)H(f-f_c)M_1(f)$.

Similarly, to recover $m_2(t)$, we multiply $r(t)$ by $\sin(2\pi f_c t)$, and then pass the resulting signal through a low-pass filter. In this case, we get an output with a spectrum equao to $(A_c/2)H(f-f_c)m_2(t)$.

Problem 3.14

When the local carriers have a phase error ϕ, we may write

$$\cos(2\pi f_c t + \phi) = \cos(2\pi f_c t)\cos\phi - \sin(2\pi f_c t)\sin\phi$$

In this case, we find that by multiplying the received signal $r(t)$ by $\cos(2\pi f_c t + \phi)$, and passing the resulting output through a low-pass filter, the corresponbding low-pass filter output in the receiver has a spectrum equal to $(A_c/2)H(f - f_c)[\cos\phi M_1(f) - \sin\phi M_2(f)]$. This indicates that there is cross-talk at the demodulator outputs.

Problem 3.15

The transmitted signal is given by

$$s(t) = A_c m_1(t)\cos(2\pi f_c t) + A_c m_2(t)\sin(2\pi f_c t)$$

$$= A_c[V_0 + m_l(t) + m_r(t)]\cos(2\pi f_c t) + A_c[m_l(t) - m_r(t)]\sin(2\pi f_c t)$$

(a) The envelope detection of $s(t)$ yields

$$y_1(t) = A_c\sqrt{(V_0 + m_l(t) + m_r(t))^2 + (m_l(t) - m_r(t))^2}$$

$$= A_c(V_0 + m_l(t) + m_r(t))\sqrt{1 + \left(\frac{m_l(t) - m_r(t)}{V_0 + m_l(t) + m_r(t)}\right)^2}$$

To minimize the distortion in the envelope detector output due to the quadrature component, we choose the DC offset V_0 to be large. We may then approximate $y_1(t)$ as

$$y_1(t) \approx A(V_0 + m_l(t) + m_r(t))$$

3.16 (a)

$$s(t) = \frac{1}{2} a \cdot A_m A_c \cos(2\pi(f_m + f_c)t) + \frac{1}{2}(1-a) A_m A_c \cos(2\pi(f_m + f_c)t)$$

$$s(t) = \frac{A_m A_c}{2}[a(\cos(2\pi f_c t)\cos(2\pi f_m t) - \sin(2\pi f_c t)\sin(2\pi f_m t))$$
$$+(1-a)(\cos(2\pi f_c t)\cos(2\pi f_m t) + \sin(2\pi f_c t)\sin(2\pi f_m t))]$$

$$s(t) = \frac{A_m A_c}{2}[\cos(2\pi f_c t)\cos(2\pi f_m t) + (1-2a)\sin(2\pi f_c t)\sin(2\pi f_m t))]$$
$$\therefore m_1(t) = \frac{A_m}{2}\cos(2\pi f_m t)$$
$$m_2(t) = \frac{A_m}{2}(1-2a)\sin(2\pi f_m t)$$

b)Let:
$$s(t) = \frac{1}{2} A_c m(t)\cos(2\pi f_c t) + \frac{1}{2} A_c m_s(t)\sin(2\pi f_c t)$$

By adding the carrier frequency:

$$s(t) = A_c[1 + \frac{1}{2} k_a m(t)]\cos(2\pi f_c t) + \frac{1}{2} k_a A_c m_s(t)\sin(2\pi f_c t)$$
where k_a is the percentage modulation.
After passing the signal through an envelope detector, the output will be:
$$|s(t)| = A_c \left\{ \left[1 + \frac{1}{2} k_a m(t)\right]^2 + \left[\frac{1}{2} k_a m_s(t)\right]^2 \right\}^{1/2}$$
$$= A_c\left[1 + \frac{1}{2} k_a m(t)\right] \cdot \left\{ 1 + \left[\frac{\frac{1}{2} k_a m_s(t)}{1 + \frac{1}{2} k_a m(t)}\right]^2 \right\}^{1/2}$$
The second factor in $|s(t)|$ is the distortion term *d(t)*. For the example in (a), this becomes:

$$d(t) = \left\{ 1 + \left[\frac{\frac{1}{2}(1-2a)\sin(2\pi f_m t)}{1 + \frac{1}{2}\cos(2\pi f_m t)}\right]^2 \right\}^{1/2}$$

c)Ideally, $d(t)$ is equal to one. However, the distortion factor increases with decreasing a. Therefore, the worst case exists when $a = 0$.

Problem 3.17

(a) $s(t) = A_c(1 + k_a m(t))\cos(2\pi f_c t)$

$$= A_c\left(1 + \frac{k_a}{1+t^2}\right)\cos(2\pi f_c t)$$

To ensure 50 percent modulation, $k_a = 1$, in which case we get

$$s(t) = A_c\left(1 + \frac{1}{1+t^2}\right)\cos(2\pi f_c t)$$

(b) $s(t) = A_c m(t)\cos(2\pi f_c t)$

$$= \frac{A_c}{1+t^2}\cos(2\pi f_c t)$$

(c) $s(t) = \frac{A_c}{2}[m(t)\cos(2\pi f_c t) - \hat{m}(t)\sin(2\pi f_c t)]$

$$= \frac{A_c}{2}\left[\frac{1}{1+t^2}\cos(2\pi f_c t) - \frac{t}{1+t^2}\sin(2\pi f_c t)\right]$$

(d) $s(t) = \frac{A_c}{2}\left[\frac{1}{1+t^2}\cos(2\pi f_c t) + \frac{t}{1+t^2}\sin(2\pi f_c t)\right]$

As an aid to the sketching of the modulated signals in (c) and (d), the envelope of either SSB wave is

$$\boldsymbol{a(t) = \frac{1}{2}\sqrt{\frac{t^2+1}{(1+t^2)^2}} = \frac{1}{2}\sqrt{\frac{1}{1+t^2}}}$$

Problem 3.18

An error Δf in the frequency of the local oscillator in the demodulation of an SSB signal, measured with respect to the carrier frequency f_c, gives rise to distortion in the demodulated signal. Let the local oscillator output be denoted by $A_c^1\cos(2\pi(f_c + \Delta f)t)$. The resulting demodulated signal is given by (for the case when the upper sideband only is transmitted)

$$v_o(t) = \frac{1}{4}A_c A_c' \, [m(t)\cos(2\pi\Delta ft) + m(t)\sin(2\pi\Delta ft)]$$

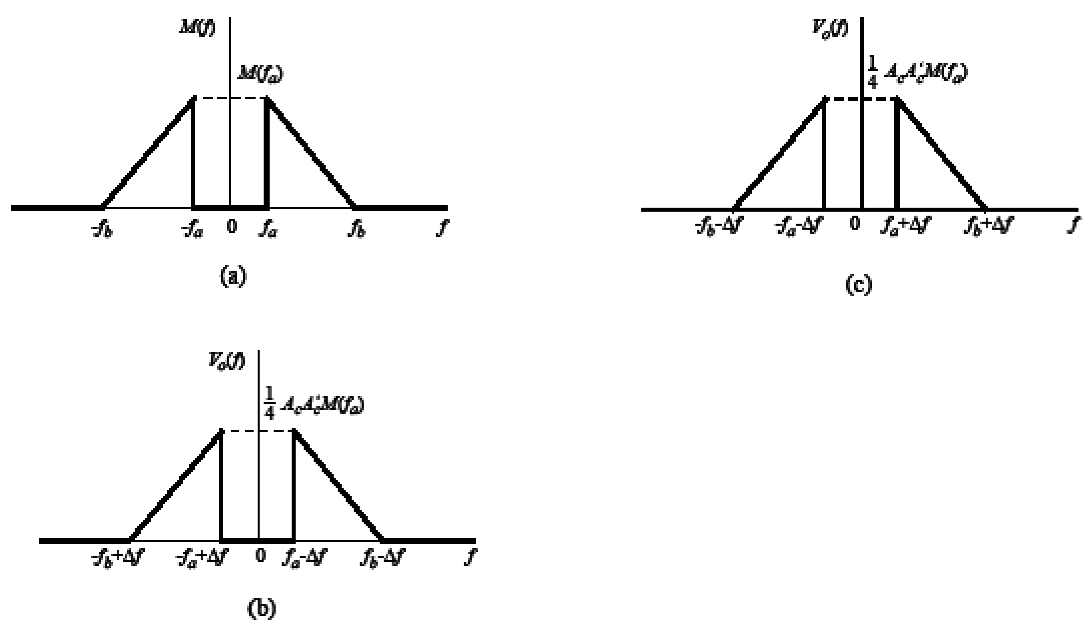
This demodulated signal represents an SSB wave corresponding to a carrier frequency Δf.

The effect of frequency error Δf in the local oscillator may be interpreted as follows:

(a) If the SSB wave $s(t)$ contains the upper sideband and the frequency error Δf is positive, or equivalently if $s(t)$ contains the lower sideband and Δf is negative, then the frequency components of the demodulated signal $v_o(t)$ are shifted inward by the amount Δf compared with the baseband signal $m(t)$, as illustrated in Fig. 1(b).

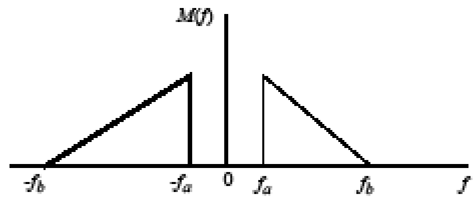
(b) If the incoming SSB wave $s(t)$ contains the lower sideband and the frequency error Δf is positive, or equivalently if $s(t)$ contains the upper sideband and Δf is negative, then the frequency components of the demodulated signal $v_o(t)$ are shifted outward by the amount Δf, compared with the baseband signal $m(t)$. This is illustrated in Fig. 1(c) for the case of a baseband signal (e.g., voice signal) with an energy gap occupying the interfal $-f_a \le f \le f_a$, in part (a) of the figure.

Figure 1

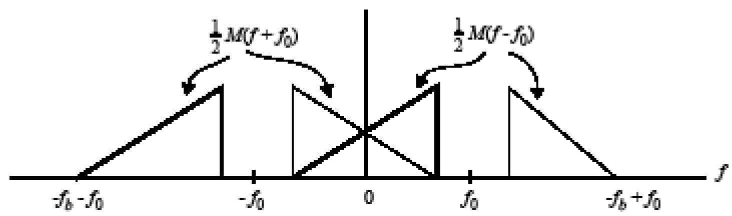


Problem 3.19

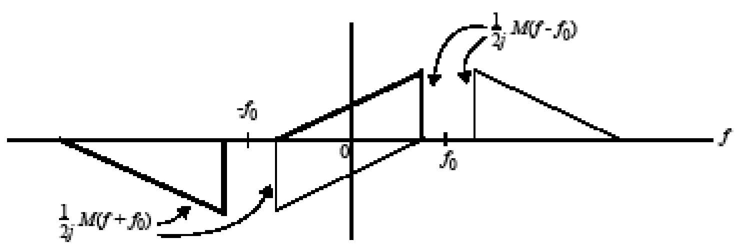
(a,b) The spectrum of the message signal is illustrated below:



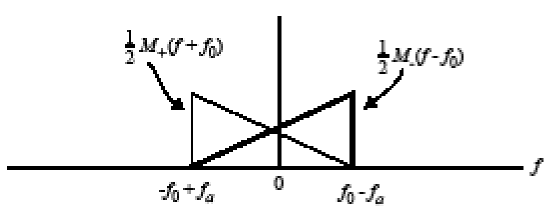
Correspondingly, the output of the upper first product modulator has the following spectrum:



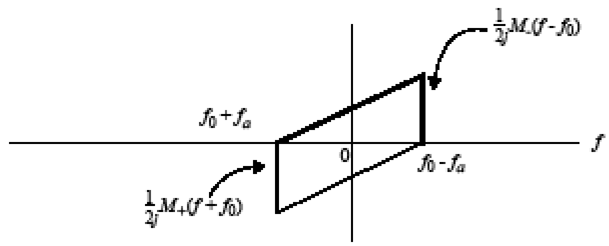
The output of the lower first product modulator has the spectrum:



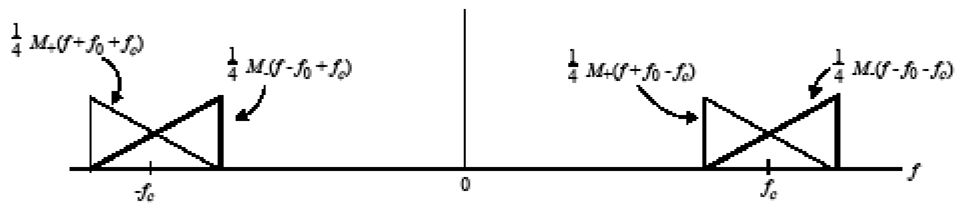
The output of the upper low pass filter has the spectrum



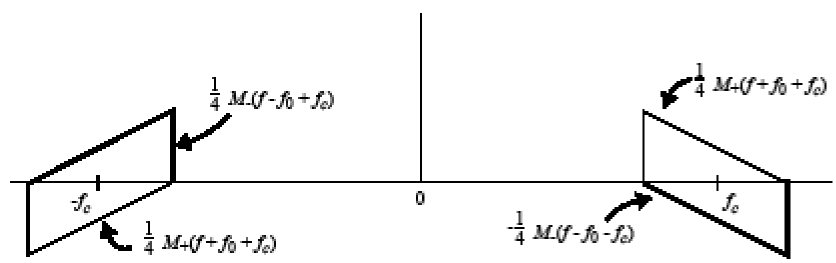
The output of the lower low pass filter has the spectrum:



The output of the upper second product modulator has the spectrum:



The output of the lower second product modulator has the spectrum:



Adding the two second product modulator outputs, their upper sidebands add constructively while their lower sidebands cancel each other.

(c) To modify the modulator to transmit only the lower sideband, a single sign change is required in one of the channels. For example, the lower first product modulator could multiply the message signal by $-\sin(2\pi f_0 t)$. Then, the upper sideband would be cancelled and the lower one transmitted.

3.20. *m(t)* contains {100,200,400} Hz

The transmitted SSB signal is: $\frac{A_c}{2}[m(t)\cos(2\pi f_c t) - \hat{m}(t)\sin(2\pi f_c t)$

Demodulation is accomplished using a product modulator and multiplying by:
$A_c^{'}\cos(2\pi f_c^{'} t)$

(a)

$$v_o(t) = \frac{1}{2}A_c A_c^{'}\cos(2\pi f_c^{'} t)[m(t)\cos(2\pi f_c t) - \hat{m}(t)\cos(2\pi f_c t)]$$

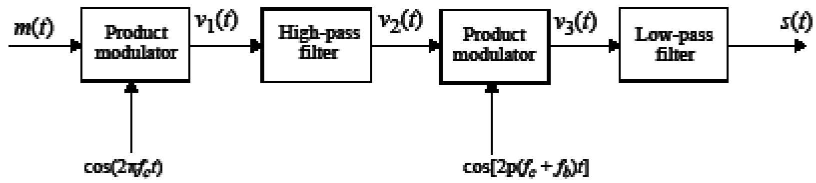
The only lowpass components will be those that are functions of only *t* and *Δf*. Higher frequency terms will be filtered out, and so can be ignored for the purposes of determining the output of the detector.

$\therefore v_o(t) = \frac{1}{4}A_c A_c^{'}[m(t)\cos(2\pi f\Delta t) - \hat{m}(t)\sin(2\pi f\Delta t)]$ by using basic trig identities.

When the upper side-band is transmitted, and *Δf*>0, the frequencies are shifted inwards by *Δf*.
$\therefore V_o(f)$ contains {99.98,199.98, 399.98} Hz

(b) When the lower side-band is transmitted, and *Δf*>0, then the baseband frequencies are shifted outwards by *Δf*.
$\therefore V_o(f)$ contains {100.02, 200.02, 400.02} Hz

Problem 3.21



(a) The first product modulator output is

$$v_1(t) = m(t)\cos(2\pi f_c t)$$

The second product modulator output is

$$v_3(t) = v_2(t)\cos[2\pi(f_c + f_b)t]$$

The amplitude spectra of $m(t)$, $v_1(t)$, $v_2(t)$, $v_3(t)$ and $s(t)$ are illustrated on the next page:
We may express the voice signal $m(t)$ as

$$m(t) = \frac{1}{2}[m_+(t) + m_-(t)]$$

where $m_+(t)$ is the pre-envelope of $m(t)$, and $m_-(t) = m_+^*(t)$ is its complex conjugate. The Fourier transforms of $m_+(t)$ and $m_-(t)$ are defined by (See Appendix 2)

$$M_+(f) = \left\{ \begin{array}{ll} 2M(f), & f > 0 \\ 0, & f < 0 \end{array} \right\}$$

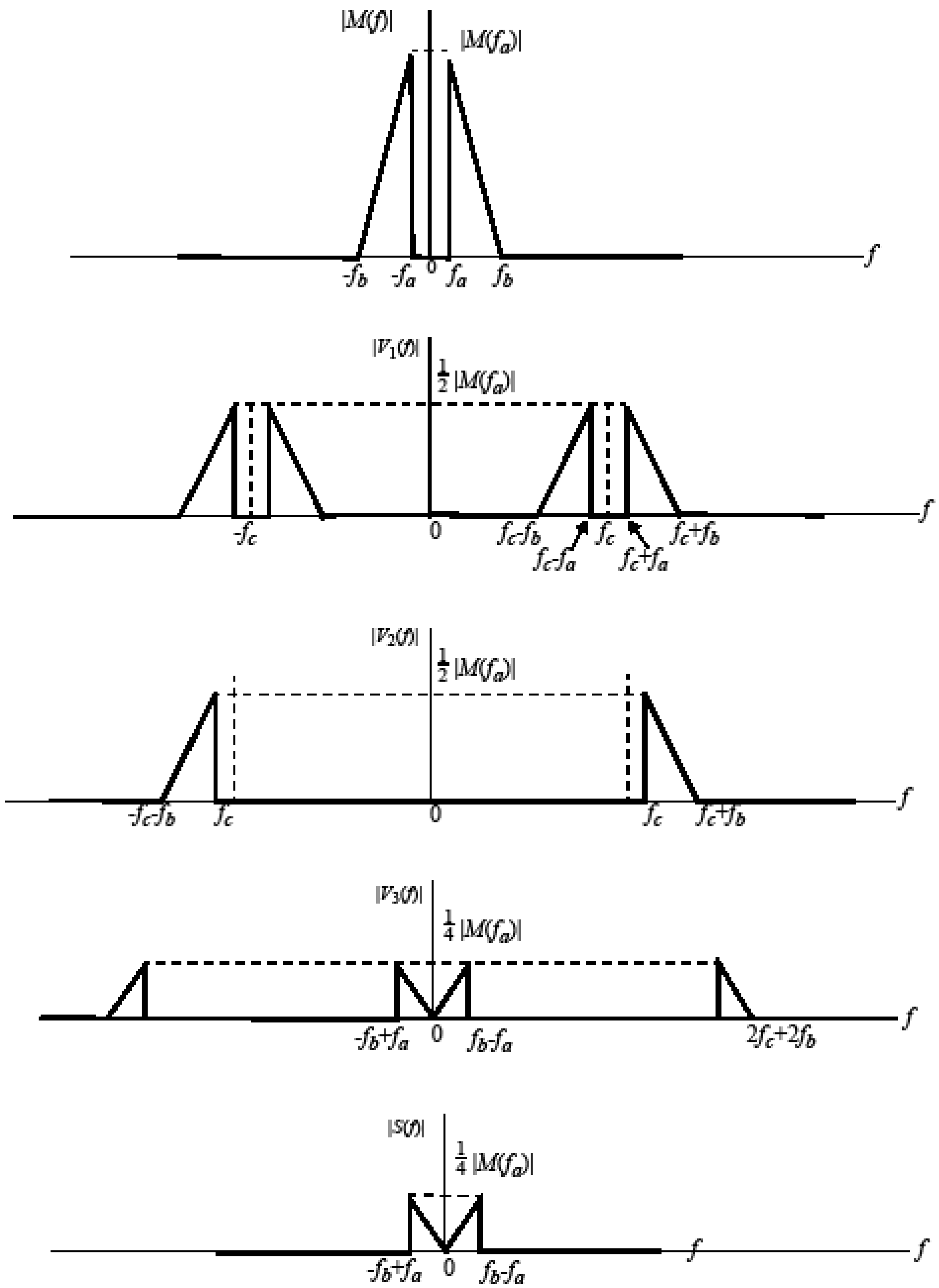
$$M_-(f) = \left\{ \begin{array}{ll} 0, & f > 0 \\ 2M(f), & f < 0 \end{array} \right\}$$

Comparing the spectrum of $s(t)$ with that of $m(t)$, we see that $s(t)$ may be expressed in terms of $m_+(t)$ and $m_-(t)$ as follows:

$$\begin{aligned} s(t) &= \frac{1}{8}m_+(t)\exp(-j2\pi f_b t) + \frac{1}{8}m_-(t)\exp(j2\pi f_b t) \\ &= \frac{1}{8}[m(t) + j\hat{m}(t)]\exp(-j2\pi f_b t) + \frac{1}{8}[m(t) - j\hat{m}(t)]\exp(j2\pi f_b t) \\ &= \frac{1}{4}m(t)\cos(2\pi f_b t) + \frac{1}{4}\hat{m}(t)\sin(2\pi f_b t) \end{aligned}$$

(b) With $s(t)$ as input, the first product modulator output is

$$v_1(t) = s(t)\cos(2\pi f_c t)$$


|M(f)|
|M(fa)|
-fb -fa 0 fa fb
f
|V1(f)|
1/2 |M(fa)|
-fc
0
fc-fb fc fc+fb
fc-fa fc+fa
f
|V2(f)|
1/2 |M(fa)|
-fc-fb fc 0 fc fc+fb
f
|V3(f)|
1/4 |M(fa)|
-fb+fa 0 fb-fa
2fc+2fb
f
|S(f)|
1/4 |M(fa)|
-fb+fa 0 fb-fa
f
f

3.22. $f_1 = f_c - \Delta f - W$

$f_2 = f_c + \Delta f$

$$v_1(t)v_2(t) = A_1A_2\cos(2\pi f_1t + \phi_1)\cos(2\pi f_2t + \phi_2)$$
$$= \frac{A_1A_2}{2}[\cos(2\pi(f_1 - f_2)t + \phi_1 - \phi_2) + \cos(2\pi(f_1 + f_2)t + \phi_1 + \phi_2)]$$

The low-pass filter will only pass the first term.

$$\therefore LFP(v_1(t)v_2(t)) = \frac{1}{2}A_1A_2[\cos(-2\pi(W + 2\Delta f)t + \phi_1 - \phi_2)]$$

Let $v_0(t)$ be the final output, before band-pass filtering.

$$v_o(t) = \frac{1}{2}A_1A_2[\cos(-2\pi\left(\frac{W + 2\Delta f}{W/\Delta f + 2}\right)t + \frac{\phi_1 - \phi_2}{W/\Delta f + 2}) \cdot A_2\cos(2\pi f_2t + \phi_2)]$$
$$= \frac{1}{2}A_1A_2^2[\cos(-2\pi\Delta ft + \frac{\phi_1 - \phi_2}{n+2} - \phi_2) \cdot \cos(2\pi f_2t + \frac{\phi_1 - \phi_2}{n+2} + \phi_2)]$$
$$= \frac{1}{4}A_1A_2^2[\cos(-2\pi(f_c + 2\Delta f) + \frac{\phi_1 - \phi_2}{n+2} - \phi_2) + \cos(-2\pi f_ct + \frac{\phi_1 - \phi_2}{n+2} + \phi_2)]$$

After band-pass filtering, retain only the second term.

$$\therefore v_o(t) = \frac{1}{4}A_1A_2^2[\cos(-2\pi f_ct + \frac{\phi_1 - \phi_2}{n+2} + \phi_2)$$

$$\frac{\phi_1}{n+2} - \frac{\phi_2}{n+2} + \phi_2 = 0$$

rearranging and solving for ϕ_2 :

$$\phi_2 = -\frac{\phi_1}{n+1}$$

(b) At the second multiplier, replace $v_2(t)$ with $v_1(t)$. This results in the following expression for the phase:

$$\frac{\phi_1}{n+2} - \frac{\phi_2}{n+2} + \phi_1 = 0$$

$$\phi_1 = \frac{\phi_2}{n+3}$$

3.23. Assume that the mixer performs a multiplication of the two signals.

$$y_1(t) \in \{1,2,3,4,5,6,7,8,9\} \text{ MHz}$$
$$y_2(t) \in \{100,200,300,400,500,600,700,800,900\} \text{ kHz}$$

This system essentially produces a DSB-SC signal centred around the frequency of $y_1(t)$.

The lowest frequencies that can be produced are:

$$y_o(t) = \frac{1}{2}[\cos(2\pi(f_1 - f_2)t) + \cos(2\pi(f_1 + f_2)t)]$$
$$f_1 = 1 \text{ MHz} \qquad f_1 - f_2 = 0.9 \text{ MHz}$$
$$f_2 = 100 \text{ kHz} \qquad f_1 + f_2 = 1.1 \text{ MHz}$$

The highest frequencies that can be produced are:

$$f_1 = 9 \text{ MHz} \qquad f_1 - f_2 = 8.1 \text{ MHz}$$
$$f_2 = 900 \text{ kHz} \qquad f_1 + f_2 = 9.9 \text{ MHz}$$

The resolution of the system is the bandwidth of the output signal. Assuming that no branch can be zeroed, the narrowest resolution occurs with a modulation frequency of 100 kHz. The widest bandwidth occurs when there is a modulation frequency of 900 kHz.

3.24 Given the presence of the filters, only the baseband signals need to be considered. All of the other product components can be discarded.

(a) Given the sum of the modulated carrier waves, the individual message signals are extracted by multiplying the signal with the required carrier.

For m_1(t), this results in the conditions:

$$\cos(\alpha_1)+\cos(\beta_1)=0$$
$$\cos(\alpha_2)+\cos(\beta_2)=0$$
$$\cos(\alpha_3)+\cos(\beta_3)=0$$
$$\therefore \alpha_i=\beta_i\pm\pi$$

For the other signals:

$m_2(t)$:

$$\cos(-\alpha_1)+\cos(-\beta_1)=0 \qquad \alpha_1=\beta_1\pm\pi$$
$$\cos(\alpha_2-\alpha_1)+\cos(\beta_2-\beta_1)=0 \qquad (\alpha_2-\alpha_1)=(\beta_2-\beta_1)\pm\pi$$
$$\cos(\alpha_3-\alpha_1)+\cos(\beta_3-\beta_1)=0 \qquad (\alpha_3-\alpha_1)=(\beta_3-\beta_1)\pm\pi$$

Similarly:

$m_3(t)$:

$$(\alpha_1-\alpha_2)=(\beta_1-\beta_2)\pm\pi$$
$$(\alpha_3-\alpha_2)=(\beta_3-\beta_2)\pm\pi$$

$m_4(t)$:

$$(\alpha_1-\alpha_3)=(\beta_1-\beta_3)\pm\pi$$
$$(\alpha_2-\alpha_3)=(\beta_2-\beta_3)\pm\pi$$

(b) Given that the maximum bandwidth of m_i(t) is W, then the separation between f_a and f_b must be $|f_a - f_b| > 2W$ in order to account for the modulated components corresponding to $f_a - f_b$.

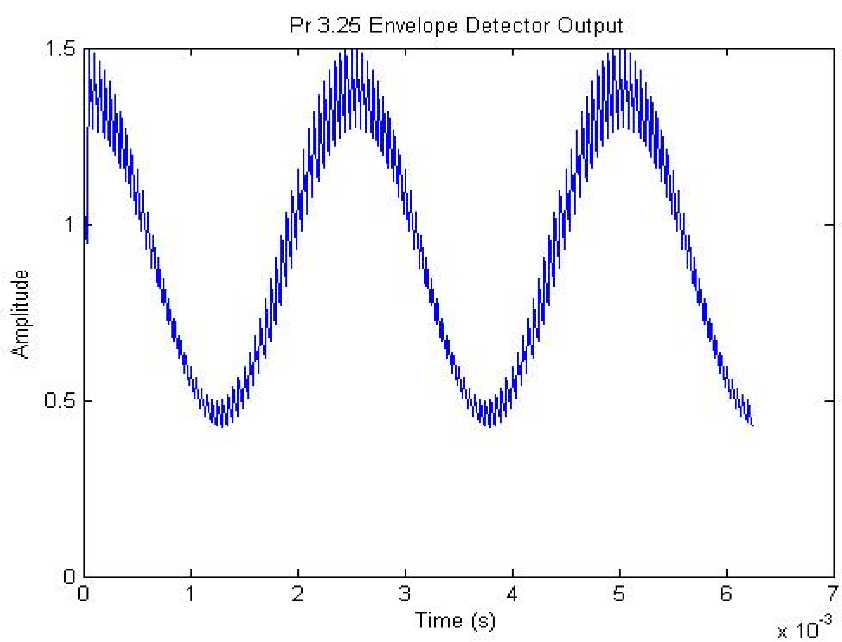
3.25 b) The charging time constant is $(r_f + R_s)C = 1\mu s$

The period of the carrier wave is $1/f_c = 50\ \mu s$.
The period of the modulating wave is $1/f_m = 0.025$ s.
$\therefore$ The time constant is much shorter than the modulating wave and therefore should track the message signal very well.

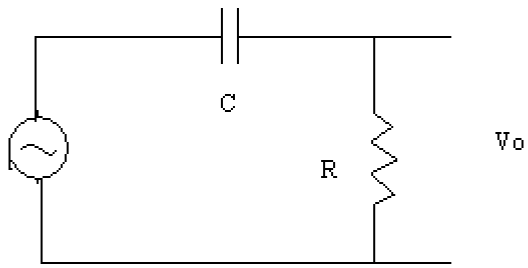
The discharge time constant is: $R_l C = 100\mu s$. This is twice the period of the carrier wave, and should provide some smoothing capability.

From a maximum voltage of V_0, the voltage V_c across the capacitor after time $t = T_s$ is:
$$V_c = V_0 \exp(-\frac{T_s}{R_l C})$$
Using a Taylor series expansion and retaining only the linear terms, will result in the linear approximation of $V_C = V_0(1 - \frac{T_s}{R_l C})$. Using this approximation, the voltage will decay by a factor of 0.94 from its initial value after a period of T_s seconds.

From the code, it can be seen that the voltage decay is close to this figure. However, it is somewhat slower than what was calculated using the linear approximation. In a real circuit, it would also be expected that the decay would be slower, as the voltage does not simply turn off, but rather decreases over time.



3.25 c)



The output of a high-pass RC circuit can be described according to:

$$V_0(t) = I(t)R$$

$$Q_c(t) = C(V_{in}(t) - V_0(t))$$

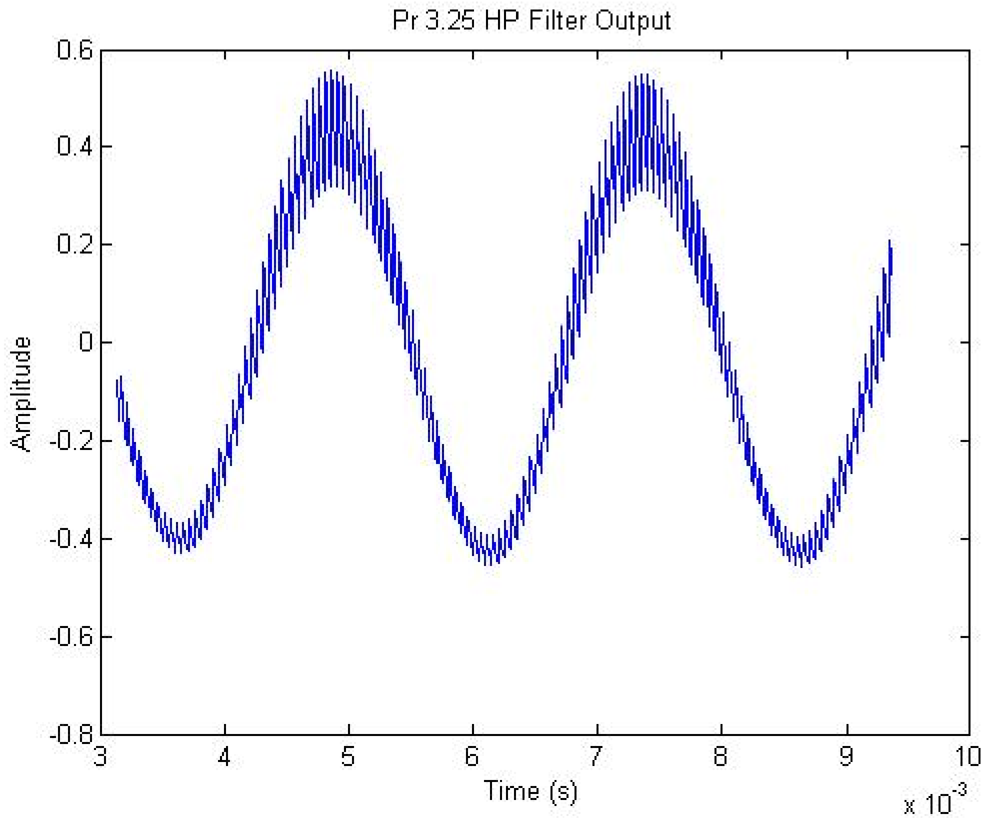
$$I(t) \quad = \frac{dQ_c}{dt}$$

$$V_0(t) = RC\left(\frac{dV_{in}(t)}{dt} - \frac{dV_0(t)}{dt}\right)$$

Using first order differences to approximate the derivatives results in the following difference equation:

$$V_0(t) = \frac{RC}{RC + T_s} V_0(t-1) + \frac{RC}{RC + T_s}(V_{in}(t) - V_{in}(t-1))$$

The high-pass filter applied to the envelope detector eliminates the DC component.



Problem 3.25. MATLAB code

```
function [y,t,Vc,Vo]=AM_wave(fc,fm,mi)

%Problem 3.25
%Inputs:    fc   Carrier Frequency
%           fm   Modulation Frequency
%           mi   modulation index

%Problem 3.25 (a)
fs=160000;    %sampling rate
deltaT=1/fs;  %sampling period

t=linspace(0,.1,.1/deltaT); %Create the list of time periods
y=(1+mi*cos(2*pi*fm*t)).*cos(2*pi*fc*t); %Create the AM wave

%Problem 3.25 (b)
%%%%Create the envelope detector%%%%

Vc=zeros(1,length(y));
Vc(1)=0; %inital voltage

for k=2:length(y)
    if (y(k)>(Vc(k-1)))
        Vc(k)=y(k);
    else
```

```
        Vc(k)=Vc(k-1)-0.023*Vc(k-1);
    end
end


%Problem 3.25 (c)
%%%Implement the high pass filter%%%
%%This implements bias removal
Vo=zeros(1,length(y));
Vo(1)=0;
RC=.001;
beta=RC/(RC+deltaT);

for k=2:length(y)
    Vo(k)=beta*Vo(k-1)+beta*(Vc(k)-Vc(k-1));

end
```

Chapter 4 Problems

Problem 4.1

For the PM case,

$$s(t) = A_c \cos[2\pi f_c t + k_p m(t)].$$

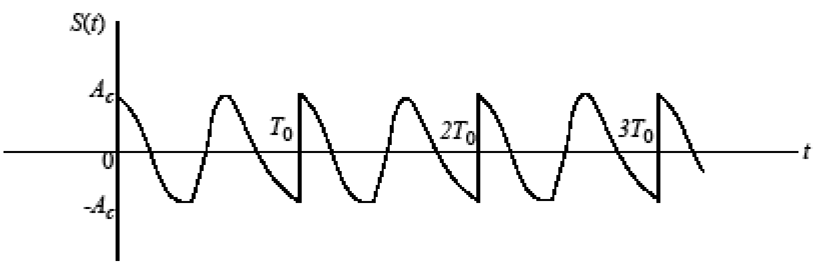
The angle equals

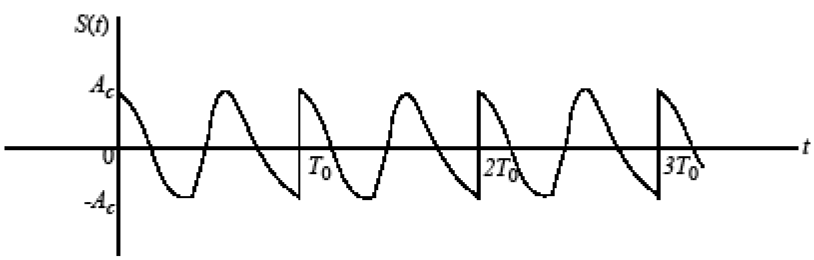
$$\theta_i(t) = 2\pi f_c t + k_p m(t).$$

The instantaneous frequency,

$$f_i(t) = f_c + \frac{Ak_p}{2\pi T_0} - \sum_n \frac{Ak_p}{2\pi}\delta(t - nT_0) \quad ,$$

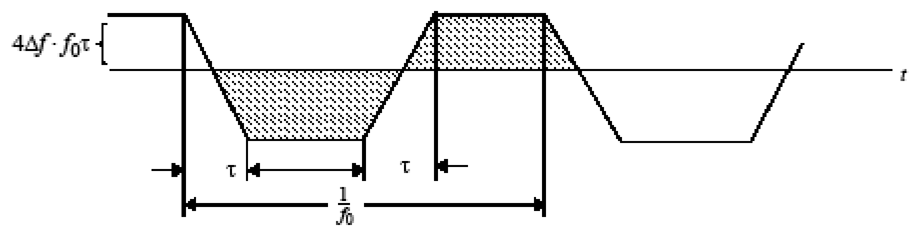
is equal to $f_c + Ak_p/2\pi T_0$ except for the instants that the message signal has discontinuities. At these instants, the phase shifts by $-k_pA/T_0$ radians.





Problem 4.2

The instantaneous frequency of the mixer output is as shown below:



The presence of negative frequency merely indicates that the phasor representing the difference frequency at the mixer output has reversed its direction of rotation.

Let N denote the number of beat cycles in one period. Then, noting that N is equal to the shaded area shown above, we deduce that

$$N = 2\left[4\Delta f \cdot f_0\tau\left(\frac{1}{2f_0} - \tau\right) + 2\Delta f \cdot f_0\tau^2\right]$$

$$= 4\Delta f \cdot \tau(1 - f_0\tau)$$

Since $f_0\tau << 1$, we have

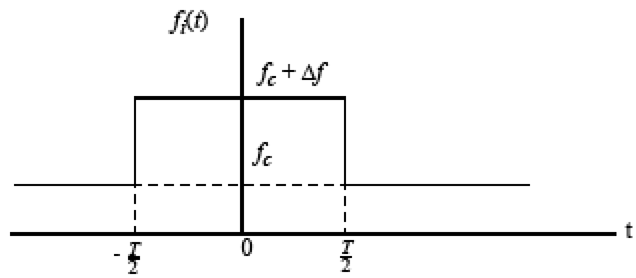
$$N \approx 4\Delta f \cdot \tau$$

Therefore, the number of beat cycles counted over one second is equal to

$$\frac{N}{1/f_0} = 4\Delta f \cdot f_0\tau.$$

Problem 4.3

The instantaneous frequency of the modulated wave $s(t)$ is as shown below:



We may thus express $s(t)$ as follows

$$s(t) = \begin{cases} \cos(2\pi f_c t), & t < -\frac{T}{2} \\ \cos[2\pi(f_c + \Delta f)t], & -\frac{T}{2} \le t \le \frac{T}{2} \\ \cos(2\pi f_c t), & \frac{T}{2} < t \end{cases}$$

The Fourier transform of $s(t)$ is therefore

$$\begin{aligned} S(f) &= \int_{-\infty}^{-T/2} \cos(2\pi f_c t)\exp(-j2\pi ft)dt \\ &\quad + \int_{-T/2}^{-T/2} \cos[2\pi(f_c + \Delta f)t]\exp(-j2\pi ft)dt \\ &\quad + \int_{T/2}^{-\infty} \cos(2\pi f_c t)\exp(-j2\pi ft)dt \\ &= \int_{-\infty}^{-\infty} \cos(2\pi f_c t)\exp(-j2\pi ft)dt \\ &\quad + \int_{-T/2}^{-T/2} \{\cos[2\pi(f_c + \Delta f)t - \cos(2\pi f_c t)]\}\exp(-j2\pi ft)dt \end{aligned}$$

The second term of Eq. (1) is recognized as the difference between the Fourier transforms of two RF pulses of unit amplitude, one having a frequency equal to $f_c + \Delta f$ and the other having a frequency equal to f_c. Hence, assuming that $f_c T >> 1$, we may express $S(f)$ as follows:

$$S(f) \approx \begin{cases} \frac{1}{2}\delta(f - f_c) + \frac{T}{2}\text{sinc}\,[T(f - f_c - \Delta f)] - \frac{T}{2}\text{sinc}\,[T(f - f_c)], & f > 0 \\ \frac{1}{2}\delta(f + f_c) + \frac{T}{2}\text{sinc}\,[T(f + f_c + \Delta f)] - \frac{T}{2}\text{sinc}\,[T(f + f_c)], & f < 0 \end{cases}$$

Problem 4.4

(a) The envelope of the FM wave $s(t)$ is

$$a(t) = A_c\sqrt{1 + \beta^2 \sin^2(2\pi f_m t)}$$

The maximum value of the envelope is

$$a_{\max} = A_c\sqrt{1 + \beta^2}$$

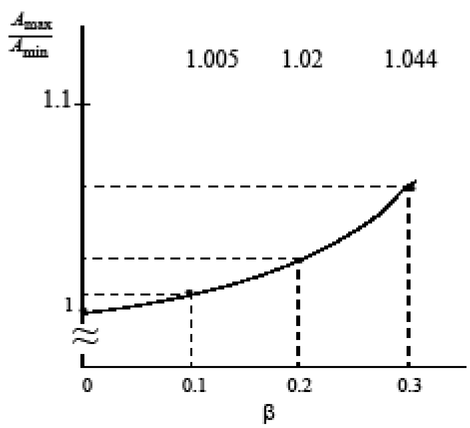
and its minimum value is

$$a_{\max} = A_c$$

Therefore,

$$\frac{a_{\max}}{a_{\min}} = \sqrt{1+\beta^2}$$

This ratio is shown plotted below for $0 < \beta < 0.3$:



(b) Expressing $s(t)$ in terms of its frequency components:

$$s(t) = A_c\cos(2\pi f_c t) + \frac{1}{2}\beta A_c\cos[2\pi(f_c+f_m)t] - \frac{1}{2}\beta A_c\cos[2\pi(f_c-f_m)t]$$

The mean power of $s(t)$ is therefore

$$P_1 = \frac{A_c^2}{2} + \frac{\beta^2 A_c^2}{8} + \frac{\beta^2 A_c^2}{8}$$

$$= \frac{A_c^2}{2}\left(1+\frac{\beta^2}{2}\right)$$

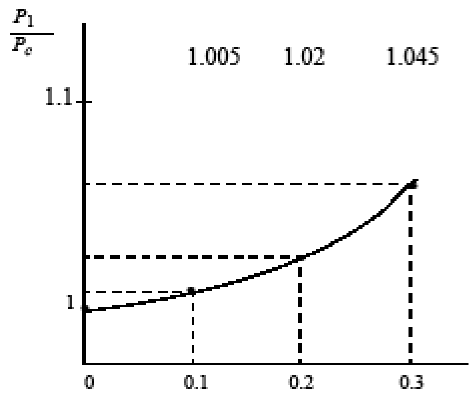
The mean power of the unmodulated carrier is

$$P_c = \frac{A_c^2}{2}$$

Therefore,

$$\frac{P_1}{P_c} = 1 + \frac{\beta^2}{2}$$

which is shown plotted below for $0 \leq \beta \leq 0.3$:



(c) The angle $\theta_i(t)$, expressed in terms of the in-phase component, $s_I(t)$, and the quadrature component $s_Q(t)$, is:

$$\theta_i(t) = 2\pi f_c t + \tan^{-1}\left[\frac{s_I(t)}{s_Q(t)}\right]$$

$$= 2\pi f_c t + \tan^{-1}[\beta \sin(2\pi f_m t)]$$

Since $\tan^{-1}(x) = x - x^3/3 + \ldots$,

$$\theta_i(t) \approx 2\pi f_c t \beta \sin(2\pi f_m t) - \frac{\beta^3}{3}\sin^3(2\pi f_m t)$$

The harmonic distortion is the power ratio of the third and first harmonics:

$$D_h = \left(\frac{\frac{1}{3}\beta^3}{\beta}\right) = \frac{\beta^4}{9}$$

For $\beta = 0.3$, $D_h = 0.09\%$.

Problem 4.5

(a) The phase-modulated wave is

$$s(t) = A_c \cos[2\pi f_c t + k_p A_m \cos(2\pi f_m t)]$$

$$= A_c \cos[2\pi f_c t + \beta_p \cos(2\pi f_m t)]$$

$$= A_c \cos(2\pi f_c t)\cos[\beta_p \cos(2\pi f_m t)] - A_c \sin(2\pi f_c t)\sin[\beta_p \cos(2\pi f_m t)] \qquad (1)$$

If $\beta_p \leq 0.5$, then

$$\cos[\beta_p \cos(2\pi f_m t)] \approx 1$$

$$\sin[\beta_p \cos(2\pi f_m t)] \approx \beta_p \cos(2\pi f_m t)$$

Hence, we may rewrite Eq. (1) as

$$s(t) \approx A_c \cos(2\pi f_c t) - \beta_p A_c \sin(2\pi f_c t)\cos(2\pi f_m t)$$

$$= A_c \cos(2\pi f_c t) - \frac{1}{2}\beta_p A_c \sin[2\pi(f_c + f_m)t]$$

$$-\frac{1}{2}\beta_p A_c \sin[2\pi(f_c - f_m)t] \qquad (2)$$

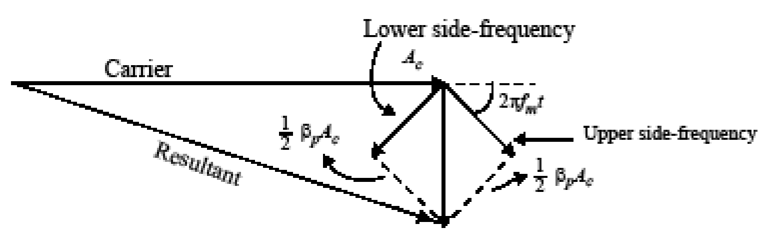
The spectrum of $s(t)$ is therefore

$$S(f) \approx \frac{1}{2}A_c[\delta(f-f_c) + \delta(f+f_c)]$$

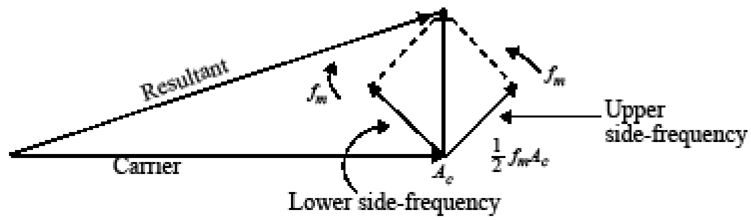
$$-\frac{1}{4j}\beta_p A_c[\delta(f-f_c-f_m) - \delta(f+f_c+f_m)]$$

$$-\frac{1}{4j}\beta_p A_c[\delta(f-f_c+f_m) - \delta(f+f_c-f_m)]$$

(b) The phasor diagram for $s(t)$ is deduced from Eq. (2) to be as follows:



The corresponding phasor diagram for the narrow-band FM wave is as follows:



Comparing these two phasor diagrams, we see that, except for a phase difference, the narrow-band PM and FM waves are of exactly the same form.

Problem 4.6

The phase-modulated wave is

$$s(t) = A_c \cos[2\pi f_c t + \beta_p \cos(2\pi f_m t)]$$

The complex envelope of $s(t)$ is

$$\tilde{s}(t) = A_c \exp[j\beta_p \cos(2\pi f_m t)]$$

Expressing $\tilde{s}(t)$ in the form of a complex Fourier series, we have

$$\tilde{s}(t) = \sum_{n=-\infty}^{\infty} c_n \exp(j2\pi n f_m t)$$

where

$$c_n = f_m \int_{-1/2f_m}^{1/2f_m} \tilde{s}(t) \exp(-2\pi n f_m t) dt$$

$$= A_c f_m \int_{-1/2f_m}^{1/2f_m} \exp[j\beta_p \cos(2\pi f_m t) - j2\pi n f_m t] dt \qquad (1)$$

Let $2\pi f_m t = \pi/2 - \phi$.

Then, we may rewrite Eq. (1) as

$$c_n = -\frac{A_c}{2\pi} \exp\left(-\frac{jn\pi}{2}\right) \int_{3\pi/2}^{-\pi/2} \exp[j\beta_p \sin(\phi) + jn\phi] d\phi$$

The integrand is periodic with respect to ϕ with a period of 2π. Hence, we may rewrite this expression as

$$c_n = \frac{A_c}{2\pi}\exp\left(-\frac{jn\pi}{2}\right)\int_{-\pi}^{-\pi}\exp[j\beta_p\sin(\phi)+jn\phi]d\phi$$

However, from the definition of the Bessel function of the first kind of order n, we have

$$J_n(x) = \frac{1}{2\pi}\int_{-\pi}^{-\pi}\exp(jx\sin\phi - jn\phi)d\phi$$

Therefore,

$$c_n = A_c\exp\left(-\frac{jn\pi}{2}\right)J_{-n}(\beta_p)$$

We may thus express the PM wave $s(t)$ as

$$s(t) = \mathrm{Re}[\tilde{s}(t)\exp(j2\pi f_c t)]$$

$$= A_c\mathrm{Re}\left[\sum_{n=-\infty}^{\infty}J_{-n}(\beta_p)\exp\left(-\frac{jn\pi}{2}\right)\exp(j2\pi nf_m t)\exp(j2\pi nf_c t)\right]$$

$$= A_c\sum_{n=-\infty}^{\infty}J_{-n}(\beta_p)\cos\left[2\pi(f_c+nf_m)t-\frac{n\pi}{2}\right]$$

The band-pass filter only passes the carrier, the first upper side-frequency, and the first lower side-frequency, so that the resulting output is

$$s_o(t) = A_cJ_0(\beta_p)\cos(2\pi f_c t)+A_cJ_{-1}(\beta_p)\cos\left[2\pi(f_c+f_m)t-\frac{\pi}{2}\right]$$

$$+A_cJ_1(\beta_p)\cos\left[2\pi(f_c-f_m)t+\frac{\pi}{2}\right]$$

$$= A_cJ_0(\beta_p)\cos(2\pi f_c t)+A_cJ_{-1}(\beta_p)\sin[2\pi(f_c+f_m)t]$$

$$-A_cJ_1(\beta_p)\sin[2\pi(f_c-f_m)t]$$

But

$$J_{-1}(\beta_p) = -J_1(\beta_p)$$

Therefore,

$$s_o(t) = A_cJ_0(\beta_p)\cos(2\pi f_c t)$$

$$-A_cJ_0(\beta_p)\{\sin[2\pi(f_c+f_m)t]+\sin[2\pi(f_c-f_m)t]\}$$

$$= A_cJ_0(\beta_p)\cos(2\pi f_c t)-2A_cJ_1(\beta_p)\cos(2\pi f_m t)\sin(2\pi f_c t)$$

The envelope of $s_o(t)$ equals

$$a(t) = A_c\sqrt{J_0^2(\beta_p) + 4J_1^2(\beta_p)\cos^2(2\pi f_m t)}$$

The phase of $s_o(t)$ is

$$\phi(t) = -\tan^{-1}\left[\frac{J_1(\beta_p)2}{J_0(\beta_p)}\right]\cos(2\pi f_m t)$$

The instantaneous frequency of $s_o(t)$ is

$$f_i(t) = f_c + \frac{1}{2\pi}\ \frac{d\phi(t)}{dt}$$

$$= f_c + \frac{2J_0(\beta_p)J_1(\beta_p)\sin(2\pi f_m t)}{J_0^2(\beta_p) + 4J_1^2(\beta_p)\cos^2(2\pi f_m t)}$$

Problem 4.7.

$$s(t) = A_c\cos(\theta(t))$$
$$\theta(t) = 2\pi f_c t + k_p m(t)$$

Let $\beta = 0.3$ for $m(t) = \cos(2\pi f_m t)$.

$$\therefore\ s(t) = A_c\cos(2\pi f_c t + \beta m(t))$$
$$= A_c[\cos(2\pi f_c t)\cos(\beta\cos(2\pi f_m t)) - \sin(2\pi f_c t)\sin(\beta\cos(2\pi f_m t))]$$

for small β:

$$\cos(\beta\cos(2\pi f_m t)) \simeq 1$$
$$\sin(\beta\sin(2\pi f_m t)) \simeq \beta\cos(2\pi f_m t)$$

$$\therefore\ s(t) = A_c\cos(2\pi f_c t) - \beta A_c\sin(2\pi f_c t)\cos(2\pi fmt)$$
$$= A_c\cos(2\pi f_c t) - \beta\frac{A_c}{2}[\sin(2\pi(f_c + f_m)t) + \sin(2\pi(f_c + f_m)t)$$

Problem 4.8

(a) From Table 4.1, we find (by interpolation) that $J_0(\beta)$ is zero for

$\beta = 2.44$,
$\beta = 5.52$,
$\beta = 8.65$,
$\beta = 11.8$,

and so on.

(b) The modulation index is

$$\beta = \frac{\Delta f}{f_m} = \frac{k_f A_m}{f_m}$$

Therefore,

$$k_f = \frac{\beta f_m}{A_m}$$

Since $J_0(\beta) = 0$ for the first time when $\beta = 2.44$, we deduce that

$$k_f = \frac{2.44 \times 10^3}{2}$$

$$= 1.22 \times 10^3 \text{ hertz/volt}$$

Next, we note that $J_0(\beta) = 0$ for the second time when $\beta = 5.52$. Hence, the corresponding value of A_m for which the carrier component is reduced to zero is

$$A_m = \frac{\beta f_m}{k_f}$$

$$= \frac{5.52 \times 10^3}{1.22 \times 10^3}$$

$$= 4.52 \text{ volts}$$

Problem 4.9

For $\beta = 1$, we have

$J_0(1) = 0.765$
$J_1(1) = 0.44$
$J_2(1) = 0.115$

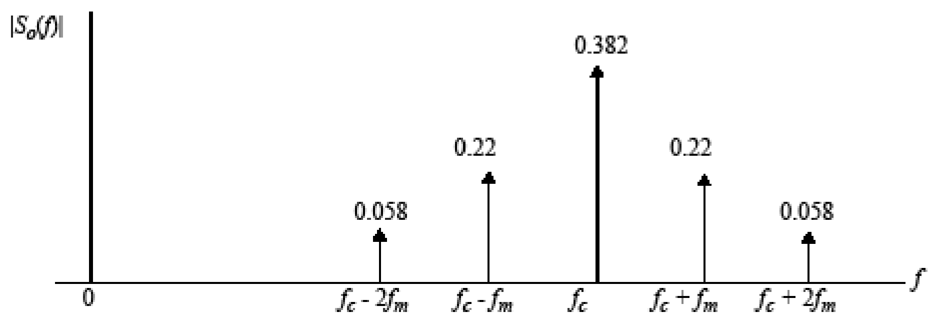
Therefore, the band-pass filter output is (assuming a carrier amplitude of 1 volt)

$$s_o(t) = 0.765\cos(2\pi f_c t)$$

$$+ 0.44\{\cos[2\pi(f_c + f_m)t] - \cos[2\pi(f_c - f_m)t]\}$$

$$+ 0.115\{\cos[2\pi(f_c + f_m)t] + \cos[2\pi(f_c - 2f_m)t]\},$$

and the amplitude spectrum (for positive frequencies) is



Problem 4.10

(a) The frequency deviation is

$$\Delta f = k_f A_m = 25 \times 10^3 \times 20 = 5 \times 10^5 \text{Hz}$$

The corresponding value of the modulation index is

$$\beta = \frac{\Delta f}{f_m} = \frac{5 \times 10^5}{10^5} = 5$$

The transmission bandwidth of the FM wave, using Carson's rule, is therefore

$$B_T = 2f_m(1 + \beta) = 2 \times 100(1 + 5) = 1200\text{kHz} = 1.2\text{MHz}$$

(b) Using the universal curve of Fig. 3.36 we find that for $\beta = 5$:

$$\frac{B_T}{\Delta f} = 3$$

Therefore,

$$B_T = 3 \times 500 = 1500\text{kHz} = 1.5\text{MHz}$$

(c) If the amplitude of the modulating wave is doubled, we find that

$$\Delta f - 1\text{MHz} \quad \text{and} \quad \beta = 10$$

Thus, using Carson's rule we obtain,

$$B_T = 2 \times 100(1 + 10) = 2200\text{kHz} = 2.2\text{MHz}$$

Using the universal curve of Fig. 3.36, we get

$$\frac{B_T}{\Delta f} = 2.75$$

and $B_T = 2.75$ MHz.

(d) If f_m is doubled, $\beta = 2.5$. Then, using Carson's rule, $B_T = 1.4$ MHz. Using the universal curve, $B_T/\Delta f = 4$, and

$$B_T = 4\Delta f = 2\text{MHz}$$

Problem 4.11

(a) The angle of the PM wave is

$$\begin{aligned}\theta_i(t) &= 2\pi f_c t + k_p m(t) \\ &= 2\pi f_c t + k_p A_m \cos(2\pi f_m t) \\ &= 2\pi f_c t + \beta_p \cos(2\pi f_m t)\end{aligned}$$

where $\beta_p = k_p A_m$. The instantaneous frequency of the PM wave is therefore

$$\begin{aligned}f_i(t) &= \frac{1}{2\pi} \frac{d\theta_i(t)}{dt} \\ &= f_c - \beta_p f_m \sin(2\pi f_m t)\end{aligned}$$

We see that the maximum frequency deviation in a PM wave varies linearly with the modulation frequency f_m.

Using Carson's rule, we find that the transmission bandwidth of the PM wave is approximately (for the case when $\beta_p >> 1$)

$$B_T \approx 2(f_m + \beta_p f_m) = 2f_m(1 + \beta_p) \approx 2f_m \beta_p.$$

This shows that B_T varies linearly with f_m.

(b) In an FM wave, the transmission bandwidth B_T is approximately equal to $2\Delta f$, if the modulation index $\beta >> 1$. Therefore, for an FM wave, B_T is effectively independent of the modulation frequency f_m.

Problem 4.12

The filter input is

$$v_1(t) = g(t)s(t)$$

$$= g(t)\cos(2\pi f_c t - \pi k t^2)$$

The complex envelope of $v_1(t)$ is

$$\tilde{v}_1(t) = g(t)\exp(-j\pi k t^2)$$

The impulse response $h(t)$ of the filter is defined in terms of the complex impulse response $\tilde{h}(t)$ as follows

$$h(t) = \mathrm{Re}[\tilde{h}(t)\exp(j2\pi f_c t)]$$

With

$$h(t) = \cos(2\pi f_c t + \pi k t^2),$$

we have

$$\tilde{h}(t) = \exp(j\pi k t^2)$$

The complex envelope of the filter output is therefore

$$\tilde{v}_0(t) = \frac{1}{2}\tilde{h}(t)H\tilde{v}_i(t)$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} g(\tau)\exp(-j\pi kt^2)\exp[j\pi k(t-\tau)]^2 d\tau$$

$$= \frac{1}{2}\exp(j\pi kt^2)\int_{-\infty}^{\infty} g(\tau)\exp(-j\pi kt\tau)d\tau$$

$$= \frac{1}{2}\exp(j\pi kt^2)G(kt)$$

Hence,

$$\tilde{v}_0(t) = \frac{1}{2}|G(kt)|$$

This shows that the envelope of the filter output is, except for scale factor of 1/2, equal to the magnitude of the Fourier transform of the input signal $g(t)$, with kt playing the role of frequency f.

Problem 4.13

The overall frequency multiplication ratio is

$$n = 2 \times 3 = 6$$

Assume that the instantaneous frequency of the FM wave at the input of the first frequency multiplier is

$$f_{i1}(t) = f_c + \Delta f\cos(2\pi f_m t)$$

The instantaneous frequency of the resulting FM wave at the output of the second frequency multiplier is therefore

$$f_{i2}(t) = nf_c + n\Delta f\cos(2\pi f_m t)$$

Thus, the frequency deviation of this FM wave is equal to

$$n\Delta f = 6 \times 100 = 60\text{kHz}$$

and its modulation index is equal to

$$\frac{n\Delta f}{f_m} = \frac{60}{5} = 12$$

The frequency separation of the adjacent side-frequencies of this FM wave is unchanged at f_m = 5 kHz.

Problem 4.14.

$$v_2 = a v_1^2$$

$$\begin{aligned} s(t) &= A_c \cos(2\pi f_c t + \beta \sin(2\pi f_m t)) \\ &= A_c \cos(2\pi f_c t + \beta m(t)) \end{aligned}$$

$$\begin{aligned} v_2 &= a \cdot s^2(t) \\ &= a \cdot \cos^2(2\pi f_c t + \beta m(t)) \\ &= \frac{a}{2} \cdot \cos(4\pi f_c t + 2\beta m(t)) \end{aligned}$$

The square-law device produces a new FM signal centred at $2f_c$ and with a frequency deviation of 2β. This doubles the frequency deviation.

Problem 4.15

(a) Let L denote the inductive component, C the capacitive component, and C_0 the capacitance of each varactor diode due to the bias voltage V_b acting alone. Then we have

$$C_0 = 100 V_b^{-1/2} pF$$

and the corresponding frequency of oscillation is

$$f_0 = \frac{1}{2\pi\sqrt{L(C + C_0/2)}}$$

Therefore,

$$10^6 = \frac{1}{2\pi\sqrt{200 \times 10^{-6}(100 \times 10^{-12} + 50 V_b^{-1/2} \times 10^{-12})}}$$

Solving for V_b, we get

$V_b = 3.52$ volts

(b) The frequency multiplication ratio is 64. Therefore, the modulation index of the FM wave at the frequency multiplier input is

$$\beta = \frac{5}{64} = 0.078$$

This indicates that the FM wave produced by the combination of L, C and the varactor diodes is a narrow-band one, which in turn means that the amplitude A_m of the modulating wave is small compared to V_b. We may thus express the instantaneous frequency of this FM wave as follows:

$$f_i(t) = \frac{1}{2\pi}\left[200 \times 10^{-6}\left\{100 \times 10^{-12} + 50 \times 10^{-12}[3.52 + A_m \sin(2\pi f_c t)]^{1/2}\right\}\right]^{-1/2}$$

$$= \frac{10^7}{2\sqrt{2\pi}}\left\{1 + 0.266\left[1 + \frac{A_m}{3.52}\sin(2\pi f_m t)\right]^{1/2}\right\}^{-1/2}$$

$$\approx \frac{10^7}{2\sqrt{2\pi}}\left\{1 + 0.266\left[1 - \frac{A_m}{7.04}\sin(2\pi f_m t)\right]\right\}^{-1/2}$$

$$= 10^6[1 - 0.03A_m \sin(2\pi f_m t)]^{-1/2}$$

$$\approx 10^6[1 + 0.015A_m \sin(2\pi f_m t)]$$

With a modulation index of 0.078, the corresponding value of the frequency deviation is

$$\Delta f = \beta f_m$$

$$= 0.078 \times 10^4 \text{Hz}$$

Therefore,

$$0.015A_m \times 10^6 = 0.078 \times 10^4$$

where A_m is in volts. Solving for A_m, we get

$$A_m = 52 \times 10^{-3} \text{ volts.}$$

Problem 4.16

The transfer function of the RC filter is

$$H(f) = \frac{j2\pi fCR}{1+j2\pi fCR}$$

If $2\pi fCR << 1$ for all frequences of interest, then we may approximate $H(f)$ as

$$H(f) \approx j2\pi fCR$$

However, multiplication by $j2\pi f$ in the frequency domain is equivalent to differentiation in the time domain. Therefore, denoting the RC filter output as $v_1(t)$, we may write

$$v_1(t) \approx CR\frac{ds(t)}{dt}$$

$$= CR\frac{d}{dt}\left\{A_c \cos\left[2\pi f_c t + 2\pi k_f \int_0^t m(t)dt\right]\right\}$$

$$= -CRA_c[2\pi f_c + 2\pi k_f m(t)]\sin\left[2\pi f_c t + 2\pi k_f \int_0^t m(t)dt\right]$$

The corresponding envelope detector output is

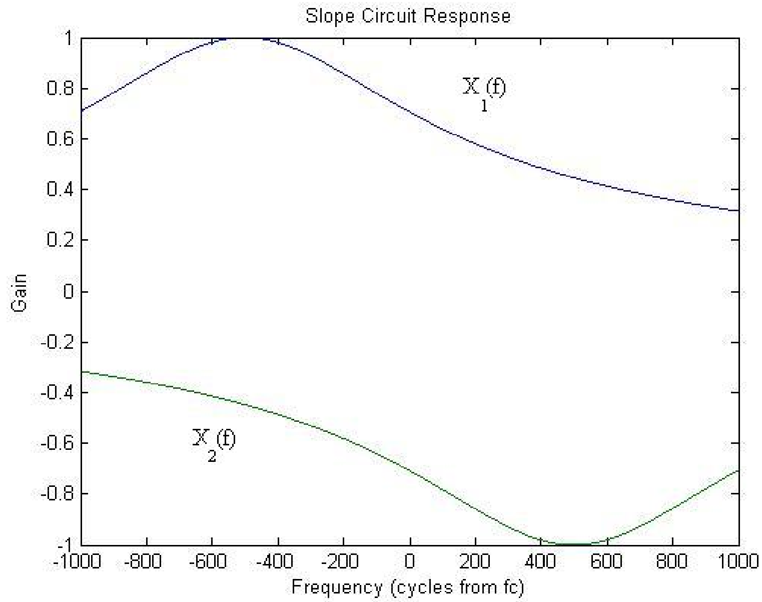
$$v_2(t) \approx 2\pi f_c CRA_c\left|1 + \frac{k_f}{f_c}m(t)\right|$$

Since $k_f|m(t)| < f_c$ for all t, then

$$v_2(t) \approx 2\pi f_c CRA_c\left|1 + \frac{k_f}{f_c}m(t)\right|$$

which shows that, except for a dc bias, the output is proportional to the modulating signal $m(t)$.

4.17. Consider the slope circuit response:



The response of $|X_1(f)|$ after the resonant peak is the same as for a single pole low-pass filter. From a table of Bode plots, the following gain response can be obtained:

$$|X_1(f)| = \frac{1}{\sqrt{1+\left(\frac{f-f_B}{B}\right)^2}}$$

Where f_B is the frequency of the resonant peak, and B is the bandwidth.

For the slope circuit, B is the filter's bandwidth or cutoff frequency. For convenience, we can shift the filter to the origin (with $\tilde{X}_1(f)$ as the shifted version).

$$|\tilde{X}_1(f)| = \frac{1}{\sqrt{1+\left(\frac{f}{B}\right)^2}}$$

$$\left.\frac{d|\tilde{X}_1(f)|}{df}\right|_{f=kB} = -\frac{k}{B(1+k^2)^{\frac{3}{2}}}$$

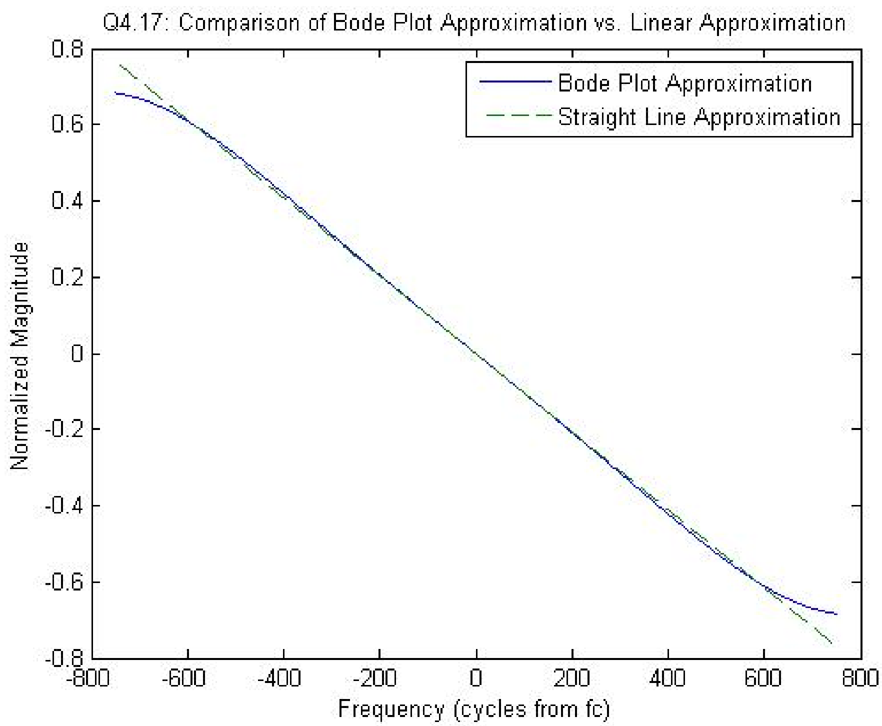
Because the filters are symmetric about the central frequency, the contribution of the second filter is identical. Adding the filter responses results in the slope at the central frequency being:

$$\left.\frac{d\,|\tilde{X}(f)|}{df}\right|_{f=kB} = -\frac{2k}{B(1+k^2)^{\frac{3}{2}}}$$

In the original definition of the slope filter, the responses are multiplied by -1, so do this here. This results in a total slope of:

$$\frac{2k}{B(1+k^2)^{\frac{3}{2}}}$$

As can be seen from the following plot, the linear approximation is very accurate between the two resonant peaks. For this plot B = 500, f_1=-750, and f_2=750.



Problem 2.18

The envelope detector input is

$$v(t) = s(t) - s(t-T)$$

$$= A_c\cos[2\pi f_c t + \phi(t)] - A_c\cos[2\pi f_c(t-T) + \phi(t-T)]$$

$$= -2A_c\sin\left[\frac{2\pi f_c(t-T) + \phi(t) + \phi(t-T)}{2}\right]\sin\left[\frac{2\pi f_c T + \phi(t) - \phi(t-T)}{2}\right] \qquad (1)$$

where

$$\phi(t) = \beta\sin(2\pi f_m t)$$

The phase difference $\phi(t)$ - $\phi(t - T)$ is

$$\phi(t) - \phi(t-T) = \beta\sin(2\pi f_m t) - \beta\sin[2\pi f_m(t-T)]$$

$$= \beta[\sin(2\pi f_m t) - \beta\sin 2\pi f_m(t+\quad)\cos(2\pi f_m T) + \cos(2\pi f_c t)\sin(2\pi f_m T)]$$

$$\approx \beta[\sin(2\pi f_m t) - \sin(2\pi f_m t) + 2\pi f_m T\cos(2\pi f_m t)]$$

$$= 2\pi\Delta f T\cos(2\pi f_m t)$$

where

$$\Delta f = \beta f_m.$$

Therefore, noting that $2\pi f_c T = \pi/2$, we may write

$$\sin\left[\frac{2\pi f_c T + \phi(t) - \phi(t-T)}{2}\right] \approx \sin[\pi f_c T + \pi\Delta f T\cos(2\pi f_m t)]$$

$$= \sin\left[\frac{\pi}{4} + \pi\Delta f T\cos(2\pi f_m t)\right]$$

$$= \sqrt{2}\cos[\pi\Delta f T\cos(2\pi f_m t)] + \sqrt{2}\sin[\pi\Delta f T\cos(2\pi f_m t)]$$

$$= \sqrt{2} + \sqrt{2}\pi\Delta f T\cos(2\pi f_m t)$$

where we have made use of the fact that $\pi\Delta f T << 1$. We may therefore rewrite Eq. (1) as

$$v(t) \approx -2\sqrt{2}A_c[1 + \pi\Delta f T\cos(2\pi f_m t)]\sin\left[\pi f_c(2t-T) + \frac{\phi(t) + \phi(t-T)}{2}\right]$$

Accordingly, the envelope detector output is

$$a(t) \approx 2\sqrt{2}A_c[1 + \pi\Delta f T\cos(2\pi f_m t)]$$

which, except for a bias term, is proportional to the modulating wave.

Problem 4.19

(a) In the time interval t - (T_1/2) to t + (T_1/2), assume there are n zero crossings. The phase difference is $\theta_i(t+T_1/2)-\theta_i(t-T_1/2) = n\pi$. Also, the angle of an FM wave is

$$\theta_i(t) = 2\pi f_c t + 2\pi k_f \int_0^t m(t)dt.$$

Since $m(t)$ is assumed constant, equal to m_1, $\theta_i(t) = 2\pi f_c t + 2\pi k_f m_1 t$. Therefore,

$$\theta_i(t+T_1/2)-\theta_i(t-T_1/2) = (2\pi f_c + 2\pi k_f m_1)[t+T_1/2-(t-T_1/2)]$$

$$= (2\pi f_c + 2\pi k_f m_1)T_1 .$$

But

$$f_i(t) = \frac{d\theta_i(t)}{dt} = 2\pi f_c + 2\pi k_f m_1 .$$

Thus,

$$\theta_i(t+T_1/2)-\theta_i(t-T_1/2) = f_i(t)T_1 .$$

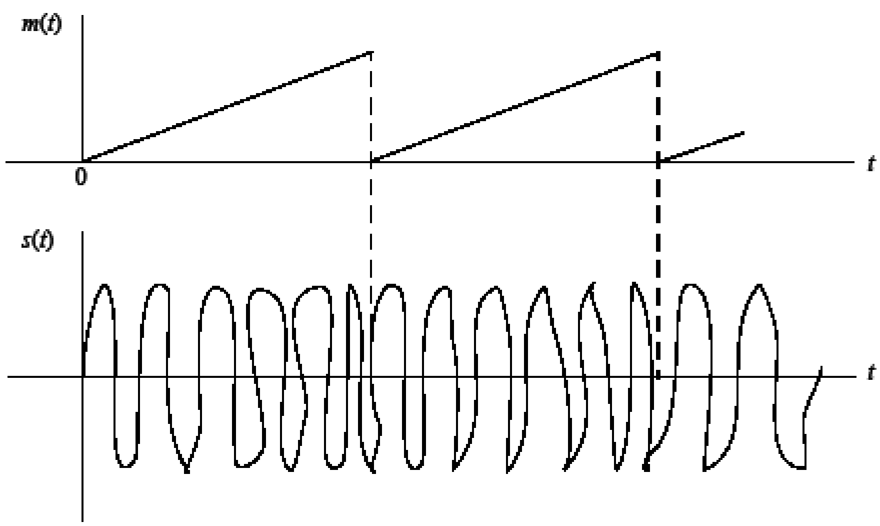
But this phase difference also equals $n\pi$. So,

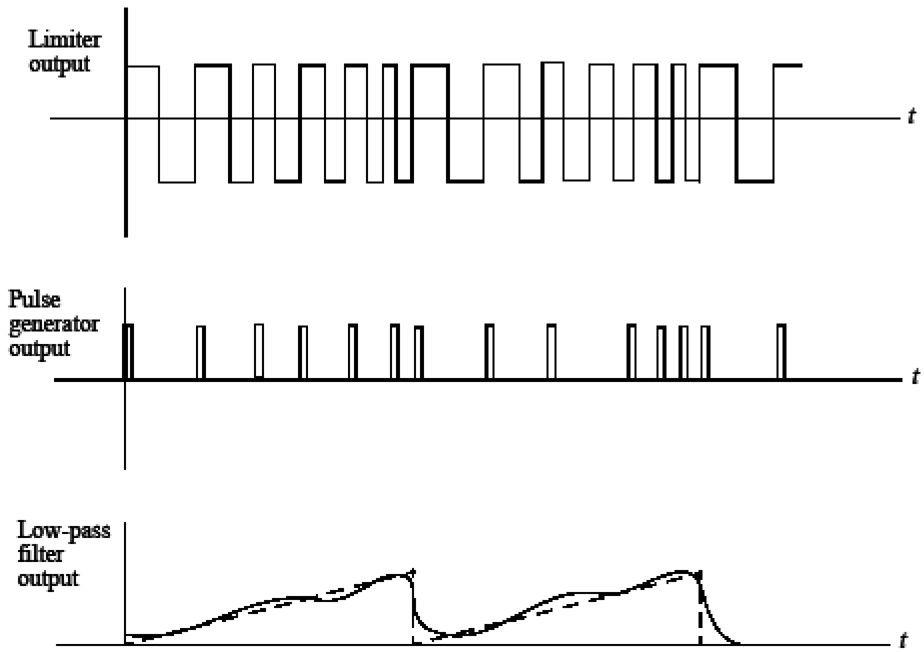
$$f_i(t)T_1 = n\pi$$

and

$$f_i(t) = n\pi/T_1$$

(b) For a repetitive ramp as the modulating wave, we have the following set of waveforms




Limiter output
t
Pulse generator output
t
Low-pass filter output
t

Problem 4.20

The complex envelope of the modulated wave $s(t)$ is

$$\tilde{s}(t) = a(t)\exp[j\phi(t)]$$

Since $a(t)$ is slowly varying compared to $\exp[j\phi(t)]$, the complex envelope $\tilde{s}(t)$ is restricted effectively to the frequency band $-B_T/2 \le f \le B_T/2$. An ideal frequency discriminator consists of a differentiator followed by an envelope detector. The output of the differentiator, in response to $\tilde{s}(t)$, is

$$\tilde{v}_o(t) = \frac{d}{dt}\tilde{s}(t)$$

$$= \frac{d}{dt}\{a(t)\exp[j\phi(t)]\}$$

$$= \frac{da(t)}{dt}\exp\left[j\phi(t) + j\frac{d\phi(t)}{dt}a(t)\exp[j\phi(t)]\right]$$

$$= a(t)\exp[j\phi(t)]\left[\frac{1}{a(t)}\ \frac{da(t)}{dt} + j\frac{j\phi(t)}{dt}\right]$$

Since $a(t)$ is slowly varying compared to $\phi(t)$, we have

$$\left|\frac{j\phi(t)}{dt}\right| >> \left|\frac{1}{a(t)}\ \frac{da(t)}{dt}\right|$$

Accordingly, we may approximate $\tilde{v}_o(t)$ as

$$\tilde{v}_o(t) = ja(t)\frac{d\phi(t)}{dt}\exp[j\phi(t)]$$

However, by definition

$$\phi(t) = 2\pi k_f \int_0^t m(t)dt$$

Therefore,

$$\tilde{v}_o(t) = j2\pi k_f a(t)m(t)\exp[j\phi(t)]$$

Hence, the envelope detector output is proportional to $a(t)m(t)$ as shown by

$$|\tilde{v}_o(t)| \approx 2\pi k_f a(t)m(t)$$

Problem 4.21

(a) The limiter output is

$$z(t) = \text{sgn}\{a(t)\cos[2\pi f_c t + \phi(t)]\}$$

Since $a(t)$ is of positive amplitude, we have

$$z(t) = \text{sgn}\{\cos[2\pi f_c t + \phi(t)]\}$$

Let

$$\psi(t) = 2\pi f_c t = \phi(t)$$

Then, we may write

$$\text{sgn}[\cos\psi] = \sum_{n=-\infty}^{\infty} c_n \exp(jn\psi)$$

$$c_n = \frac{1}{2\pi}\text{sgn}[\cos\psi]\exp(-jn\psi)d\psi$$

$$= \frac{1}{2\pi}\int_{-\pi}^{-\pi/2}(-1)\exp(-jn\psi)d\psi + \frac{1}{2\pi}\int_{-\pi}^{-\pi/2}(+1)\exp(-jn\psi)d\psi$$

$$+ \frac{1}{2\pi}\int_{-\pi/2}^{-\pi}(-1)\exp(-jn\psi)d\psi$$

If $n \neq 0$, then

$$c_n = \frac{1}{2\pi(-jn)}\left[\exp\left(\frac{jn\pi}{2}\right) + \exp(jn\pi) + \left(\frac{-jn\pi}{2}\right) - \exp\left(\frac{jn\pi}{2}\right) - \exp\left(-\frac{jn\pi}{2}\right) + \exp\left(\frac{-jn\pi}{2}\right)\right]$$

$$= \frac{1}{\pi n}\left[2\sin\left(\frac{n\pi}{2}\right) - \sin(n\pi)\right]$$

$$= \begin{cases} \frac{2}{\pi n}(-1)^{(n-1)/2}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

If $n = 0$, we find from Eq. (1) that $c_n = 0$. Therefore,

$$\text{sgn}[\cos\psi] = \frac{2}{\pi}\sum_{\substack{n=-\infty \\ n \text{ odd}}}^{\infty} \frac{1}{n}(-1)^{(n-1)/2}\exp(jn\pi)$$

$$= \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\cos[\psi(2k+1)]$$

We may thus express the limiter output as

$$z(t) = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \cos[2\pi f_c t(2k+1) + \phi(t)(2k+1)] \qquad (2)$$

(b) Consider the term

$$\cos[2\pi f_c t(2k+1) + \phi(t)(2k+1)] = \mathrm{Re}\{\exp j2\pi f_c t(2k+1) \exp[j\phi(t)(2k+1)]\}$$

$$= \mathrm{Re}\left\{\exp j2\pi f_c t(2k+1)[\exp(j\phi(t))]^{2k+1}\right\}$$

The function $\exp[j\phi(t)]$, representing the complex envelope of the FM wave with unit amplitude, is effectively low-pass in nature. Therefore, this term represents a band-pass signal centered about $\pm f_c(2k+1)$. Furthermore, the Fourier transform of $\{\exp[j\phi(t)]\}2k+1$ is equal to that of $\exp[j\phi(t)]^{2k+}$ convolved with itself $2k$ times. Therefore, assuming that $\exp[j\phi(t)]$ is limited to the interval $-B_T/2 \le f \le B_T/2$, we find that $(\exp[j\phi(t)])^{2k+1}$ is limited to the interval $-(B_T/2)(2k+1) \le f \le (B_T/2)(2k+1)$.

Assuming that $f_c > B_T$, as is usually the case, we find that none of the terms corresponding to values of k greater than zero will overlap the spectrum of the term corresponding to $k = 0$. Thus, if the limiter output is applied to a band-pass filter of bandwidth B_T and mid-band frequency f_c, all terms, except the term corresponding to $k = 0$ in Eq. (2), are removed by the filter. The resulting filter output is therefore

$$y(t) = \frac{4}{\pi} \cos[2\pi f_c t + \phi(t)]$$

We thus see that by using the amplitude limiter followed by a band-pass filter, the effect of amplitude variation, represented by $a(t)$ in the modulated wave $s(t)$, is completely removed.

Problem 4.22

Consider an incoming narrow-band signal of bandwidth 10 kHz, and mid-band frequency which may lie in the range 0.535-1.605 MHz. It is required to translate this signal to a fixed frequency band centered at 0.455 MHz. The problem is to determine the range of tuning that must be provided in the local oscillator.

Let f_c denote the mid-band frequency of the incoming signal, and f_l denote the local oscillator frequency. Then we may write

$$0.535 < f_c < 1.605$$

and

$$f_c - f_l = 0.455$$

where both f_c and f_l are expressed in MHz. That is,

$$f_l = f_c - 0.455$$

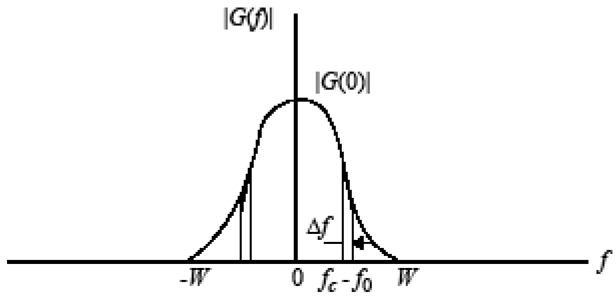
When $f_c = 0.535$ MHz, we fet $f_l = 0.08$ MHz; and when $f_c = 1.605$ MHz, we get $f_l = 1.15$ MHz. Thus the required range of tuning of the local oscillator is 0.08 - 1.15 MHz.

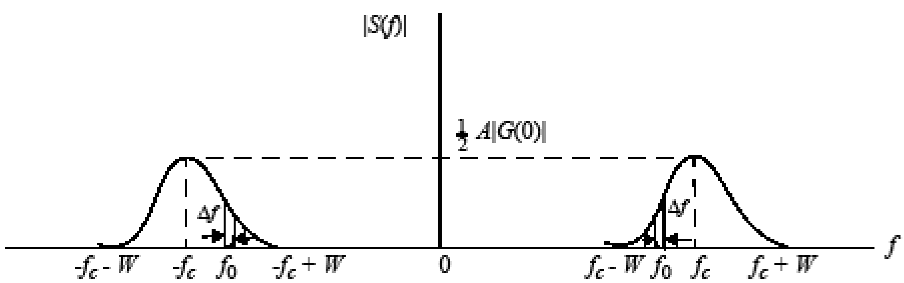
Problem 4. 23

Let $s(t)$ denote the multiplier output, as shown by

$$s(t) = Ag(t)\cos(2\pi f_c t)$$

where f_c lies in the range f_0 to $f_0 + W$. The amplitude spectra of $s(t)$ and $g(t)$ are related as follows:





With $v(t)$ denoting the band-pass filter output, we thus find that the Fourier transform of $v(t)$ is approximately given by

$$V(f) \approx \frac{1}{2}AG(f_c - f_0), \quad f_0 - \frac{\Delta f}{2} \le |f| \le f_0 + \frac{\Delta f}{2}$$

The rms meter output is therefore (by using Rayleigh's energy theorem)

$$\begin{aligned} V_{\text{rms}} &= \left[\int_{-\infty}^{\infty} v^2(t)dt\right]^{1/2} \\ &= \left[\int_{-\infty}^{\infty} |V(f)|^2 df\right]^{1/2} = \left[2\left(\frac{1}{4}A^2|G(f_c - f_o)|^2\right)\Delta f\right]^{1/2} \\ &= \frac{A}{\sqrt{2}}|G(f_c - f_o)|\sqrt{\Delta f} \end{aligned}$$

Problem 4.24

The amplitude spectrum corresponding to the Gaussian pulse

$$p(t) = c\exp\left[-\pi c^2 t^2\right] * rect[t/T]$$

is given by the magnitude of its Fourier transform.

$$\begin{aligned}\left|P(f)\right| &= \left|\mathbf{F}\left[c\exp\left(-\pi c^2 t^2\right)\right]\mathbf{F}\left[rect\left(t/T\right)\right]\right| \\ &= c\exp\left[-\pi\ f^2/c^2\right]\ \left|T\mathrm{sinc}\left[fT\right]\right|\end{aligned}$$

where we have used the convolution theorem

Problem 4.25

The Carson rule bandwidth for GSM is

$$B_T = 2\left(\Delta f + W\right)$$

where the peak deviation is given by

$$\Delta f = \frac{k_f c}{2\pi} = \frac{1}{4}B\sqrt{2\pi/\log(2)} = 0.75B$$

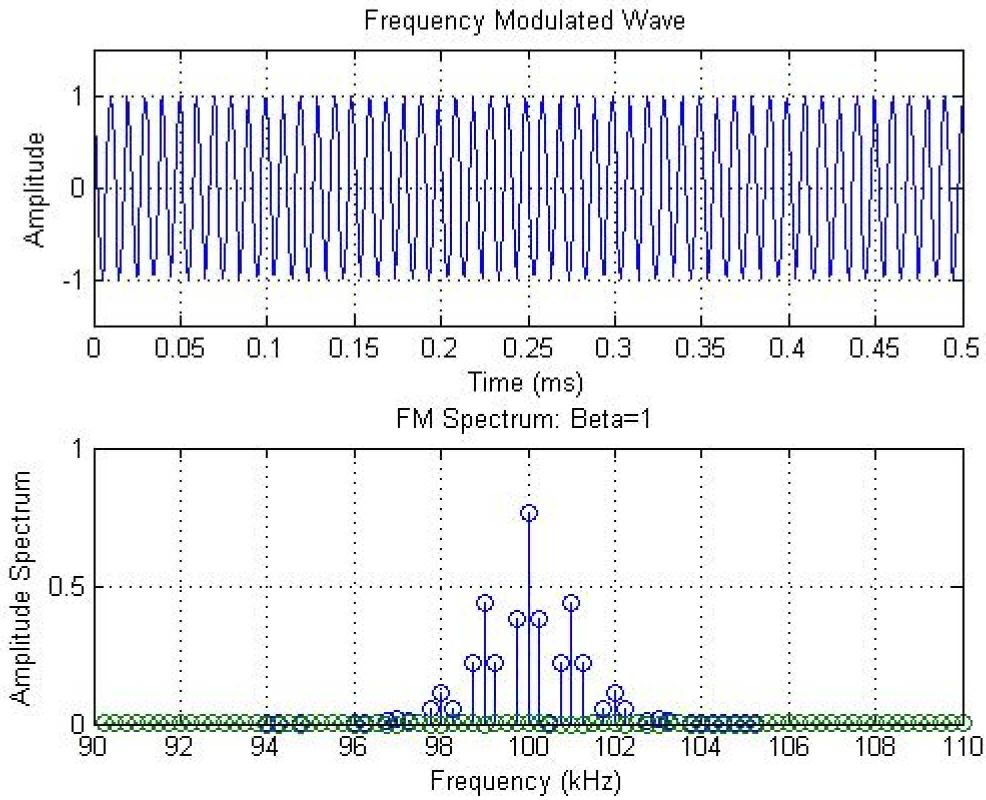
With $BT = 0.3$ and $T = 3.77$ microseconds, the peak deviation is 59.7 kHz
From Figure 4.22, the one-sided 3-dB bandwidth of the modulating signal is approximately 50 kHz. Combining these two results, the Carson rule bandwidth is

$$\begin{aligned}B_T &= 2\left(59.7 + 50\right) \\ &= 219.4\ \text{kHz}\end{aligned}$$

The 1-percent FM bandwidth is given by Figure 4.9 with $\beta = \frac{\Delta f}{W} = \frac{59.7}{50} = 1.19$. From the vertical axis we find that $\frac{B_T}{\Delta f} = 6$, which implies $B_T = 6(59.7) = 358.2$ kHz.

Problem 4.26.

a)



Beta	# of side frequencies
1	1
2	2
5	8
10	14

b)By experimentation, a modulation index of 2.408, will force the amplitude of the carrier to be about zero. This corresponds to the first root of $J_0(\beta)$, as predicted by the theory.

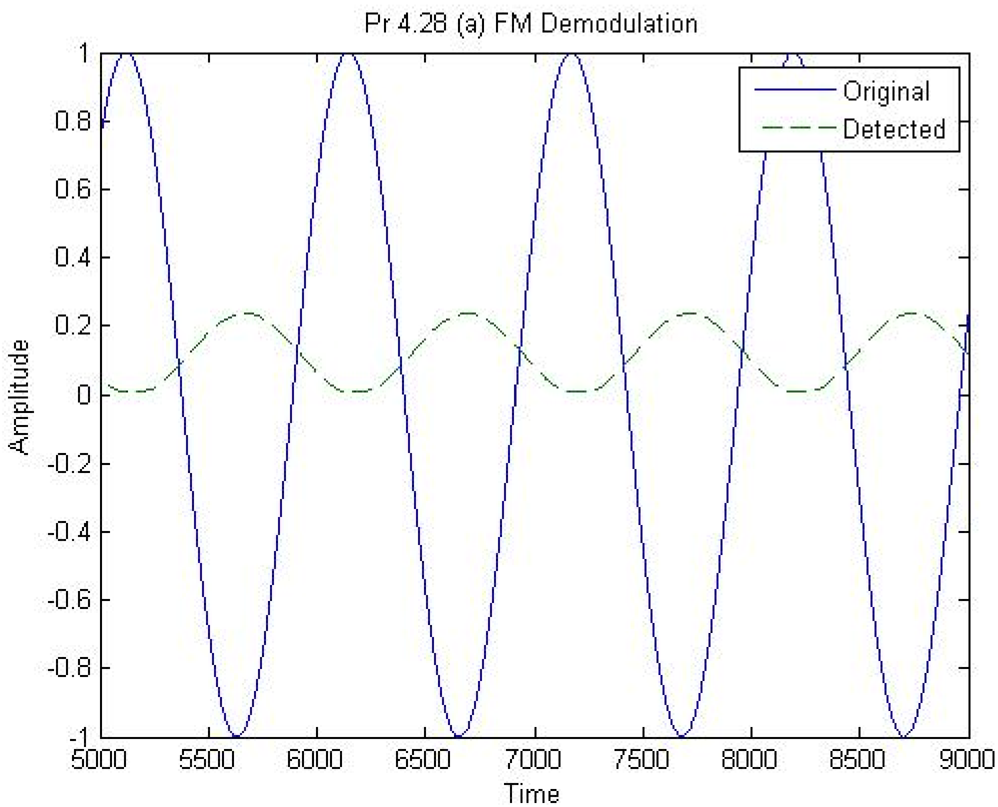
Problem 4.27.

a)Using the original MATLAB script, the rms phase error is 6.15 %
b)Using the plot provided, the rms phase error is 19.83%

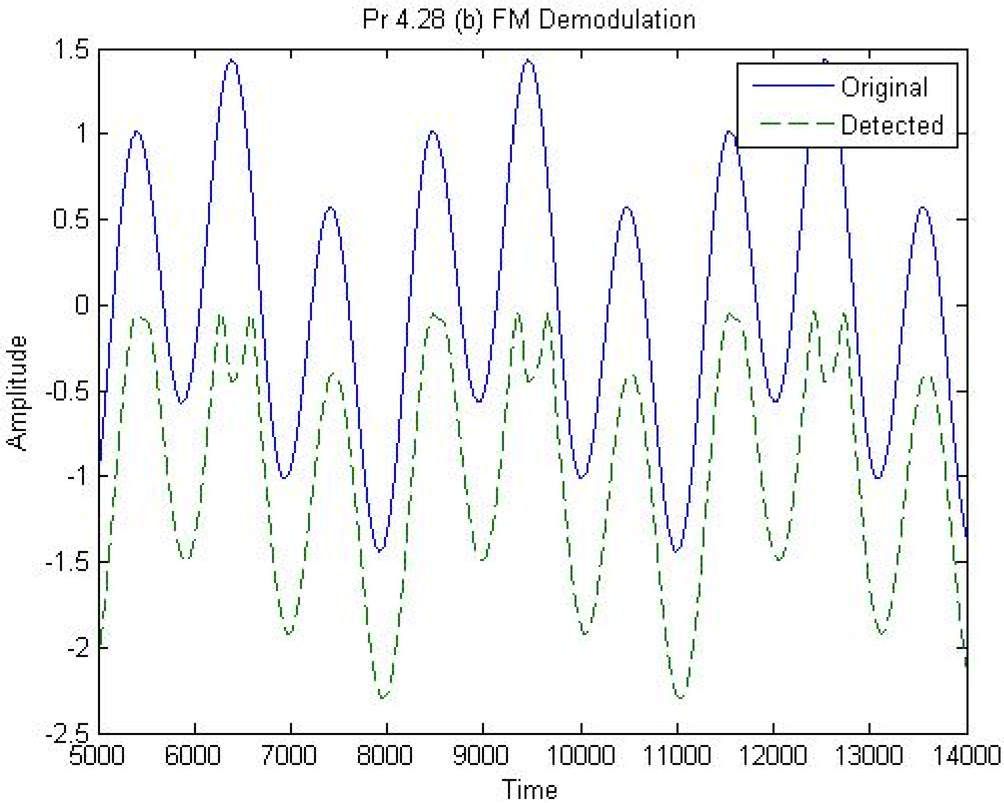
Problem 4.28

a)The output of the detected signal is multiplied by -1. This results from the fact that m(t)=cos(t) is integrated twice. Once to form the transmitted signal and once by the envelope detector.

In addition, the signal also has a DC offset, which results from the action of the envelope detector. The change in amplitude is the result of the modulation process and filters used in detection.



b)If $s(t) = \sin(2\pi f_m t) + 0.5\cos\left(2\pi \frac{f_m}{3} t\right)$, then some form of clipping is observed.



The above signal has been multiplied by a constant gain factor in order to highlight the differences with the original message signal.

c)The earliest signs of distortion start to appear above about fm =4.0 kHz. As the message frequency may no longer lie wholly within the bandwidth of either the differentiator or the low-pass filter. This results in the potential loss of high-frequency message components.

4.29. By tracing the individual steps of the MATLAB algorithm, it can be seen that the resulting sequence is the same as for the 2nd order PLL.

$e(t)$ is the phase error $\phi_e(t)$ in the theoretical model.

The theoretical model of the VCO is:

$$\phi_2(t) = 2\pi k_v \int_0^t v(t)dt$$

and the discrete-time model is:

$$\text{VCOState} = \text{VCOState} + 2\pi k_v (t-1)T_s$$

which approximates the integrator of the theoretical model.

The loop filter is a PI-controller, and has the transfer function:

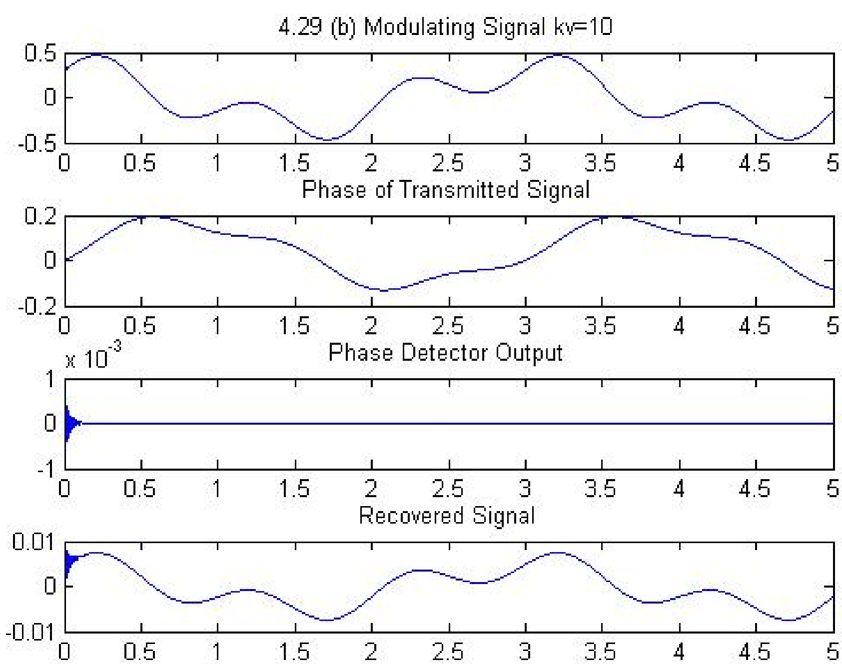
$$H(f) = 1 + \frac{a}{jf}$$

This is simply a combination of a sum plus an integrator, which is also present in the MATLAB code:

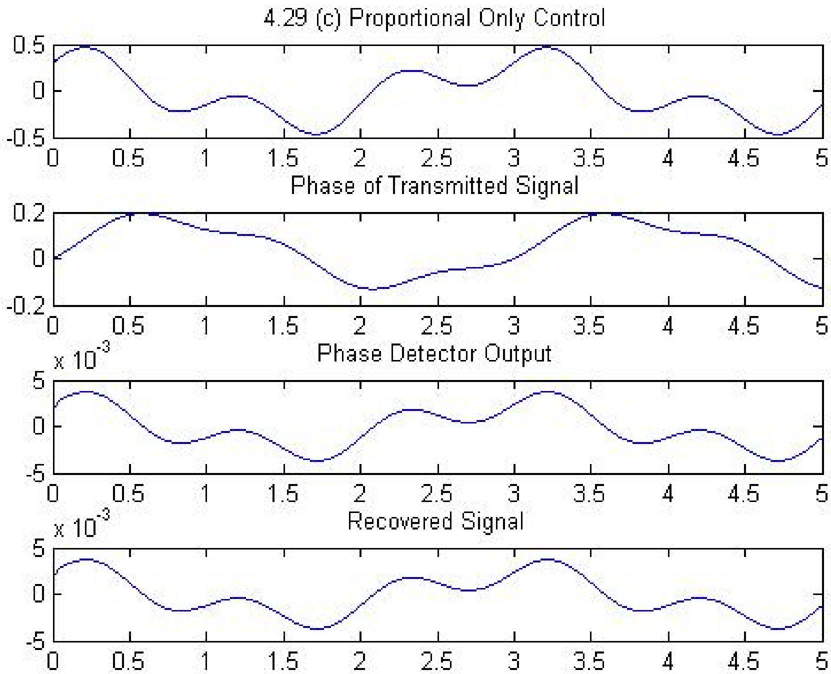
$\text{Filterstate} = \text{Filterstate} + e(t)$ Integrator

$v(t) = \text{Filterstate} + e(t)$ Integrator +input

b)For smaller kv, the lock-in time is longer, but the output amplitude is greater.



c)The phase error increases, and tracks the message signal.



d)For a single sinusoid, the track is lost if $f_m \geq K_0$ where $K_0 = k_f k_v A_c A_v$

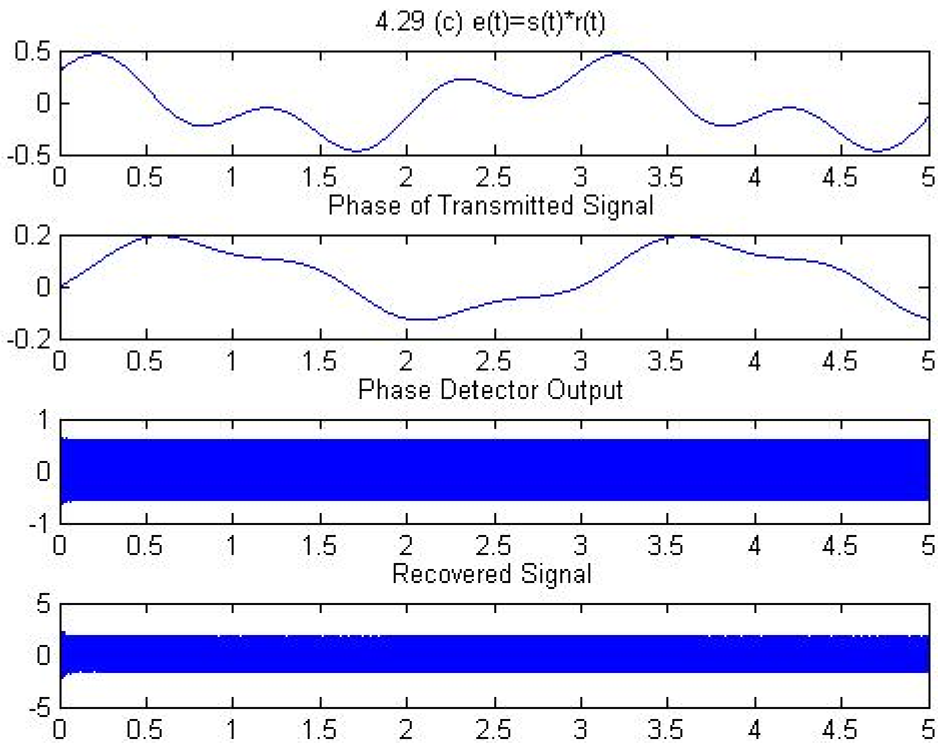
For this question, K_0=100 kHz, but tracking degrades noticeably around 60-70 kHz.

e)No useful signal can be extracted.

By multiplying s(t) and r(t), we get:

$$\frac{A_c A_v}{2}\left[\sin(k_f\phi - \text{VCOState}) + \sin(4\pi f_c t + k_f\phi + \text{VCOState})\right]$$

This is substantially different from the original error signal, and cannot be seen as an adequate approximation. Of particular interest is the fact that this equation is substantially more sensitive to changes in ϕ than the previous one owing to the presence of the gain factor k_v


4.29 (c) e(t)=s(t)*r(t)
0.5
0
-0.5
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
Phase of Transmitted Signal
0.2
0
-0.2
Phase Detector Output
1
0
-1
Recovered Signal
5
0
-5

Chapter 5 Problems

5.1. (a) Given $f(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} \exp(-\frac{(x-\mu_x)^2}{2\sigma_x^2})$

and $\exp(-\pi t^2) \rightleftarrows \exp(-\pi f^2)$, then by applying the time-shifting and scaling properties:

$$
\begin{aligned}
F(f) &= \frac{1}{\sqrt{2\pi\sigma_x^2}} \left|\sqrt{2\pi\sigma_x^2}\right| \exp(-\pi(\sqrt{2\pi\sigma_x^2})^2 \pi f^2) \exp(j2\pi f \mu_x) \\
&= \exp(-\pi^2 2\sigma_x^2 f^2 + j\mu_x 2\pi f) \qquad \text{and let } \nu = 2\pi f \\
&= \exp(j\nu\mu_x - \frac{1}{2}\nu^2\sigma_x^2)
\end{aligned}
$$

(b)The value of μ_x does not affect the moment, as its influence is removed.

Use the Taylor series approximation of $\phi_x(x)$, given $\mu_x = 0$.

$$\phi_x(\nu) = \exp(-\frac{1}{2}\nu^2\sigma_x^2)$$

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^2}{n!}$$

$$E[X^n] = \left.\frac{d^n\phi_x(\nu)}{d\nu^n}\right|_{\nu=0}$$

$$\therefore\ \phi_x(\nu) = \sum_{k=0}^{\infty} \left(-\frac{1}{2}\right)^k \frac{\sigma_x^{2k}\nu^{2k}}{k!}$$

For any odd value of n, taking $\frac{d^n\phi_x(\nu)}{d\nu^n}$ leaves the lowest non-zero derivative as ν^{2k-n}. When this derivative is evaluated for ν=0, then $E[X^n]$=0.

For even values of n, only the terms in the resulting derivative that correspond to $\nu^{2k-n} = \nu^0$ are non-zero. In other words, only the even terms in the sum that correspond to $k = n/2$ are retained.

$$\therefore\ E[X^n] = \frac{n!}{(n/2)!}\sigma_x^2$$

5.2. (a) All the inputs for x ≤ 0 are mapped to y = 0. However, the probability that x > 0 is unchanged. Therefore the probability density of x ≤ 0 must be concentrated at y=0.

(b) Recall that $\int_{-\infty}^{\infty} f_x x) dx = 1$ where $f_x(x)$ is an even function. Because $f_y(y)$ is a probability distribution, its integral must also equal 1.

$$\therefore \int_{0}^{\infty} f_x(x)dx = 0.5 \quad \text{and} \quad \int_{0^+}^{\infty} f_y(y)dy = 0.5$$

Therefore, the integral over the delta function must be 0.5. This means that the factor *k* must also be 0.5.

5.3 (a) $p_y(y) = p_y(y \mid x_0)P(x_0) + p_y(y \mid x_1)P(x_1)$

Assume: $P(x_0) = P(x_1) = 0.5$

$$\therefore\ p_y(y) = \frac{1}{2}[p_y(y \mid x_0) + p_y(y \mid y_1)$$

$$p_y(y) = \frac{1}{2\sqrt{2\pi\sigma^2}}[\exp(-\frac{(y+1)^2}{2\sigma^2}) + \exp(-\frac{(y-1)^2}{2\sigma^2})]$$

(b) $P(y \geq \alpha) = \int_{\alpha}^{\infty} p_y(y)dy$

Use the cumulative Gaussian distribution,

$$\Phi_{\mu,\sigma^2}(y) = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-\frac{(y-\mu)^2}{2\sigma^2})dy$$

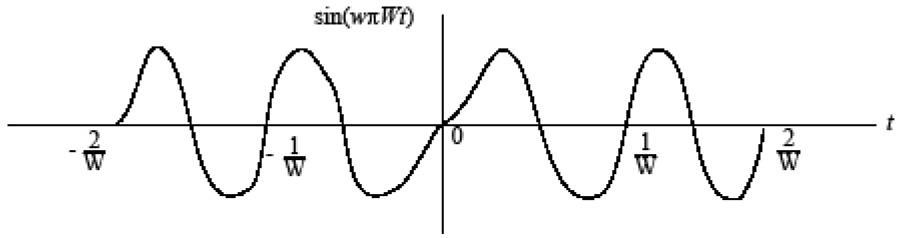
$$\therefore\ P(y \geq \alpha) = \frac{1}{2}[\Phi_{-1,\sigma^2}(-\alpha) + \Phi_{1,\sigma^2}(-\alpha)]$$

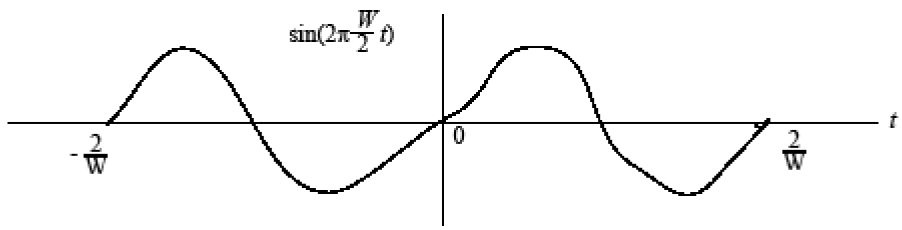
But, $\Phi_{\mu,\sigma^2}(y) = \frac{1}{2}[1 + erf(\frac{y-\mu}{\sigma\sqrt{2}})]$

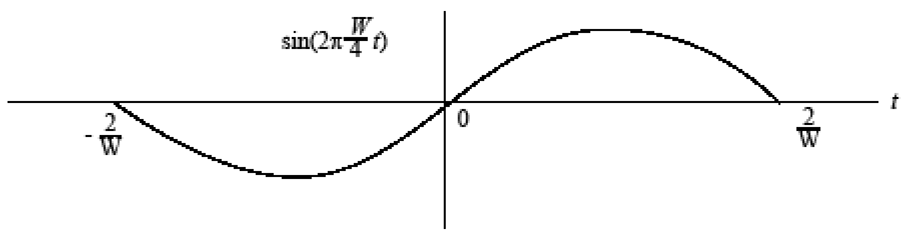
$$\therefore\ P(y \geq \alpha) = \frac{1}{2}[2 + erf\left(\frac{-\alpha+1}{\sigma\sqrt{2}}\right) + erf\left(\frac{-\alpha-1}{\sigma\sqrt{2}}\right)]$$

Problem 5.4

As an illustration, three particular sample functions of the random process $X(t)$, corresponding to $F = W/4$, $W/2$, and W, are plotted below:







To show that $X(t)$ is nonstationary, we need only observe that every waveform illustrated above is zero at $t = 0$, positive for $0 < t < 1/2W$, and negative for $-1/2W < t < 0$. Thus, the probability density function of the random variable $X(t_1)$ obtained by sampling $X(t)$ at $t1 = 1/4W$ is identically zero for negative argument, whereas the probability density function of the random variable $X(t_2)$ obtained by sampling $X(t)$ at $t = -1/4W$ is nonzero only for negative arguments. Clearly, therefore,

$f_{X(t_1)}(x_1) \neq f_{X(t_2)}(x_2)$, and the random process $X(t)$ is nonstationary.

Problem 5.5

If, for a complex random process $Z(t)$

$$R_Z(\tau) = \mathbf{E}\left[Z^*(t)Z(t+\tau)\right]$$

then

(i) The mean square of a complex process is given by

$$\begin{aligned} R_Z(0) &= \mathbf{E}\left[Z^*(t)Z(t)\right] \\ &= \mathbf{E}\left[\left|Z(t)\right|^2\right] \end{aligned}$$

(ii) We show $R_Z(\tau)$ has conjugate symmetry by the following

$$\begin{aligned} R_Z(-\tau) &= \mathbf{E}\left[Z^*(t)Z(t-\tau)\right] \\ &= \mathbf{E}\left[Z^*(s+\tau)Z(s)\right] \\ &= \mathbf{E}\left[Z(s)Z(s+\tau)\right]^* \\ &= R_Z^*(\tau) \end{aligned}$$

where we have used the change of variable $s = t - \tau$.

(iii) Taking an approach similar to that of Eq. (5.67)

$$\begin{aligned} 0 &\le \mathbf{E}\left[\left|\left(Z(t) \pm Z(t+\tau)\right)\right|^2\right] \\ &= \mathbf{E}\left[\left(Z(t) \pm Z(t+\tau)\right)\left(Z^*(t) \pm Z^*(t+\tau)\right)\right] \\ &= \mathbf{E}\left[Z(t)Z^*(t) \pm Z(t)Z^*(t+\tau) \pm Z^*(t)Z(t+\tau) + Z(t+\tau)Z^*(t+\tau)\right] \\ &= \mathbf{E}\left[\left|Z(t)\right|^2\right] \pm \mathbf{E}\left[Z(t)Z^*(t+\tau)\right] \pm \mathbf{E}\left[Z^*(t)Z(t+\tau)\right] + \mathbf{E}\left[\left|Z(t+\tau)\right|^2\right] \\ &= 2\mathbf{E}\left[\left|Z(t)\right|^2\right] \pm 2\,\mathrm{Re}\left\{\mathbf{E}\left[Z^*(t)Z(t+\tau)\right]\right\} \\ &= 2R_Z(0) \pm 2\,\mathrm{Re}\left\{R_Z(\tau)\right\} \end{aligned}$$

Thus $\left|\mathrm{Re}\left\{R_Z(\tau)\right\}\right| \le R_Z(0)$.

Problem 5.6 (a)

$$E[Z(t_1)Z^*(t_2)]$$
$$= E[(A\cos(2\pi f_1 t_1 + \theta_1) + jA\cos(2\pi f_2 t_1 + \theta_2)) \cdot (A\cos(2\pi f_1 t_2 + \theta_1) + jA\cos(2\pi f_2 t_2 + \theta_2))]$$

Let $\omega_1 = 2\pi f_1$ $\omega_2 = 2\pi f_2$

After distributing the terms, consider the first term:

$$A^2 E[\cos(\omega_1 t_1 + \theta_1)\cos(\omega_1 t_2 + \theta_1)]$$
$$= \frac{A^2}{2} E[\cos(\omega_1(t_1 - t_2)) + \cos(\omega_1(t_1 + t_2) + 2\theta_1)]$$

The expectation over θ_1 goes to zero, because θ_1 is distributed uniformly over $[-\pi,\pi]$. This result also applies to the term $A^2[\cos(\omega_2 t_1 + \theta_2)\cos(\omega_2 t_2 + \theta_2)]$. Both cross-terms go to zero.

$$\therefore\ R(t_1, t_2) = \frac{A^2}{2}[\cos(\omega_1(t_1 - t_2)) + \cos(\omega_2(t_1 - t_2))]$$

(b) If $f_1 = f_2$, only the cross terms may be different:

$$E[jA^2(\cos(\omega_1 t_1 + \theta_2)\cos(\omega_1 t_2 + \theta_1) + \cos(\omega_1 t_1 + \theta_2)\cos(\omega_1 t_2 + \theta_1)]$$

But, unless $\theta_1 = \theta_2$, the cross-terms will also go to zero.

$$\therefore\ R(t_1, t_2) = A^2\cos(\omega_1(t_1 - t_2))$$

(c) If $\theta_1 = \theta_2$, then the cross-terms become:

$$-jA^2 E[\cos((\omega_1 t_1 - \omega_2 t_2)) + \cos((\omega_1 t_1 + \omega_2 t_2) + 2\theta_1) + jA^2 E[\cos((\omega_2 t_1 - \omega_1 t_2)) + \cos((\omega_1 t_1 + \omega_2 t_2) + 2\theta_1)]$$

After computing the expectations, the cross-terms simplify to:

$$\frac{jA^2}{2}[\cos(\omega_2 t_1 - \omega_1 t_2) - \cos(\omega_1 t_1 - \omega_2 t_2)]$$

$$\therefore\ R_Z(t_1, t_2) = \frac{A^2}{2}[\cos(\omega_1(t_1 - t_2)) + \cos(\omega_2(t_1 - t_2)) + j\cos(\omega_2 t_1 - \omega_1 t_2) - j\cos(\omega_1 t_1 - \omega_2 t_2)]$$

Problem 5.7

(a) The expected value of $Z(t_1)$ is

$$E[Z(t_1)] = \cos(2\pi t_1)E[X] + \sin(2\pi t_1)E[Y]$$

Since $E[X] = E[Y] = 0$, we deduce that

$$E[Z(t_1)] = 0$$

Similarly, we find that

$$E[Z(t_2)] = 0$$

Next, we note that

$$\begin{aligned}\text{Cov}[Z(t_1)Z(t_2)] &= E[Z(t_1)Z(t_2)] \\ &= E\{[X\cos(2\pi t_1) + Y\sin(2\pi t_1)][X\cos(2\pi t_2) + Y\sin(2\pi t_2)]\} \\ &= \cos(2\pi t_1)\cos(2\pi t_2)E[X^2] \\ &\quad + [\cos(2\pi t_1)\sin(2\pi t_2) + \sin(2\pi t_1)\cos(2\pi t_2)]E[XY] \\ &\quad + \sin(2\pi t_1)\sin(2\pi t_2)E[Y^2]\end{aligned}$$

Noting that

$$E[X^2] = \sigma_X^2 + \{E[X]\}^2 = 1$$

$$E[Y^2] = \sigma_Y^2 + \{E[Y]\}^2 = 1$$

$$E[XY] = 0$$

we obtain

$$\begin{aligned}\text{Cov}[Z(t_1)Z(t_2)] &= \cos(2\pi t_1)\cos(2\pi t_2) + \sin(2\pi t_1)\sin(2\pi t_2) \\ &= \cos[2\pi(t_1 - t_2)]\end{aligned}$$

Since every weighted sum of the samples of the process $Z(t)$ is Gaussian, it follows that $Z(t)$ is a Gaussian process. Furthermore, we note that

$$\sigma^2_{Z(t_1)} = E[Z^2(t_1)] = 1$$

This result is obtained by putting $t_1 = t_2$ in Eq. (1). Similarly,

$$\sigma^2_{Z(t_2)} = E[Z^2(t_2)] = 1$$

Therefore, the correlation coefficient of $Z(t_1)$ and $Z(t_2)$ is

$$\rho = \frac{\mathrm{Cov}[Z(t_1)Z(t_2)]}{\sigma_Z(t_1)\sigma_Z(t_2)}$$

$$= \cos[2\pi(t_1 - t_2)]$$

Hence, the joint probability density function of $Z(t_1)$ and $Z(t_2)$

$$f_{Z(t_1),\,Z(t_2)}(z_1, z_2) = C\exp[-Q(z_1, z_2)]$$

where

$$C = \frac{1}{2\pi\sqrt{1-\cos^2[2(t_1 - t_2)]}}$$

$$= \frac{1}{2\pi\sin[2\pi(t_1 - t_2)]}$$

$$Q(z_1, z_2) = \frac{1}{2\sin[2\pi(t_1 - t_2)]}\left\{z_1^2 - 2\cos[2\pi(t_1 - t_2)]z_1, z_2 + z_2^2\right\}$$

(b) We note that the covariance of $Z(t_1)$ and $Z(t_2)$ depends only on the time difference t_1 - t_2. The process $Z(t)$ is therefore wide-sense stationary. Since it is Gaussian it is also strictly stationary.

Problem 5.8

(a) Let

$$X(t) = A + Y(t)$$

where A is a constant and $Y(t)$ is a zero-mean random process. The autocorrelation function of $X(t)$ is

$$R_X(\tau) = E[X(t+\tau)X(t)]$$

$$= E\{[A + Y(t+\tau)][A + Y(t)]\}$$

$$= E[A^2 + AY(t+\tau) + AY(t) + Y(t+\tau)Y(t)]$$

$$= A^2 + R_Y(\tau)$$

which shows that $R_X(\tau)$ contains a constant component equal to A^2.

(b) Let

$$X(t) = A_c\cos(2\pi f_c t + \theta) + Z(t)$$

where $A_c \cos(2\pi f_c t + \theta)$ represents the sinusoidal component of $X(t)$ and θ is a random phase variable. The autocorrelation function of $X(t)$ is

$$
\begin{aligned}
R_X(\tau) &= E[X(t+\tau)X(t)] \\
&= E\{[A_c \cos(2\pi f_c t + 2\pi f_c \tau + \theta) + Z(t+\tau)][A_c \cos(2\pi f_c t + \theta) + Z(t)]\} \\
&= E[A_c^2 \cos(2\pi f_c t + 2\pi f_c \tau + \theta)\cos(2\pi f_c t + \theta)] \\
&\quad + E[Z(t+\tau) A_c \cos(2\pi f_c t + \theta)] \\
&\quad + E[A_c \cos(2\pi f_c t + 2\pi f_c \tau + \theta) Z(t)] \\
&\quad + E[Z(t+\tau)Z(t)] \\
&= (A_c^2/2)\cos(2\pi f_c \tau) + R_Z(\tau)
\end{aligned}
$$

whih shows that $R_X(\tau)$ contains a sinusoidal component of the same frequency as $X(t)$.

Problem 5.9

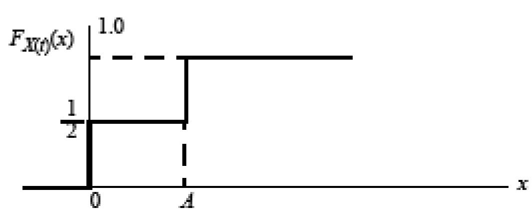
(a) We note that the distribution function of $X(t)$ is

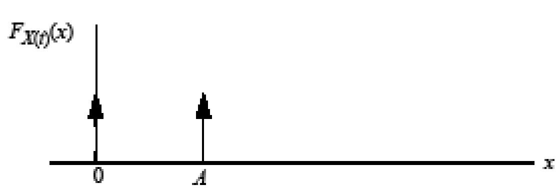
$$
F_{X(t)}(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{2}, & 0 \le x \le A \\ 1, & A < x \end{cases}
$$

and the corresponding probability density function is

$$
F_{X(t)}(x) = \frac{1}{2}\delta(x) + \frac{1}{2}\delta(x - A)
$$

which are illustrated below:





(b) By ensemble-averaging, we have

$$E[X(t)] = \int_{-\infty}^{\infty} x f_{X(t)}(x)dx$$

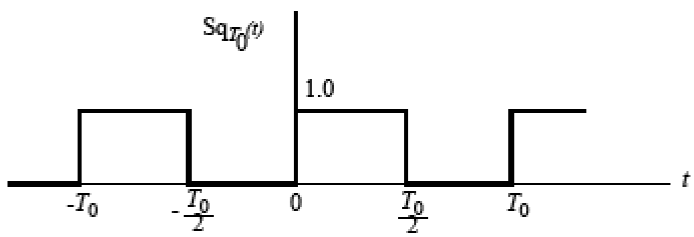
$$= \int_{-\infty}^{\infty} x\left[\frac{1}{2}\delta(x) + \frac{1}{2}\delta(x-A)\right]dx$$

$$= \frac{A}{2}$$

The autocorrelation function of $X(t)$ is

$$R_X(\tau) = E[X(t+\tau)X(t)]$$

Define the square function $\mathrm{Sq}_{T_0}(t)$ as the square-wave shown below:



Then, we may write

$$R_X(\tau) = E[A\mathrm{Sq}_{T_0}(t-t_d+\tau)\cdot A\mathrm{Sq}_{T_0}(t-t_d)]$$

$$= A^2\int_{-\infty} \mathrm{Sq}_{T_0}(t-t_d+\tau)\mathrm{Sq}_{T_0}(t-t_d)f_{T_d}(t_d)dt_d$$

$$= A^2\int_{-T_0/2}^{T_0/2} \mathrm{Sq}_{T_0}(t-t_d+\tau)\mathrm{Sq}_{T_0}(t-t_d)\cdot\frac{1}{T_0}dt_d$$

$$= \frac{A^2}{2}\left(1-2\frac{|\tau|}{T_0}\right), |\tau| \le \frac{T_0}{2}.$$

Since the wave is periodic with period T_0, $R_X(\tau)$ must also be periodic with period T_0.

(c) On the time-averaging basis, we note by inspection of Fig. P1.6 that the mean is

$$<x(t)> = \frac{A}{2}$$

Next, the autocorrelation function

$$<x(t+\tau)x(t)> = \frac{1}{T_0}\int_{-T_0/2}^{(T_0/2)} x(t+\tau)x(t)dt$$

has its maximum value of $A^2/2$ at $\tau = 0$, and decreases linearly to zero at $\tau = T_0/2$. Therefore,

$$<x(t+\tau)x(t)> \;=\; \frac{A^2}{2}\left(1-2\frac{|\tau|}{T_0}\right), \; |\tau| \le \frac{T_0}{2}.$$

Again, the autocorrelation must be periodic with period T_0.

(d) We note that the ensemble-averaging and time-averaging procedures yield the same set of results for the mean and autocorrelation functions. Therefore, $X(t)$ is ergodic in both the mean and the autocorrelation function. Since ergodicity implies wide-sense stationarity, it follows that $X(t)$ must be wide-sense stationary.

Problem 5.10

(a) For $|\tau| > T$, the random variables $X(t)$ and $X(t + \tau)$ occur in different pulse intervals and are therefore independent. Thus,

$$E[X(t)X(t+\tau)] \;=\; E[X(t)]E[X(t+\tau)], \qquad |\tau| > T.$$

Since both amplitudes are equally likely, we have $E[X(t)] \;=\; E[x(t+\tau)] \;=\; A/2$. Therefore, for $|\tau| > T$,

$$R_X(\tau) \;=\; \frac{A^2}{4}.$$

For $|\tau| \ge T$., the random variables occur in the same pulse interval if $t_d < T - |\tau|$. If they do oc
the same pulse interval,

$$E[X(t)X(t+\tau)] \;=\; \frac{1}{2}A^2 + \frac{1}{2}0^2 \;=\; \frac{A^2}{2}.$$

We thus have a conditional expectation:

$$E[X(t)X(t+\tau)] \;=\; A^2/2, \qquad t_d < T - |\tau|$$

$$\;=\; A^2/4, \quad \text{otherwise.}$$

Averaging over t_d, we get

$$R_X(\tau) \;=\; \int_0^{T-|\tau|} \frac{A^2}{2T}dt_d + \int_{T-|\tau|}^{T} \frac{A^2}{2T}dt_d$$

$$\;=\; \frac{A^2}{4}\left(1-\frac{|\tau|}{T}\right) + \frac{A^2}{4}, \qquad |\tau| \le T$$

(b) The power spectral density is the Fourier transform of the autocorrelation function. The Fourier transform of

$$g(\tau) \;=\; 1-\frac{|\tau|}{T}, \quad |\tau| \le T$$

$$\;=\; 0, \quad \text{otherwise,}$$

is given by

$$G(f) = T\,\text{sinc}^2(fT)\ .$$

Therefore,

$$S_x(f) = \frac{A^2T}{4}\text{sinc}^2(fT).$$

We next note that

$$\frac{A^2}{4}\int_{-\infty}^{\infty}\delta(f)df = \frac{A^2}{4},$$

$$\frac{A^2}{4}\int_{-\infty}^{\infty}T\,\text{sinc}^2(fT)df = \frac{A^2}{4},$$

$$\int_{-\infty}^{\infty}S_x(f)df = R_X(0) = \frac{A^2}{2}.$$

It follows therefore that half the power is in the dc component.

Problem 5.11

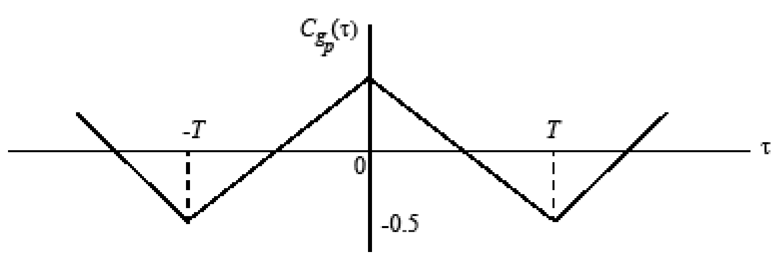
Since

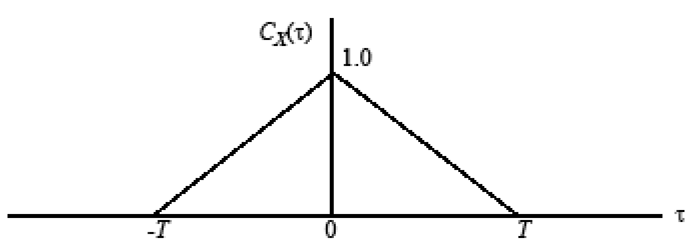
$$Y(t) = g_p(t) + X(t) + \sqrt{3/2}$$

and $g_p(t)$ and $X(t)$ are uncorrelated, then

$$C_Y(\tau) = C_{g_p}(\tau) + C_X(\tau)$$

where $C_{g_p}(\tau)$ is the autocovariance of the periodic component and $C_X(\tau)$ is the autocovariance of the random component. $C_Y(\tau)$ is the plot in Fig. P1.8 shifted down by 3/2, removing the dc component. $C_{g_p}(\tau)$ and $C_X(\tau)$ are plotted below:





Both $g_p(t)$ and $X(t)$ have zero mean.

(a) The average power of the periodic component $g_p(t)$ is therefore,

$$\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} g_p^2(t)dt = C_{g_p}(0) = \frac{1}{2}$$

(b) The average power of the random component $X(t)$ is

$$E[X^2(t)] = C_X(0) = 1$$

Problem 5.12

(a) $R_{XY}(\tau) = E[X(t+\tau)Y(t)]$

Replacing τ with $-\tau$:

$$R_{XY}(-\tau) = E[X(t-\tau)Y(t)]$$

Next, replacing $t - \tau$ with t, we get

$$R_{XY}(-\tau) = E[X(t+\tau)X(t)]$$

$$= R_{YX}(\tau)$$

(b) Form the non-negative quantity

$$E[\{X(t+\tau) \pm Y(t)\}^2] = E[X^2(t+\tau) \pm 2X(t+\tau)Y(t) + Y^2(t)]$$

$$= E[X^2(t+\tau) \pm 2EX(t+\tau)Y(t)] + E[Y^2(t)]$$

$$= R_X(0) \pm 2R_{XY}(\tau) + R_Y(0)$$

Hence,

$$R_X(0) \pm 2R_{XY}(\tau) + R_Y(0) \geq 0$$

or

$$|R_{XY}(\tau)| \leq \frac{1}{2}[R_X(0) + R_Y(0)]$$

Problem 5.13

(a) The cascade connection of the two filters is equivalent to a filter with impulse response.

$$h(t) = \int_{-\infty}^{\infty} h_1(u)h_2(t-u)du$$

The autocorrelation function of $Y(t)$ is given by

$$R_Y(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)R_X(\tau-\tau_1+\tau_2)d\tau_1 d\tau_2$$

(b) The cross-correlation function of $V(t)$ and $Y(t)$ is

$$R_{VY}(\tau) = E[V(t+\tau)Y(t)]$$

The $Y(t)$ and $V(t+\tau)$ are related by

$$Y(t) = \int_{-\infty}^{\infty} V(\lambda)h_2(t-\lambda)d\lambda$$

Therefore,

$$R_{VY}(\tau) = E\left[V(t+\tau)\int_{-\infty}^{\infty} V(\lambda)h_2(t-\lambda)d\lambda\right]$$

$$= \int_{-\infty}^{\infty} h_2(t-\lambda)E[V(t+\tau)V(\lambda)]d\lambda$$

$$= \int_{-\infty}^{\infty} h_2(t-\lambda)R_V(t+\tau-\lambda)d\lambda$$

Substituting λ for t - λ:

$$R_{VY}(\tau) = \int_{-\infty}^{\infty} h_2(\lambda)R_V(t+\lambda)d\lambda$$

The autocorrelation function $R_V(\tau)$ is related to the given $R_X(\tau)$ by

$$R_V(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_1(\tau_1)h_1(\tau_2)R_X(\tau-\tau_1+\tau_2)d\tau_1 d\tau_2$$

Problem 5.14

(a) The cross-correlation function $R_{YX}(\tau)$ is

$$R_{YX}(\tau) = E[Y(t+\tau)X(t)]$$

The $Y(t)$ and $X(t)$ are related by

$$Y(t) = \int_{-\infty}^{\infty} X(u)h(t-u)du$$

Therefore,

$$R_{YX}(\tau) = E\left[\int_{-\infty}^{\infty} X(u)X(t)h(t+\tau-u)du\right]$$

$$= \int_{-\infty}^{\infty} h(t+\tau-u)E[X(u)X(t)]du$$

$$= \int_{-\infty}^{\infty} h(t+\tau-u)R_X(u-t)du$$

Replacing $t + \tau - u$ by u:

$$R_{YX}(\tau) = \int_{-\infty}^{\infty} h(u)R_X(\tau-u)du$$

(b) Since $R_{XY}(\tau) = R_{YX}(-\tau)$, we have

$$R_{XY}(\tau) = \int_{-\infty}^{\infty} h(u)R_X(-\tau-u)du$$

Since $R_X(\tau)$ is an even function of τ:

$$R_{XY}(\tau) = \int_{-\infty}^{\infty} h(u)R_X(t+u)du$$

Replacing u by $-u$:

$$R_{XY}(\tau) = \int_{-\infty}^{\infty} h(-u)R_X(t-u)du$$

(c) If $X(t)$ is a white noise process with zero mean and power spectral density $N_0/2$, we may write

$$R_X(\tau) = \frac{N_0}{2}\delta(\tau)$$

Therefore,

$$R_{YX}(\tau) = \frac{N_0}{2}\int_{-\infty}^{\infty} h(u)\delta(\tau-u)du$$

Using the sifting property of the delta function:

$$R_{YX}(\tau) = \frac{N_0}{2}h(\tau)$$

That is,

$$h(\tau) = \frac{2}{N_0}R_{YX}(\tau)$$

This means that we may measure the impulse response of the filter by applying a white noise of power spectral density $N_0/2$ to the filter input, cross-correlating the filter output withthe input, and then multiplying the result by $2/N_0$.

Problem 5.15

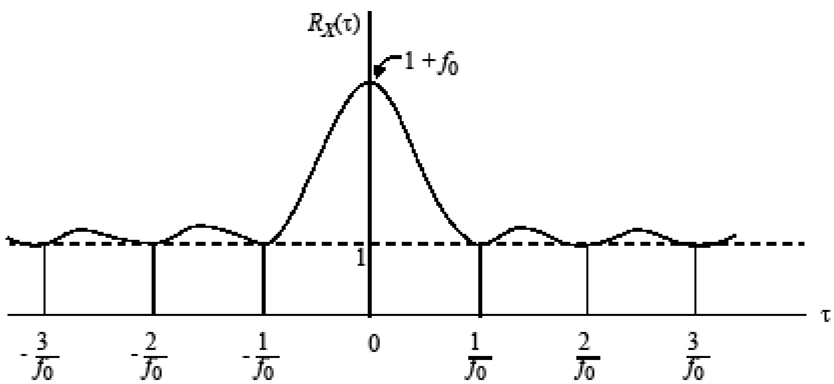
(a) The power spectral density consists of two components:

(1) A delta function $\delta(f)$ at the origin, whose inverse Fourier transform is one.
(2) A triangular component of unit amplitude and width $2f_0$, centered at the origin; the inverse Fourier transform of this component is $f_0\text{sinc}2(f_0\tau)$.

Therefore, the autocorrelation function of $X(t)$ is

$$R_X(\tau) = 1 + f_0\,\text{sinc}^2(f_0\tau)$$

which is sketched below:



(b) Since $R_X(\tau)$ contains a constant component of amplitude 1, it follows that the dc power contained in $X(t)$ is 1.

(c) The mean-square value of $X(t)$ is given by

$$E[X^2(t)] = R_X(0)$$

$$= 1 + f_0$$

The ac power contained in $X(f)$ is therefore equal to f_0.

(d) If the sampling rate is f_0/n, where n is an integer, the samples are uncorrelated. They are not, however, statistically independent. They would be statistically independent if $X(t)$ were a Gaussian process.

Problem 5.16

The autocorrelation function of $n_2(t)$ is

$$R_{N_2}(t_1, t_2) = E[n_2(t_1)n_2(t_2)]$$

$$= E\{[n_1(t_1)\cos(2\pi f_c t_1 + \theta) - n_1(t_1)\sin(2\pi f_c t_1 + \theta)]\}$$

$$\bullet [n_1(t_2)\cos(2\pi f_c t_2 + \theta) - n_1(t_2)\sin(2\pi f_c t_2 + \theta)]\}$$

$$= E[n_1(t_1)n_1(t_2)\cos(2\pi f_c t_1 + \theta)\cos(2\pi f_c t_2 + \theta)]$$

$$- \; n_1(t_1)n_1(t_2)\cos(2\pi f_c t_1 + \theta)\sin(2\pi f_c t_2 + \theta)$$

$$- n_1(t_1)n_1(t_2)\sin(2\pi f_c t_1 + \theta)\cos(2\pi f_c t_2 + \theta)$$

$$+ n_1(t_1)n_1(t_2)\sin(2\pi f_c t_1 + \theta)\sin(2\pi f_c t_2 + \theta)]$$

$$= E\{n_1(t_1)n_1(t_2)\cos[2\pi f_c(t_1 - t_2)]\}$$

$$- n_1(t_1)n_1(t_2)\sin[2\pi f_c(t_1 + t_2) + 2\theta]\}$$

$$= E[n_1(t_1)n_1(t_2)]\cos[2\pi f_c(t_1 - t_2)]$$

$$- E[n_1(t_1)n_1(t_2)] \bullet E\{\sin[2\pi f_c(t_1 + t_2) + 2\theta]\}$$

Since θ is a uniformly distributed random variable, the second term is zero, giving

$$R_{N_2}(t_1, t_2) = R_{N_1}(t_1, t_2)\cos[2\pi f_c(t_1 - t_2)]$$

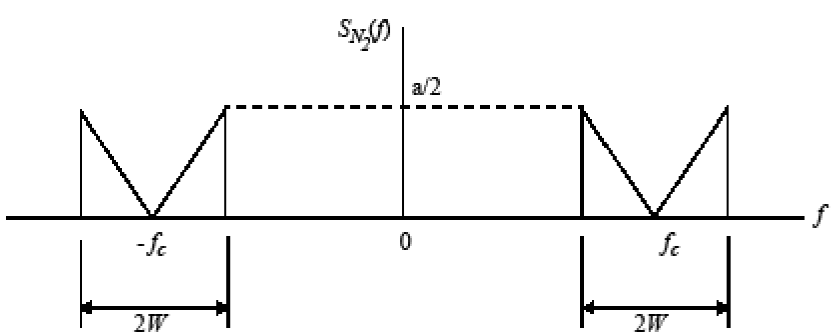
Since $n_1(t)$ is stationary, we find that in terms of $\tau = t_1 - t_2$:

$$R_{N_2}(\tau) = R_{N_1}(\tau)\cos(2\pi f_c \tau)$$

Taking the Fourier transforms of both sides of this relation:

$$S_{N_2}(f) = \frac{1}{2}[S_{N_2}(f+f_c) + S_{N_1}(f-f_c)]$$

With $S_{N_1}(f)$ as defined in Fig. P1.13, we find $S_{N_2}(f)$ is as shown below:



Problem 5.17

The power spectral density of the random telegraph wave is

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau)\exp(-j2\pi f\tau)d\tau$$

$$= \int_{-\infty}^{0} \exp(2\nu\tau)\exp(-j2\pi f\tau)d\tau$$

$$+\int_{0}^{\infty} \exp(-2\nu\tau)\exp(-2j\pi f\tau)d\tau$$

$$= \frac{1}{2(\nu - j\pi f)}[\exp(2\nu\tau - j2\pi f\tau)]$$

$$-\frac{1}{2(\nu + j\pi f)}[\exp(-2\nu\tau - j2\pi f\tau)]_0^{\infty}$$

$$= \frac{1}{2(\nu - j\pi f)} + \frac{1}{2(\nu + j\pi f)}$$

$$= \frac{\nu}{\nu^2 + \pi^2 f^2}$$

The transfer function of the filter is

$$H(f) = \frac{1}{1 + j2\pi fRC}$$

Therefore, the power spectral density of the filter output is

$$S_Y(f) = |H(f)|^2 S_X(f)$$

$$= \frac{\nu}{[1 + (2\pi fRC)^2](\nu^2 + \pi^2 f^2)}$$

To determine the autocorrelation function of the filter output, we first expand $S_Y(f)$ in partial fractions as follows:

$$S_Y(f) = \frac{\nu}{1 - 4R^2C^2\nu^2}\left[-\frac{1}{(1/2RC)^2 + \pi^2 f^2} + \frac{1}{\nu^2 + \pi^2 f^2}\right]$$

Recognizing that

$$\exp(-2\nu|t|) \rightleftharpoons \frac{\nu}{\nu^2 + \pi^2 f^2}$$

$$\exp(-|t|/RC) \rightleftharpoons \frac{1/2RC}{(1/2RC)^2 + \pi^2 f^2}$$

we obtain the desired result:

$$R_Y(\tau) = \frac{\nu}{1 - 4R^2C^2\nu^2}\left[\frac{1}{\nu}\exp(-2\nu|\tau|) - 2RC\exp\left(-\frac{\tau}{RC}\right)\right]$$

Problem 5.18

The autocorrelation function of $X(t)$ is

$$R_X(\tau) = E[X(t+\tau)X(t)]$$

$$= A^2 E[\cos(2_\pi Ft + 2_\pi F_\tau - \theta)\cos(2_\pi Ft - \theta)]$$

$$= \frac{A^2}{2} E[\cos(4_\pi Ft + 2_\pi F_\tau - 2\theta) + \cos(2_\pi F_\tau)]$$

Averaging over θ, and hoting that θ is uniformly distributed over 2π radians, we get

$$R_X(\tau) = \frac{A^2}{2} E[\cos(2\pi F\tau)]$$

$$= \frac{A^2}{2} \int_{-\infty}^{\infty} f_F(f)\cos(2\pi f\tau)\,df \qquad (1)$$

Next, we note that $R_X(\tau)$ is related to the power spectral density by

$$R_X(\tau) = \int_{-\infty}^{\infty} S_X(f)\cos(2\pi f\tau)\,df \qquad (2)$$

Therefore, comparing Eqs. (1) and (2), we deduce that the power spectral density of $X(t)$ is

$$S_X(f) = \frac{A^2}{2} f_F(f)$$

When the frequency assumes a constant value, f_c (say), we have

$$f_F(f) = \frac{1}{2}\delta(f - f_c) + \frac{1}{2}\delta(f + f_c)$$

and, correspondingly,

$$S_X(f) = \frac{A^2}{4}\delta(f - f_c) + \frac{A^2}{4}\delta(f + f_c)$$

Problem 5.19

Let σ_X^2 denote the variance of the random variable X_k obtained by observing the random process $X(t)$ at time t_k. The variance σ_X^2 is related to the mean-square value of X_k as follows

$$\sigma_X^2 = E[X_k^2] - \mu_X^2$$

where $\mu_X = E[X_k]$. Since the process $X(t)$ has zero mean, it follows that

$$\sigma_X^2 = E[X_k^2]$$

Next we note that

$$E[X_k^2] = \int_{-\infty}^{\infty} S_X(f)df$$

We may therefore define the variance σ_X^2 as the total area under the power spectral density $S_X(f)$ as

$$\sigma_X^2 = \int_{-\infty}^{\infty} S_X(f)df \qquad (1)$$

Thus with the mean $\mu_X = 0$ and the variance σ_X^2 defined in Eq. (1), we may express the probability density function X_k as follows

$$f_{X_k}(x) = -\frac{1}{\sqrt{2\pi}\sigma_X}\exp\left(-\frac{x^2}{2\sigma_X^2}\right)$$

Problem 5.20

The input-output relation of a full-wave rectifier is defined by

$$Y(t_k) = |X(t_k)| = \begin{cases} X(t_k), & X(t_k) \geq 0 \\ -X(t_k), & X(t_k) \leq 0 \end{cases}.$$

The probability density function of the random variable $X(t_k)$, obtained by observing the input random process at time t_k, is defined by

$$f_{X(t_k)}(x) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{x^2}{2\sigma^2}\right)$$

To find the probability density function of the random variable $Y(t_k)$, obtained by observing the output random process, we need an expression for the inverse relation defining $X(t_k)$ in terms of $Y(t_k)$. We note that a given value of $Y(t_k)$ corresponds to 2 values of $X(t_k)$, of equal magnitude and opposite sign. We may therefore write

$$X(t_k) = -Y(t_k), \qquad X(t_k) < 0$$
$$X(t_k) = Y(t_k), \qquad X(t_k) > 0.$$

In both cases, we have

$$\left|\frac{dX(t_k)}{dY(t_k)}\right| = 1.$$

The probability density function of $Y(t_k)$ is therefore given by

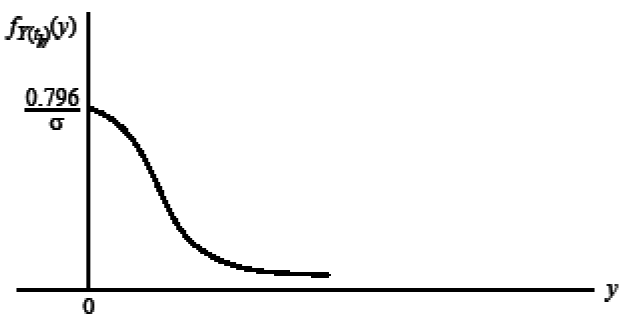
$$f_{Y(t_k)}(y) = f_{X(t_k)}(x = -y) \cdot \frac{dX(t_k)}{dY(t_k)} + f_{X(t_k)}(x = y) \cdot \frac{dX(t_k)}{dY(t_k)}$$

$$= \sqrt{\frac{2}{\pi}}\ \frac{1}{\sigma}\exp -\frac{y^2}{2\sigma^2}$$

We may therefore write

$$f_{Y(t_k)}(y) = \begin{cases} \sqrt{\frac{2}{\pi}}\ \frac{1}{\sigma}\exp -\frac{y^2}{2\sigma^2}, & y \geq 0 \\ 0, & y < 0. \end{cases}$$

which is illustrated below:



Problem 5.21

(a) The probability density function of the random variable $Y(t_k)$, obtained by observing the rectifier output $Y(t)$ at time t_k, is

$$f_{Y(t_k)}(y) = \begin{cases} \dfrac{1}{\sqrt{2\pi y}\sigma_X},\exp\left(-\dfrac{y^2}{2\sigma_X}\right), & y \ge 0 \\ 0, & y < 0. \end{cases}$$

where $\sigma_X^2 = E[X^2(t_k)] - \{E[X(t_k)]\}^2$

$$= E[X^2(t_k)]$$

$$= R_X(0)$$

The mean value of $Y(t_k)$ is therefore

$$E[Y(t_k)] = \int_{-\infty}^{\infty} f_{Y(t_k)}(y)dy$$

$$= \frac{1}{\sqrt{2\pi y}\sigma_X}\int_0^{\infty} \sqrt{y}\exp\left(-\frac{y^2}{2\sigma_X^2}\right)dy$$

Put

$$\frac{y}{\sigma_X^2} = u^2$$

Then, we may rewrite Eq. (1) as

$$E[Y(t_k)] = \sqrt{\frac{2}{\pi}}\sigma_X^2\int_0^{\infty} u^2\exp\left(-\frac{u^2}{2}\right)du$$

$$= \sigma_X^2$$

$$= R_X(0)$$

(b) The autocorrelation function of $Y(t)$ is

$$R_Y(\tau) = E[Y(t+\tau)Y(t)]$$

Since $Y(t) = X^2(t)$, we have

$$R_Y(\tau) = E[X^2(t+\tau)X^2(t)]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1^2 f_{X(t_k+\tau),X(t_k)}(x_1, x_2)(dx_1)dx_2 \qquad (2)$$

The $X(t_k + \tau)$ and $X(t_k)$ are jointly Gaussian with a joint probability density function defined by

$$f_{X(t_k+\tau),X(t_k)}(x_1, x_2) = \frac{1}{2\pi\sigma_X^2\sqrt{1-\rho_X^2(\tau)}}\exp\left[-\frac{x_{1-2\rho_X(\tau)x_1,}^2 x_2 + x_2^2}{2\sigma_X^2(1-\rho_X^2(\tau))}\right]$$

where $\sigma_X^2 = R_X(0)$,

$$\rho_X(\tau) = \frac{\mathrm{Cov}[X(t_k+\tau)X(t_k)]}{\sigma_X^2},$$

$$= \frac{R_X(\tau)}{R_X(0)}$$

Rewrite Eq. (2) in the form:

$$R_Y(\tau) = \frac{1}{2\pi\sigma_X^2\sqrt{1-\rho_X^2(\tau)}}\int_{-\infty}^{\infty} x_2^2\exp\left(-\frac{x_2^2}{2\sigma_X^2}\right)g(x_2)dx_2$$

where

$$g(x_2) = \int_{-\infty}^{\infty} x_1^2\exp\left\{-\frac{[x_1-\rho_X(\tau)x_2]^2}{2\sigma_X^2[1-\rho_X^2(\tau)]}\right\}dx_1$$

Let

$$u = \frac{x_1-\rho_X(\tau)x_2}{\sigma_X\sqrt{1-\rho_X^2(\tau)}}$$

Then, we may express $g(x_2)$ in the form

$$g(x_2) = \sigma_X\sqrt{1-\rho_X^2(\tau)}\int_{-\infty}^{\infty}\exp\left(-\frac{u^2}{2}\right)\left\{\rho_X^2(\tau)x_2^2 + \sigma_X^2[1-\rho_X^2(\tau)]u^2 + 2\sigma_X\rho_X\sqrt{1-\rho_X^2(\tau)}ux_2\right\}du$$

However, we note that

$$\int_{-\infty}^{\infty} \exp\left(-\frac{u^2}{2}\right) du = \sqrt{2\pi}$$

$$\int_{-\infty}^{\infty} u \exp\left(-\frac{u^2}{2}\right) du = 0$$

$$\int_{-\infty}^{\infty} u^2 \exp\left(-\frac{u^2}{2}\right) du = \sqrt{2\pi}$$

Hence,

$$g(x_2) = \sigma_X \sqrt{2\pi[1-\rho_X^2(\tau)]}\left\{\rho_X^2(\tau)x_2^2 + \sigma_X^2[1-\rho_X^2(\tau)]\right\} dx_2$$

Thus, from Eq. (3):

$$R_Y(\tau) = \frac{1}{\sqrt{2\pi}\sigma_X}\int_{-\infty}^{\infty} x_2^2 \exp\left(-\frac{x_2^2}{2\sigma_X^2}\right)\left\{\rho_X^2(\tau)x_2^2 + \sigma_X^2[1-\rho_X^2(\tau)]\right\} dx_2$$

Using the results:

$$\int_{-\infty}^{\infty} x_2^2 \exp\left(-\frac{x_2^2}{2\sigma_X^2}\right) dx_2 = \sqrt{2\pi}\sigma_X^3$$

$$\int_{-\infty}^{\infty} x_2^4 \exp\left(-\frac{x_2^2}{2\sigma_X^2}\right) dx_2 = 3\sqrt{2\pi}\sigma_X^5$$

we obtain,

$$R_Y(\tau) = 3\sigma_X^4\rho_X^2(\tau) + \sigma_X^4[1-\rho_X^2(\tau)]$$

$$= \sigma_X^4[1+2\rho_X^2(\tau)]$$

Since $\sigma_X^2 = R_X(0)$

$$\rho_X(\tau) = \frac{R_X(\tau)}{R_X(0)}$$

we obtain

$$R_Y(\tau) = R_X^2(0)\left[1 + 2\frac{R_X^2(\tau)}{R_X^2(0)}\right]$$

$$= R_X^2(0) + 2R_X^2(\tau)$$

The autocovariance function of $Y(t)$ is therefore

$$C_Y(\tau) = R_Y(\tau) - \{E[Y(t_k)]\}^2$$

$$= R_X^2(0) + 2R_X^2(\tau)$$

$$= 2R_X^2(\tau)$$

Problem 5.22

(a) The random variable $Y(t_1)$ obtained by observing the filter output of impulse response $h_1(t)$, at time t_1, is given by

$$Y(t_1) = \int_{-\infty}^{\infty} X(t_1 - \tau)h_1(\tau)d\tau$$

The expected value of $Y(t_1)$ is

$$m_{Y_1} = E[Y(t_1)]$$

$$= H_1(0)m_X$$

where

$$H_1(0) = \int_{-\infty}^{\infty} h_1(\tau)d\tau$$

The random variable $Z(t_2)$ obtained by observing the filter output of impulse response $h_2(t)$, at time t_2, is given by

$$Z(t_2) = \int_{-\infty}^{\infty} X(t_2 - u)h_2(u)du$$

The expected value of $Z(t_2)$ is

$$m_{Z_2} = E[z(t_2)]$$

$$= H_2(0)m_X$$

where

$$H_2(0) = \int_{-\infty}^{\infty} h_2(u)du$$

The random variable $Z(t_2)$ obtained by observing the filter output of impulse response $h_2(t)$, at time t_2, is given by

$$Z(t_2) = \int_{-\infty}^{\infty} X(t_2 - u)h_2(u)du$$

The expected value of $Z(t_2)$ is

$$m_{Z_2} = E[z(t_2)]$$

$$= H_2(0)m_X$$

where

$$H_2(0) = \int_{-\infty}^{\infty} h_2(u)du$$

The covariance of $Y(t_1)$ and $Z(t_2)$ is

$$\mathrm{Cov}[Y(t_1)Z(t_2)] = E[(Y(t_1) - \mu_{Y_1})(Z(t_2) - \mu_{Y_2})]$$

$$= E\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(X(t_1 - \tau) - \mu_X)(X(t_2 - u) - \mu_X)(h_1)(\tau)h_2(u)(d\tau)du\right]$$

$$= E\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(X(t_1 - \tau) - \mu_X)(X(t_2 - u) - \mu_X)\right]h_1(\tau)h_2(u)(d\tau)du$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C_X(t_1 - t_2 - \tau + u)h_1(\tau)h_2(u)(d\tau)du$$

where $C_X(\tau)$ is the autocovariance function of $X(t)$. Next, we note that the variance of $Y(t_1)$ is

$$\sigma_{Y_1}^2 = E\left[(Y(t_1) - \mu_{Y_1})^2\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C_X(\tau - u)h_1(\tau)h_1(u)d\tau du$$

and the variance of $Z(t_2)$ is

$$\sigma_{Z_2}^2 = E\left[(Z(t_2) - \mu_{Z_1})^2\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C_X(\tau - u)h_2(\tau)h_2(u)d\tau du$$

The correlation coefficient of $Y(t_1)$ and $Z(t_2)$ is

$$\rho = \frac{\mathrm{cov}[Y(t_1)Z(t_2)]}{\sigma_{Y_1}\sigma_{Z_1}}$$

Since $X(t)$ is a Gaussian process, it follows that $Y(t_1)$ and $Z(t_2)$ are jointly Gaussian with a probability density function given by

$$f_{Y(t_1), Z(t_2)}(y_1, z_2) = K\exp[-Q(y_1, z_2)]$$

where

$$K = \frac{1}{2\pi\sigma_{Y_1}\sigma_{z_2}\sqrt{1-\rho^2}}$$

$$Q(y_1, z_2) = \frac{1}{2(1-\rho^2)}\left[\left(\frac{y_1-\mu_{Y_1}}{\sigma_{Y_1}}\right)^2 - 2\rho\left(\frac{y_1-\mu_{Y_1}}{\sigma_{Y_1}}\right)\left(\frac{z_2-\mu_{Z_2}}{\sigma_{Z_2}}\right) + \left(\frac{z_2-\mu_{Z_2}}{\sigma_{Z_2}}\right)^2\right]$$

(b) The random variables $Y(t_1)$ and $Z(t_2)$ are uncorrelated if and only if their covariance is zero. Since $Y(t)$ and $Z(t)$ are jointly Gaussian processes, it follows that $Y(t_1)$ and $Z(t_2)$ are statistically independent if Cov[$Y(t_1)$ and $Z(t_2)$] is zero. Therefore, the necessary and sufficient condition for $Y(t_1)$ and $Z(t_2)$ to be statistically independent is that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C_X(t_1 - t_2 - \tau + u)h_1(\tau)h_2(u)d\tau du = 0$$

for choices of t_1 and t_2.

Problem 5.23

(a) The filter output is

$$Y(t) = \int_{-\infty}^{\infty} h(\tau)X(t-\tau)d\tau$$

$$= \frac{1}{T}\int_0^T X(t-\tau)d\tau$$

Put $T - t = u$. Then, the sample value of $Y(t)$ at $t = T$ equals

$$Y = \frac{1}{T}\int_0^T X(u)du$$

The mean of Y is therefore

$$E[Y] = E\left[\frac{1}{T}\int_0^T X(u)du\right]$$

$$= \frac{1}{T}\int_0^T E[X(u)]du$$

$$= 0$$

The variance of Y is

$$\sigma_Y^2 = E[Y^2] - \{E[Y]\}^2$$

$$= R_Y(0)$$

$$= \int_{-\infty}^{\infty} S_Y(f)df$$
$$= \int_{-\infty}^{\infty} S_X(f)|H(f)|^2 df$$

But

$$H(f) = \int_{-\infty}^{\infty} h(t)\exp(-j2\pi ft)dt$$

$$= \frac{1}{T}\int_0^T \exp(-j2\pi ft)dt$$

$$= \frac{1}{T}\left[\frac{\exp(-j2\pi ft)}{-j2\pi ft}\right]_0^T$$

$$= \frac{1}{j2\pi fT}[1 - \exp(-j2\pi fT)]$$

$$= \text{sinc}(fT)\exp(-j2\pi fT)$$

Therefore,

$$\sigma_Y^2 = \int_{-\infty}^{\infty} S_X(f)\,\text{sinc}^2(fT)df$$

(b) Since the filter input is Gaussian, it follows that Y is also Gaussian. Hence, the probability density function of Y is

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma_Y}\exp\left(-\frac{y^2}{2\sigma_Y^2}\right)$$

where σ_Y^2 is defined above.

Problem 5.24

(a) The power spectral density of the noise at the filter output is given by

$$S_N(f) = \frac{N_0}{2}\left|\frac{j2\pi fL}{R + j2\pi fL}\right|^2$$

$$S_N(f) = \frac{N_0}{2}\ \frac{(j2\pi fL/R)^2}{1 + (2\pi fL/R)^2}$$

$$= \frac{N_0}{2}\left[1 - \frac{1}{1 + (2\pi fL/R)^2}\right]$$

The autocorrelation function of the filter output is therefore

$$R_N(\tau) = \frac{N_0}{2}\left[\delta(\tau) - \frac{R}{2L}\exp\left(1 - \frac{R}{L}|\tau 1|\right)\right]$$

(b) The mean of the filter output is equal to $H(0)$ times the mean of the filter input. The process at the filter input has zero mean. The value $H(0)$ of the filter's transfer function $H(f)$ is zero. It follows therefore that the filter output also has a zero mean.

The mean-square value of the filter output is equal to $R_N(0)$. With zero mean, it follows therefore that the variance of the filter output is

$$\sigma_N^2 = R_N(0)$$

Since $R_N(\tau)$ contains a delta function $\delta(\tau)$ centered on $\tau = 0$, we find that, in theory σ_N^2 is infinitely large.

Problem 5.25

(a) The noise equivalent bandwidth is

$$W_N = \frac{1/2}{|H(0)|^2}\int_{-\infty}^{\infty}|H(f)|^2 df$$

$$= \frac{1}{2}\int_{-\infty}^{\infty}\frac{df}{1+(f/f_0)^{2n}}$$

$$= \int_{-\infty}^{\infty}\frac{df}{1+(f/f_0)^{2n}}$$

$$= \frac{f_0}{\text{sinc}(1/2n)}$$

(b) When the filter order n approaches infinity, we have

$$W_N = f_0 \lim_{n\to\infty}\frac{1}{\text{sinc}(1/2n)}$$
$$= f_0$$

Problem 5.26

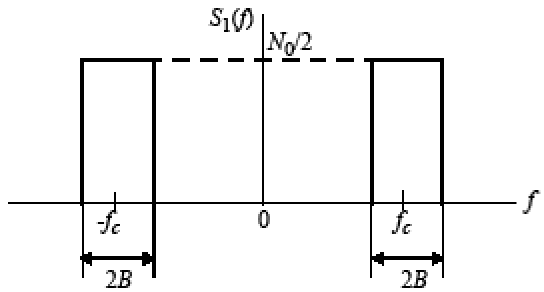
The process $X(t)$ defined by

$$X(t) = \sum_{k=-\infty}^{\infty} h(t-\tau_k),$$

where $h(t - \tau_k)$ is a current pulse at time τ_k, is stationary for the following simple reason. There is no distinguishing origin of time.

Problem 5.27

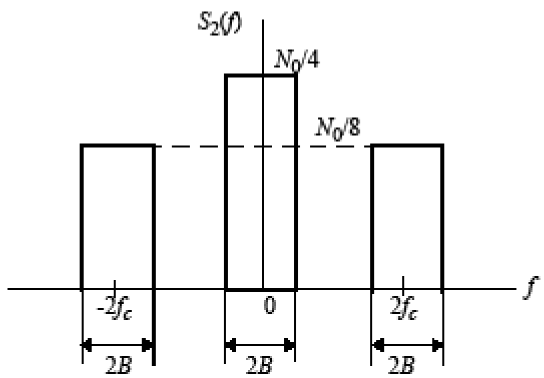
(a) Let $S_1(f)$ denote the power spectral density of the noise at the first filter output. The dependence of $S_1(f)$ on frequency is illustrated below:



Let $S_2(f)$ denote the power spectral density of the noise aat the mixer output. Then, we may write

$$S_2(f) = \frac{1}{4}[S_1(f+f_c) + S_1(f-f_c)]$$

which is illustrated below:



The power spectral density of the noise $n(t)$ at the second filter output is therefore defined by

$$S_o(f) = \begin{cases} \frac{N_0}{4}, & -B < f < B \\ 0, & \text{otherwise} \end{cases}$$

The autocorrelation function of the noise $n(t)$ is

$$R_o(\tau) = \frac{N_0 B}{2}\operatorname{sinc}(2B\tau)$$

(b) The mean value of the noise at the system output is zero. Hence, the variance and mean-square value of this noise are the same. Now, the total area under $S_o(f)$ is equal to $(N_0/4)(2B) = N_0B/2$. The variance of the noise at the system output is therefore $N_0B/2$.

(c) The maximum rate at which $n(t)$ can be sampled for the resulting samples to be uncorrelated is $2B$ samples per second.

Problem 5.28

(a) The autocorrelation function of the filter output is

$$R_X(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)R_W(\tau - \tau_1 + \tau_2)d\tau_1 d\tau_2$$

Since $R_W(\tau) = (N_0/2)\delta(\tau)$, we find that the impulse response $h(t)$ of the filter must satisfy the condition:

$$R_X(\tau) = \frac{N_0}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)\delta(\tau - \tau_1 + \tau_2)d\tau_1 d\tau_2$$
$$= \frac{N_0}{2}\int_{-\infty}^{\infty} h(\tau_1 + \tau_2)h(\tau_2)d\tau_2$$

(b) For the filter output to have a power spectral density equal to $S_X(f)$, we have to choose the transfer function $H(f)$ of the filter such that

$$S_X(f) = \frac{N_0}{2}|H(f)|^2$$

or

$$|H(f)| = \sqrt{\frac{2S_X(f)}{N_0}}$$

c)For a given filter, $H(f)$, let $\alpha = \ln|H(f)|$

and the Paley-Wiener criterion for causality is: $\int_{-\infty}^{\infty} \frac{|\alpha(f)|}{1+(2\pi f)^2}df < \infty$

For the filter of part (b)

$$\alpha(f) = \frac{1}{2}\left[\ln(2) + \ln(S_x(f) - \ln(N_0)\right]$$

The first and the last terms have no impact on the absolute integrability of the previous expression, and so do not matter as far as evaluating the above criterion. This leaves the only condition:

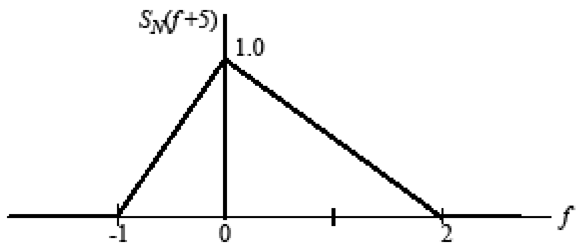
$$\int_{-\infty}^{\infty} \frac{|\ln S_x(f)|}{1+(2\pi f)^2}df < \infty$$

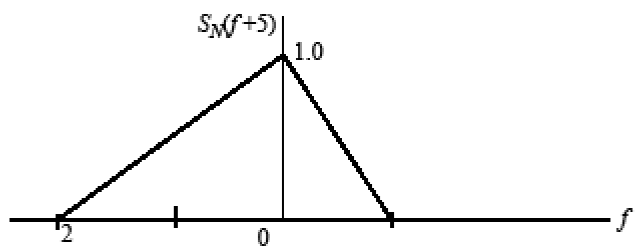
Problem 5.29

(a) The power spectral density of the in-phase component or quadrature component is defined by

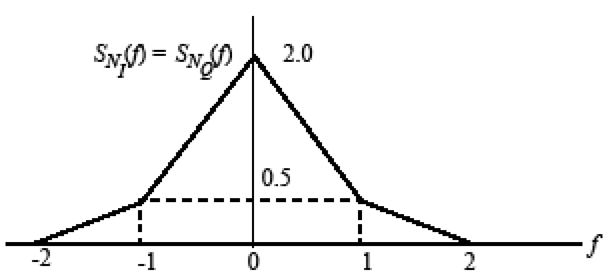
$$S_{N_I}(f) = S_{N_Q}(f) = \begin{cases} S_N(f+f_c) + S_N(f-f_c) & -B \le f \le B \\ 0 & \text{otherwise} \end{cases}$$

We note that, for -2 $\le f \le$ 2, the $S_N(f + 5)$ and $S_N(f - 5)$ are as shown below:





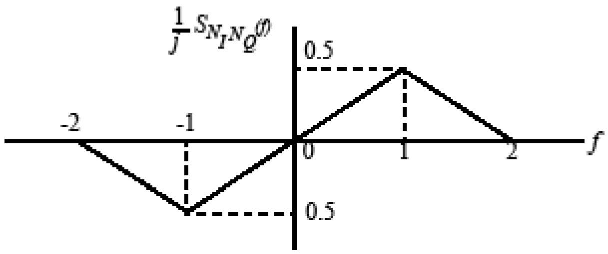
We thus find that $S_{N_I}(f)$ or $S_{N_Q}(f)$ is as shown below:



(b) The cross-spectral density $S_{N_I N_Q}(f)$ is defined by

$$S_{N_I N_Q}(f) = \begin{cases} j[S_N(f+f_c) - S_N(f-f_c)], & -B \le f \le B \\ 0, & \text{otherwise} \end{cases}$$

We therefore find that $S_{N_I N_O}(f)/j$ is as shown below:



Problem 5.30

(a) Express the noise $n(t)$ in terms of its in-phase and quadrature components as follows:

$$n(t) = n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t)$$

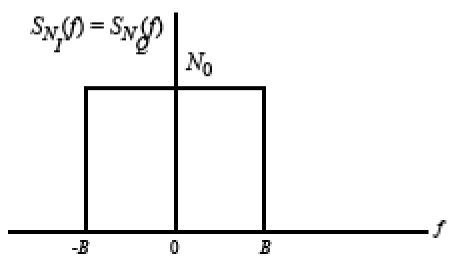
The envelope of $n(t)$ is

$$r(t) = \sqrt{n_I^2(t) + n_Q^2(t)}$$

which is Rayleigh-distributed. That is

$$f_R(r) = \begin{cases} \dfrac{r}{\sigma^2}\exp-\dfrac{r^2}{2\sigma^2}, & r \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

To evaluate the variance σ^2, we note that the power spectral density of $n_I(t)$ or $n_Q(t)$ is as follows



Since the mean of $n(t)$ is zero, we find that

$$\sigma^2 = 2N_0B$$

Therefore,

$$f_R(r) = \begin{cases} \dfrac{r}{2N_0B}\exp\left(-\dfrac{r^2}{4N_0B}\right), & r \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

(b) The mean value of the envelope is equal to $\sqrt{\pi N_0 B}$, and its variance is equal to 0.858 N_0B.

Problem 5.31

(a) Consider the part of the analyzer in Fig. 1.19 defining the in-phase component $n_I(t)$, reproduced here as Fig. 1:

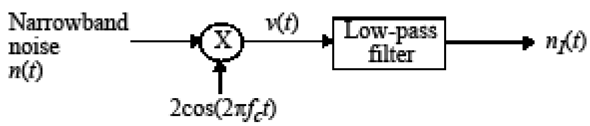


Figure 1

For the multiplier output, we have

$$v(t) = 2n(t)\cos(2\pi f_c t)$$

Applying Eq. (1.55) in the textbook, we therefore get

$$S_V(f) = [S_N(f-f_c) + S_N(f+f_c)]$$

Passing $v(t)$ through an ideal low-pass filter of bandwidth B, defined as one-half the bandwidth of the narrowband noise $n(t)$, we obtain

$$S_{N_I}(f) = \left\{ \begin{array}{ll} S_V(f) & \text{for } -B \le f \le B \\ 0 & \text{otherwise} \end{array} \right\}$$

$$= \left\{ \begin{array}{ll} S_N(f-f_c) + S_N(f+f_c) & \text{for } -B \le f \le B \\ 0 & \text{otherwise} \end{array} \right\} \qquad (1)$$

For the quadrature component, we have the system shown in Fig. 2:

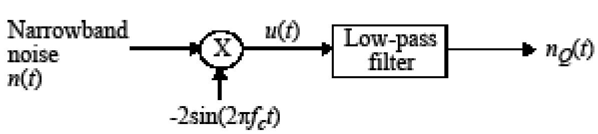


Fig. 2

The multiplier output $u(t)$ is given by

$$u(t) = -2n(t)\sin(2\pi f_c t)$$

Hence,

$$S_U(f) = [S_N(f-f_c) + S_N(f+f_c)]$$

and

$$S_{N_Q}(f) = \begin{cases} S_U(f) & \text{for } -B \le f \le B \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} S_N(f-f_c) + S_N(f+f_c) & \text{for } -B \le f \le B \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Accordingly, from Eqs. (1) and (2) we have

$$S_{N_I}(f) = S_{N_Q}(f)$$

(b) Applying Eq. (1.78) of the textbook to Figs. 1 and 2, we obtain

$$S_{N_I N_Q}(f) = |H(f)|^2 S_{VU}(f) \tag{3}$$

where

$$|H(f)| = \begin{cases} 1 & \text{for } -B \le f \le B \\ 0 & \text{otherwise} \end{cases}$$

Applying Eq. (1.23) of the textbook to the problem at hand:

$$R_{VU}(\tau) = 2R_N(\tau)\sin(2\pi f_c \tau) = \frac{1}{j}R_N(\tau)\left(e^{j2\pi f_c \tau} - e^{-j2\pi f_c \tau}\right)$$

Applying the Fourier transform to both sides of this relation:

$$S_{VU}(t) = \frac{1}{j}(S_N(f-f_c) - S_N(f+f_c)) \tag{4}$$

Substituting Eq. (4) into (3):

$$S_{N_I N_Q}(f) = \begin{cases} j[S_N(f+f_c) - S_N(f-f_c)] & \text{for } -B \le f \le B \\ 0 & \text{otherwise} \end{cases}$$

which is the desired result.

Problem 5.32

If the power spectral density $S_N(f)$ of narrowband noise $n(t)$ is symmetric about the midband frequency f_c we then have

$$S_N(f-f_c) = S_N(f+f_c) \quad \text{for} \quad -B \le f \le B$$

From part (b) of Problem 1.28, the cross-spectral densities between the in-phase noise component $n_I(t)$ and quadrature noise component $n_Q(t)$ are zero for all frequencies:

$$S_{N_I N_Q}(f) = 0 \quad \text{for} \quad \text{all } f$$

This, in turn, means that the cross-correlation functions $R_{N_I N_Q}(\tau)$ and $R_{N_Q N_I}(\tau)$ are both zero, that is,

$$E[N_I(t_k+\tau)N_Q(t_k)] = 0$$

which states that the random variables $N_I(t_k + \tau)$ and $N_Q(t_k)$, obtained by observing $n_I(t)$ at time $t_k + \tau$ and observing $n_Q(t)$ at time t_k, are orthogonal for all t.

If the narrow-band noise $n(t)$ is Gaussian, with zero mean (by virtue of the narrowband nature of $n(t)$), then it follows that both $N_I(t_k + \tau)$ and $N_Q(t_k)$ are also Gaussian with zero mean. We thus conclude the following:

- $N_I(t_k + \tau)$ and $N_Q(t_k)$ are both uncorrelated
- Being Gaussian and uncorrelated, $N_I(t_k + \tau)$ and $N_Q(t_k)$ are therefore statistically independent.

That is, the in-phase noise component $n_I(t)$ and quadrature noise component $n_Q(t)$ are statistically independent.

Problem 5.33

(a) The receiver position is given by $x(t) = x_0 + vt$ Thus the signal observed by the receiver is

$$\begin{aligned}
r(t,x) &= A(x)\cos\left[2\pi f_c\left(t-\frac{x}{c}\right)\right] \\
&= A(x)\cos t\left[2\pi f_c\left(t-\frac{x_0+vt}{c}\right)\right] \\
&= A(x)\cos\left[2\pi\left(f_c-\frac{f_c v}{c}\right)t - f_c\frac{x_0}{c}\right]
\end{aligned}$$

The Doppler shift of the frequency observed at the receiver is $f_D = \dfrac{f_c v}{c}$.

(b) The expectation is given by

$$\begin{aligned}
\mathbf{E}\left[\exp\left(j2\pi f_n\tau\right)\right] &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\exp\left(j2\pi f_D\tau\cos\psi_n\right)d\psi_n \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi}\exp\left(j2\pi f_D\tau\sin\psi_n\right)d\psi_n \\
&= J_0\left(2\pi f_D\tau\right)
\end{aligned}$$

where the second line comes from the symmetry of cos and sin under a $-\pi/2$ translation.

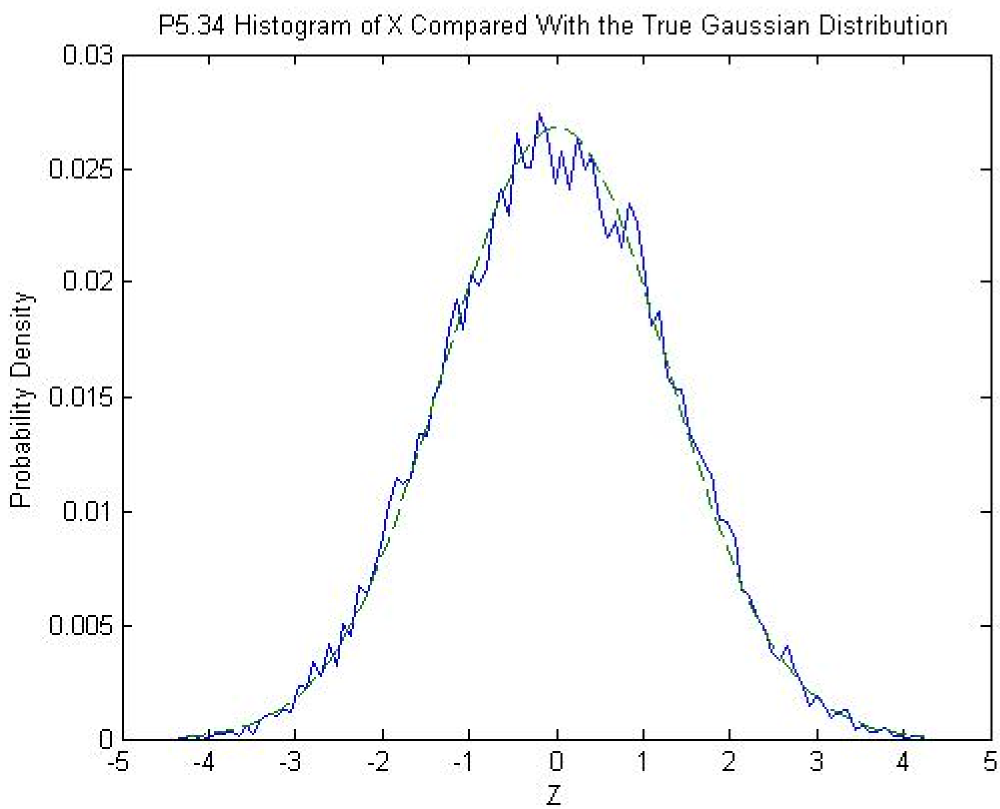
Eq. (5.174) follows directly from this upon noting that, since the expectation result is real-valued, the right-hand side of Eq.(5.173) is equal to its conjugate.

Problem 5.34

The histogram has been plotted for 100 bins. Larger numbers of bins result in larger errors, as the effects of averaging are reduced.

Distance	Relative Error
0σ	0.94%
1σ	2.6 %
2σ	4.8 %
3σ	47.4%
4σ	60.7%

The error increases further out from the centre. It is also important to note that the random numbers generated by this MATLAB procedure can never be greater than 5. This is very different from the Gaussian distribution, for which there is a non-zero probability for any real number.



5.34 Code Listing

```
%Problem 5.34
%Set the number of samples to be 20,000
N=20000
M=100;
Z=zeros(1,20000);

for i=1:N
    for j=1:5
        Z(i)=Z(i)+2*(rand(1)-0.5);
    end
end
sigma=sqrt(var(Z-mean(Z)));

%Calculate a histogram of Z
[X,C]=hist(Z,M);
l=linspace(C(1),C(M),M);

%Create a gaussian function with the same variance as Z
G=1/(sqrt(2*pi*sigma^2))*exp(-(l.^2)/(2*sigma^2));
delta2=abs(l(1)-l(2));
X=X/(20000*delta2);
```

5.35 (a) For the generated sequence:

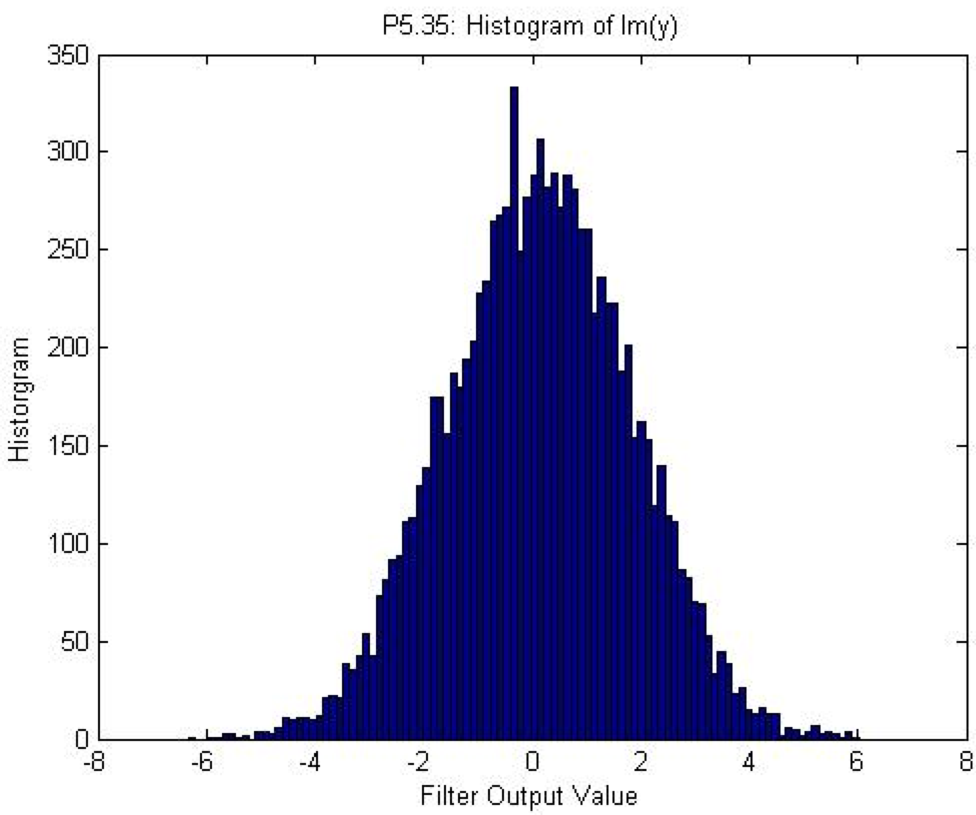
$$\hat{\mu}_y = -0.0343 + j0.0493$$
$$\hat{\sigma}_y^2 = 5.597$$

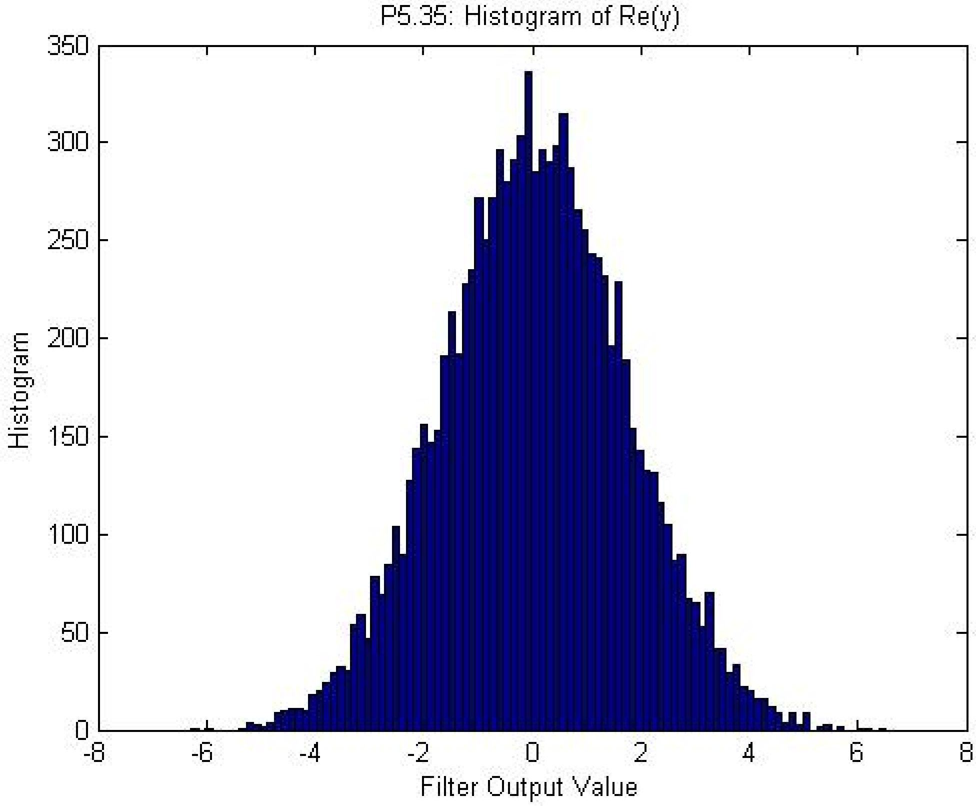
The theoretical values are: $\mu_y = 0$ (by inspection).

The theoretical value of σ_y^2 =5.56. See 5.35 (c) for the calculation.

5.35 (b)

From the plots, it can be seen that both the real and imaginary components are approximately Gaussian. In addition, from statistics, the sum of tow zero-mean Gaussian signals is also Gaussian distributed. As a result, the filter output must also be Gaussian.





5.35 (c)

$$y(n) = ay(n-1) + w(n)$$
$$Y(z) = aY(z)z^{-1}$$

$$\therefore H(z) = \frac{1}{1-az^{-1}} \rightleftarrows h(n) = a^{|n|}u(n)$$

$$R_h(z) = H(z)H(z^{-1}) = \frac{1}{(1-az^{-1})(1-az)}$$

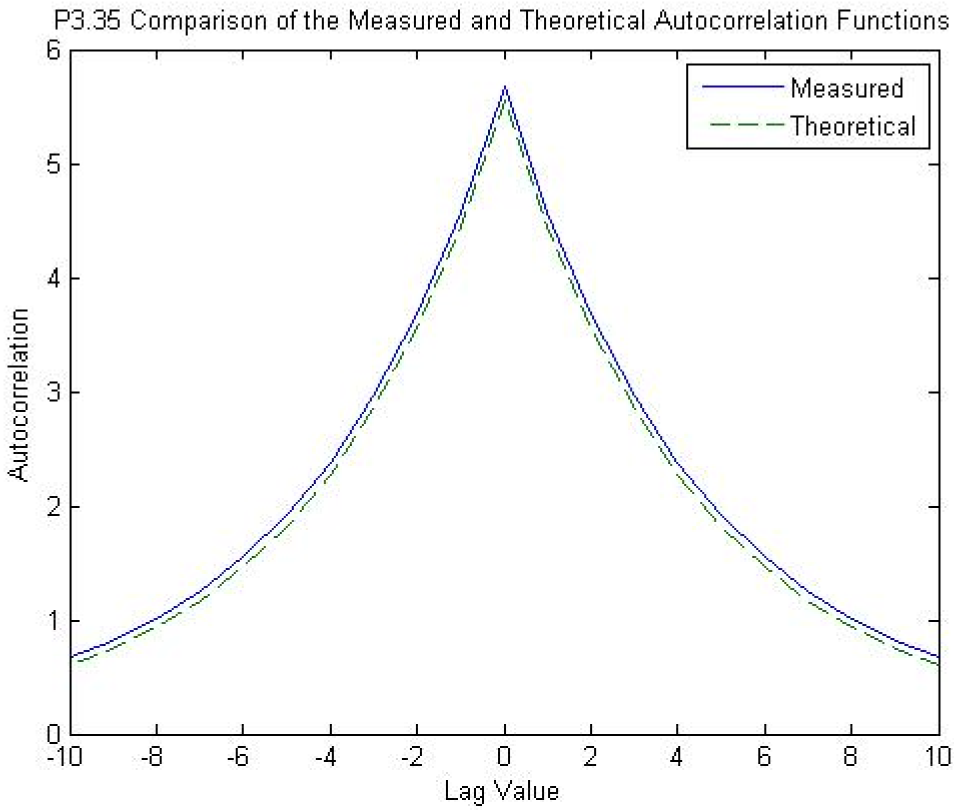
$$= \frac{a}{1-a^2}\frac{z^{-1}}{1-az^{-1}} + \frac{1}{1-a^2}\frac{1}{1-az}$$

But, $R_y(z) = R_h(z)R_w(z)$

Taking the inverse z-transform:

$$r_y(n) = \frac{\sigma_w^2}{1-a^2}a^{|n|} \qquad -\infty < n < \infty$$

From the plots, the measured and observed autocorrelations are almost identical.


P3.35 Comparison of the Measured and Theoretical Autocorrelation Functions
Measured
Theoretical
Autocorrelation
Lag Value
0
1
2
3
4
5
6
-10
-8
-6
-4
-2
0
2
4
6
8
10

Chapter 6 Solutions

Problem 6.3

After passing the received signal through a narrow-band filter of bandwidth 8kHz centered on f_c = 200kHz, we get

$$x(t) = A_c m(t)\cos(2\pi f_c t) + n'(t)$$

$$= A_c m(t)\cos(2\pi f_c t) + n'_I(t)\cos(2\pi f_c t) - n'_I(t)\sin(2\pi f_c t)$$

$$= (A_c m(t) + n_I(t))\cos(2\pi f_c t) - n'_Q(t)\sin(2\pi f_c t)$$

where $n'(t)$ is the narrow-band noise produced at the filter output, and $n'_I(t)$ and $n'_Q(t)$ are its in-phase and quadrature components. Coherent detection of $x(t)$ yields the output

$$y(t) = A_c m(t) + n'_I(t)$$

The average power of the modulated wave is

$$\frac{A_c^2 P}{4} = 10W$$

where P is the average power of $m(t)$. To calculate the average power of the in-phase noise component $n'_I(t)$, we refer to the spectra shown in Fig. 1:

- Part (a) of Fig. 1 shows the power spectral density of the noise $n(t)$, and a superposition of the frequency response of the narrow-band filter.
- Part (b) shows the power spectral density of the noise $n'_I(t)$ produced at the filter output.
- Part (c) shows the power spectral density of the in-phase component $n'_I(t)$ of $n'(t)$.

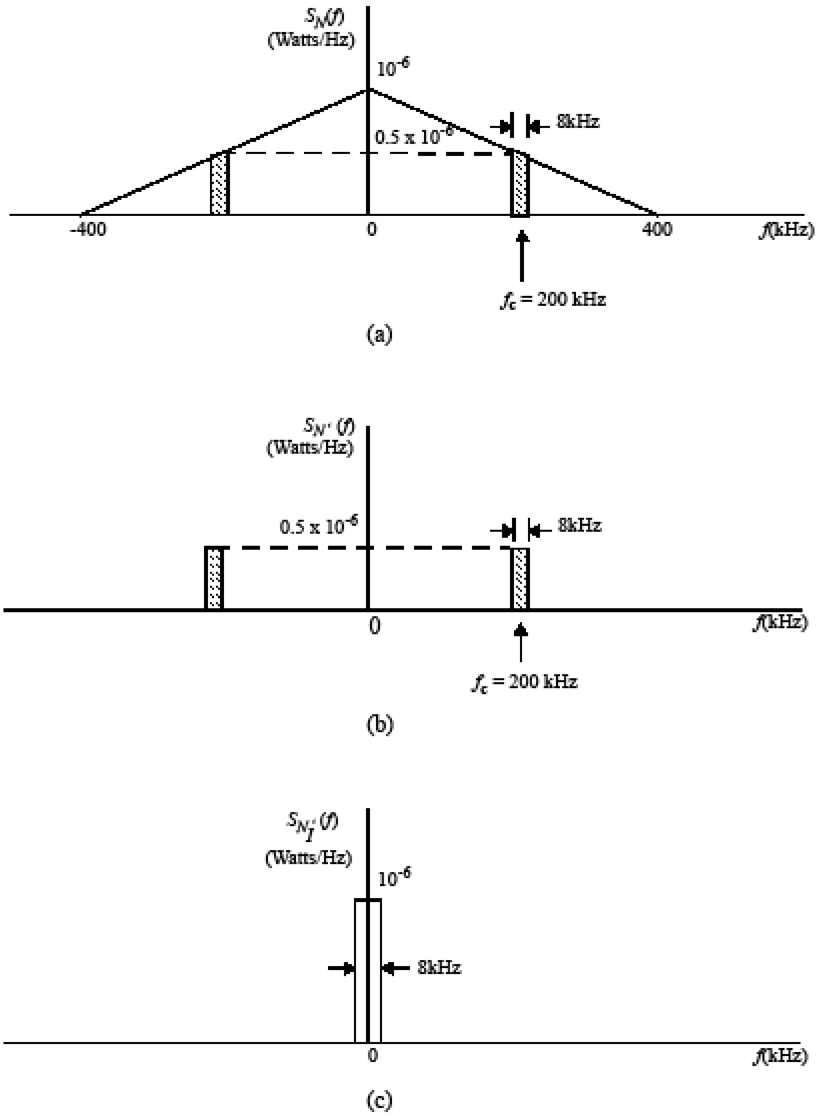
Note that since the bandwidth of the filter is small compared to the carrier frequency f_c, we have approximated the spectral characteristic of $n'(t)$ to be flat at the level of 0.5 x 10^{-6} watts/Hz. Hence, the average power of $n'_I(t)$ is (from Fig. 1c):

$$(10^{-6} \text{ watts/Hz})\ (8 \times 10^3) = 0.008 \text{ watts}$$

The output signal-to-noise ratio (SNR) is therefore

$$\frac{10}{0.008} = 1,250$$

Expressing this result in decibels, we have an output SNR of 31 dB.



$S_N(f)$ (Watts/Hz)
10^{-6}
0.5×10^{-6}
8kHz
-400
0
400
f(kHz)
f_c = 200 kHz
(a)
$S_{N'}(f)$ (Watts/Hz)
0.5×10^{-6}
8kHz
0
f(kHz)
f_c = 200 kHz
(b)
$S_{N_I'}(f)$ (Watts/Hz)
10^{-6}
8kHz
0
f(kHz)
(c)

Figure 1

Problem 6.4

we note that the quadrature components of a narrow-band noise have

$$R_{N_I}(\tau) = R_{N_Q}(\tau) = R_N(\tau)\cos(2\pi f_c\tau) + \hat{R}_N(\tau)\sin(2\pi f_c\tau)$$

where $R_N(\tau)$ is the autocorrelation of the narrow-band noise, $\hat{R}_N(\tau)$ is the Hilbert transform of $R_N(\tau)$, and f_c is the band center. The cross-correlation of the quadrature components are

$$R_{N_IN_Q}(\tau) = -R_{N_IN_Q}(\tau) = R_N(\tau)\sin(2\pi f_c\tau) - \hat{R}_N(\tau)\cos(2\pi f_c\tau)$$

(a) For a DSBSC system,

$$R_{N_I}(\tau) = R_{N_Q}(\tau) = R_N(\tau)\cos(2\pi f_c\tau) + \hat{R}_N(\tau)\sin(2\pi f_c\tau)$$

$$R_{N_IN_Q}(\tau) = -R_{N_QN_I}(\tau) = R_N(\tau)\sin(2\pi f_c\tau) - \hat{R}_N(\tau)\cos(2\pi f_c\tau)$$

where f_c is the carrier frequency, and $R_N(\tau)$ is the autocorrelation function of the narrow-band noise on the interval $f_c - W \le f \le f_c + W$.

(b) For an SSB system using the lower sideband,

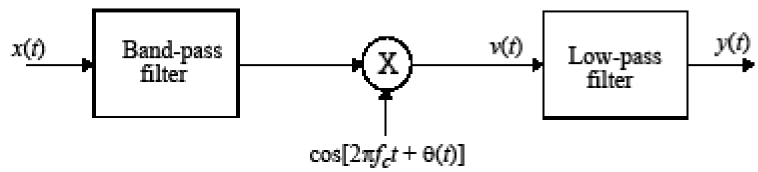
$$R_{N_I}(\tau) = R_{N_Q}(\tau) = R_N(\tau)\cos\left(2\pi\left(f_c - \frac{W}{2}\right)\tau\right) - \hat{R}_N(\tau)\cos\left(2\pi\left(f_c - \frac{W}{2}\right)\tau\right)$$

$$R_{N_IN_Q}(\tau) = -R_{N_QN_I}(\tau) = R_N(\tau)\sin\left(2\pi\left(f_c - \frac{W}{2}\right)\tau\right) - \hat{R}_N(\tau)\cos\left(2\pi\left(f_c - \frac{W}{2}\right)\tau\right)$$

where in this case, $R_N(\tau)$ is the autocorrelation of the narrow-band noise on the interval $f_c - W \le f \le f_c$.

(c) For an SSB system with only the upper sideband transmitted, the correlations are similar to (b) above, except that $\left(f_c - \frac{W}{2}\right)$ is replaced by $\left(f_c + \frac{W}{2}\right)$, and the narrow-band noise is on the interval $f_c \le f \le f_c + W$.

Problem 6.5



The signal at the mixer input is equal to $s(t) + n(t)$, where $s(t)$ is the modulated wave, and $n(t)$ is defined by

$$n(t) = n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t)$$

with

$$E[n_I^2(t)] = E[n_Q^2(t)] = N_0 B_T$$

The $s(t)$ is defined by for DSB-SC modulation

$$s(t) = A_c m(t)\cos(2\pi f_c t)$$

The mixer output is

$$v(t) = [s(t) + n(t)]\cos[2\pi f_c t + \theta(t)]$$

$$= \{[A_c m(t) + n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t)\}\cos 2\pi f_c t + \theta(t)]$$

$$= \frac{1}{2}[A_c m(t) + n_I(t)\{\cos[\theta(t)]\} + \cos[4\pi f_c t + \theta(t)]$$

$$+ \frac{1}{2}A_c n_Q(t)\{\sin[\theta(t)] - \sin[4\pi f_c t + \theta(t)]\}$$

The postdetection low-pass filter removes the high frequency components of $v(t)$, producing the output

$$y(t) = \frac{1}{2}[[A_c m(t) + n_I(t)]\cos[\theta(t)] + \frac{1}{2}A_c n_Q(t)\sin[\theta(t)] \qquad (1)$$

When the phase error $\theta(t)$ is zero, we find that the message signal component of the receiver output is $\frac{1}{2}A_c m(t)$. The error at the receiver output is therefore

$$e(t) = y(t) - \frac{A_c}{2}m(t)$$

The mean-square value of this error is

$$\varepsilon = E[e^2(t)]$$
$$= E\left[\left(y(t) - \frac{A_c}{2}m(t)\right)^2\right] \qquad (2)$$

Substituting Eq. (1) into (2), expanding the expectation, and noting that the processes $m(t)$, $\theta(t)$, $n_I(t)$ and $n_Q(t)$ are all independent of one another, we get

$$\varepsilon = \frac{A_c^2}{4}E[m^2(t)]E[(\cos^2\theta(t))] + \frac{1}{4}E[n_I^2(t)]E[\cos^2\theta(t)]$$

$$+\frac{1}{4}E[n_Q^2(t)]E[\sin^2\theta(t)]$$

$$+\frac{A_c^2}{4}E[m^2(t)] - \frac{A_c^2}{2}E[m^2(t)]E[\cos\theta(t)]$$

We now note that

$$E[n_I^2(t)] = E[n_Q^2(t)] = \sigma_N^2$$

$$E[n_I^2(t)]E[\cos^2\theta(t)] + E[n_Q^2(t)]E[\sin^2\theta(t)] = \sigma_N^2$$

Therefore,

$$\varepsilon = \frac{A_c^2}{2}E[m^2(t)]E\{[1-\cos\theta(t)]^2\} + \frac{\sigma_N^2}{4}$$

$$= \frac{A_c^2 P}{4}E\{[1+\cos\theta(t)]^2\} + \frac{\sigma_N^2}{4}$$

where $P = E[m^2(t)]$.

For small values of $\theta(t)$, we may use the approximation

$$1-\cos\theta(t) \approx \frac{\sigma_N^2}{2}$$

Hence,

$$\varepsilon = \frac{A_c^2 P}{16}E[\theta^4(t)] + \frac{\sigma_N^2}{4}$$

Since $\theta(t)$ is Gaussian-distributed with zero mean and variance σ_θ^2, we have

$$E[\theta^4(t)] = 3\sigma_\theta^4$$

The mean-square error for the case of a DSBSC system is therefore

$$\varepsilon = \frac{3A_c^2 P\sigma_\theta^4}{16} + \frac{\sigma_N^2}{4}$$

Problem 6.6

Problem 6.7

(a) If the probability

$$P(|n_s(t)|) > \varepsilon A_c|1 + k_a m(t)| \le \delta_1,$$

then, with a probability greater than 1 - δ_1, we may say that

$$y(t) \approx \left\{[A_c + A_c k_a m(t) + n_c(t)]^{1/2}\right\}^{1/2}$$

That is, the probability that the quadrature component $n_s(t)$ is negligibly small is greater than 1 - δ_1.

(b) Next, we note that if $k_a m(t) < -1$, then we get overmodulation, so that even in the absence of noise, the envelope detector output is badly distorted. Therefore, in order to avoid overmodulation, we assume that k_a is adjusted relative to the message signal $m(t)$ such that the probability

$$P(A_c + A_c k_a m(t) + n_c(t) < 0) = \delta_2$$

Then, the probability of the event

$$y(t) \approx A_c[1 + k_a m(t) + n_c(t)]$$

for any value of t, is greater than (1 - δ_1) (1 - δ_2).

(c) When δ_1 and δ_2 are both small compared with unity, we find that the probability of the event

$$y(t) \approx A_c[1 + k_a m(t)] + n_c(t)$$

for any value of t, is very close to unity. Then, the output of the envelope detector is approximately the same as the corresponding output of a coherent detector.

Problem 6.8

The received signal is

$$x(t) = A_c \cos(2\pi f_c t) + n(t)$$

$$= A_c \cos(2\pi f_c t) + n(t)\cos(2\pi f_c t) - n_s(t)\sin(2\pi f_c t)$$

$$= [A_c + n_c(t)]\cos(2\pi f_c t) - n_s(t)\sin(2\pi f_c t)$$

The envelope detector output is therefore

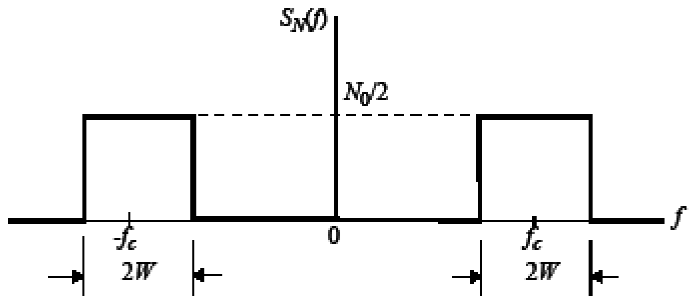
$$a(t) = \left\{ [A_c + n_c(t)]^2 + n_s^2(t) \right\}^{1/2}$$

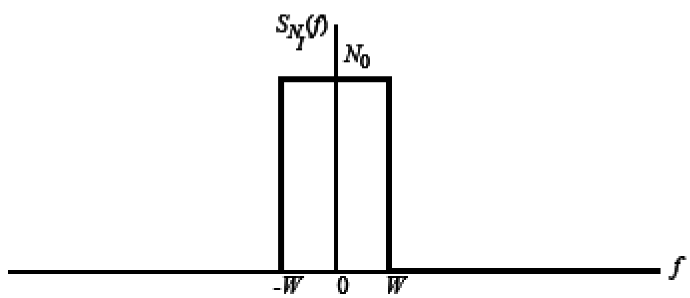
For the case when the carrier-to-noise ratio is high, we may approximate this result as

$$a(t) \approx A_c + n_c(t)$$

The term A_c represents the useful signal component. The output signal power is thus A_c^2.

The power spectral densities of $n(t)$ and $n_I(t)$ are as shown below:





The output noise power is $2N_0W$. The output signal-to-noise ratio is therefore

$$(\text{SNR})_0 = \frac{A_c^2}{2N_0W}$$

Problem 6.9

Problem 6.10

(a) Following a procedure similar to that described for the case of an FM system, we find that the input of the phase detector is

$$v(t) = A_c \cos[2\pi f_c t + \theta(t)]$$

where

$$\theta(t) = k_p m(t) + \frac{n_Q(t)}{A_c}$$

with $n_Q(t)$ denoting the quadrature noise component. The output of the phase discriminator is therefore,

$$y(t) = k_p m(t) + \frac{n_Q(t)}{A_c}$$

The message signal component of $y(t)$ is equap to $k_p m(t)$. Hence, the average output signal power is $k_p^2 P$, where P is the message signal power.

With the post detection low-pass filter following the phase detector restricted to the message bandwidth w, we find that the average output noise power is $2WN_0/A_c^2$.

Hence, the output signal-to-noise ratio of the PM system is

$$(\text{SNR})_0 = \frac{k_p^2 P A_c^2}{2WN_0}$$

(b) The channel signal-to-noise ratio of the PM system is the same as that of the corresponding FM system. That is,

$$(\text{SNR})_0 = \frac{A_c^2}{2WN_0}$$

The figure of merit of the PM system is therefore equal to $k_p^2 P$.

For the case of sinusoidal modulation we have

$$m(t) = A_m \cos(2\pi f_m t)$$

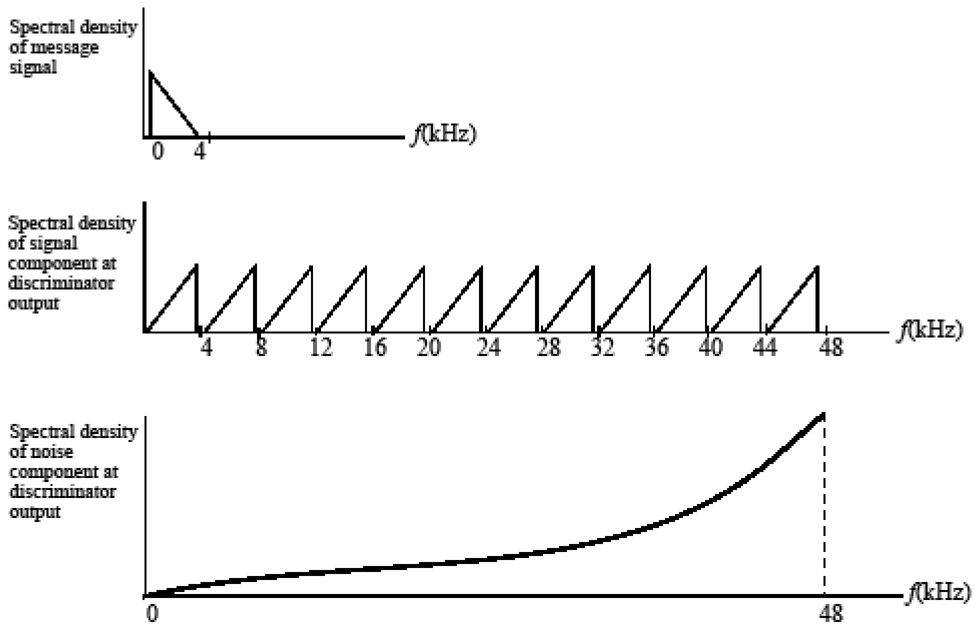
Hence,

$$P = \frac{A_m^2}{2}$$

The corresponding value of the figure of merit for a PM system is thus equal to $\frac{1}{2}\beta_p^2$, where $\beta_p = k_p A_m$. On the other hand, the figure of merit for an FM system with sinusoidal modulation is equal to $\frac{3}{2}\beta^2$. We see therefore that for a specified phase deviation, the FM system is 3 times as good as the PM system.

Problem 6.11

(a) The power spectral densities of the original message signal, and the signal and noise components at the frequency discriminator output (for positive frequencies) are illustrated below:



(b) Each SSB modulated wave contains only the lower sideband. Let A_k and kf_0 denote the amplitude and frequency of the carrier used to generate the kth modulated wave, where $f_0 = 4$ kHz, and $k = 1,2,...,12$. Then, we find that the kth modulated wave occupies the frequency interval $(k\text{-}1)$f0 $\leq |f| \leq kf_0$. We may define this modulated wave by

$$s_k(t) = \frac{A_k}{2}m(t)\cos(2\pi k f_0 t) + \frac{A_k}{2}\hat{m}(t)\sin(2\pi k f_0 t)$$

where $m(t)$ is the original message signal, and $\hat{m}(t)$ is its Hilbert transform. Therefore, the average power of $s_k(t)$ is $A_k^2 P/4$, where P is the mean power of $m(t)$. We may express the output signal-to-noise ratio for the kth SSB modulated wave as follows:

$$(\text{SNR})_0 = \frac{3A_c^2 k_f^2 (A_k^2 P/4)}{2N_0[k^3 f_0^3 - (k-1)^3 f_0^3]}$$

$$= \frac{3A_c^2 A_k^2 k_f^2 P}{8N_0 f_0^3 (3k^2 - 3k + 1)}$$

where A_c is the carrier amplitude of the FM wave. For equal signal-to-noise ratios, we must therefore choose the A_k so as to satisfy the condition

$$\frac{A_k^2}{3k^2 - 3k + 1} = \text{constant} \quad \text{for} \quad k = 1, 2, \cdots, 12.$$

Problem 6.12

The envelope $r(t)$ and phase $\psi(t)$ of the narrow-band noise $n(t)$ are defined by

$$r(t) = \sqrt{n_I^2(t) + n_Q^2(t)}$$

$$\psi(t) = \tan^{-1}\left(\frac{n_Q(t)}{n_I(t)}\right)$$

For a positive-going click to occur, we therefore require the following:

$n_I(t) \approx -A_c$

$n_Q(t)$ has a small positive value

$$\frac{d}{dt}\tan^{-1}\left(\frac{n_Q(t)}{n_I(t)}\right) > 0$$

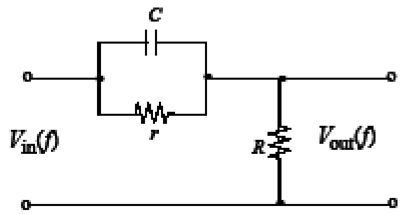
Correspondingly, for a negative-going block to occur, we require

$\boldsymbol{n_I(t) \approx A_c}$

$\boldsymbol{n_Q(t)}$ **has a small negative value**

$$\boldsymbol{\frac{d}{dt}\tan^{-1}\left(\frac{n_Q(t)}{n_I(t)}\right) < 0}$$

Problem 6.13



Let $H(f)$ be $V_{out}(f)/V_{in}(f)$, or the transfer function of the filter. At low frequencies, the capacitor behaves as an open circuit. Then,

$$H(f) \approx \frac{R}{r+R} \approx \frac{R}{r}$$

Thus, the low frequencies of the input are frequency-modulated. At high frequencies, the capacitor behaves as a short circuit in relation to the resistor. Then,

$$H(f) \approx \frac{R}{R + \frac{1}{j2\pi fC}} \approx j2\pi fCR,$$

and

$$v_{out}(t) \approx RC\frac{d}{dt}v_{in}(t)$$

Frequency modulating the derivative of a waveform is equivalent to phase modulating the waveform. Thus, the high frequencies of the input are phase modulated.

Problem 6.24

(a) For the average power of the emphasized signal to be the same as the average power of the original message signal, we must choose the transfer function $H_{pe}(f)$ of the pre-emphasis filter so as to satisfy the relation

$$\int_{-\infty}^{\infty} S_M(f)df = \int_{-\infty}^{\infty} |H_{pe}|^2 S_M(f)df$$

With

$$S_M(f) = \begin{cases} \dfrac{S_0}{1+(f/f_0)^2}, & -W \le f \le W \\ 0, & \text{elsewhere.} \end{cases}$$

$$H_{pe}(f) = k\left(1+\frac{jf}{f_0}\right)$$

we have

$$\int_{-W}^{W} \frac{df}{1+(f/f_0)^2} = k^2\int_{-W}^{W} df$$

Solving for k, we get

$$k = \left[\frac{f_0}{W}\tan^{-1}\left(\frac{W}{f_0}\right)\right]^{1/2} \qquad (1)$$

(b) The improvement in output signal-to-noise ratio obtained by using pre-emphasis in the transmitter and de-emphasis in the receiver is defined by the ratio

$$D = \frac{2W^3}{3\int_{-W}^{W} f^2 |H_{de}(f)|^2 df}$$

$$= \frac{2W^3}{3\int_{-W}^{W} \frac{f^2}{k^2} \frac{df}{1+(f/f_0)^2}}$$

$$= \frac{k^2(W/f_0)^3}{3[(W/f_0) - \tan^{-1}(W/f_0)]} \qquad (2)$$

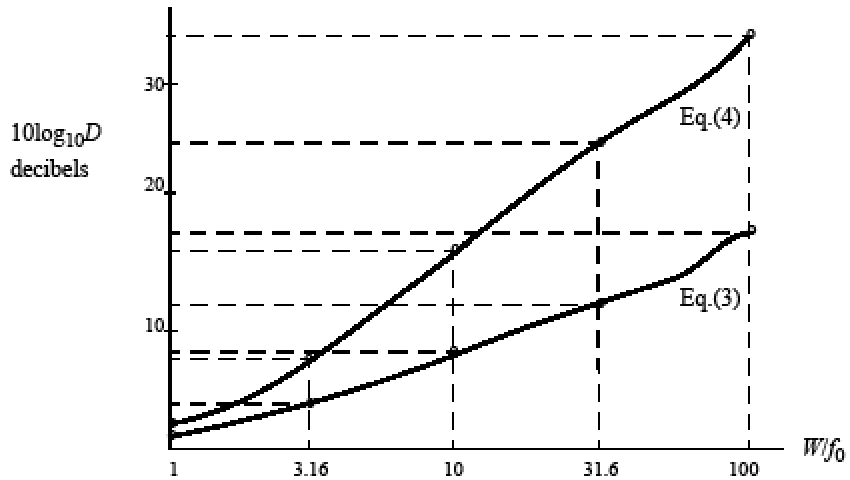
Substituting Eq. (1) in (2), we get

$$D = \frac{(W/f_0)^2 \tan^{-1}(W/f_0)}{3[(W/f_0) - \tan^{-1}(W/f_0)]} \qquad (3)$$

This result applies to the case when the rms bandwidth of the FM system is maintained the same with or without pre-emphasis. When, however, there is no such constraint, we find from Example 4 of Chapter 6 that the corresponding value of D is

$$D = \frac{(W/f_0)^3}{3[(W/f_0) - \tan^{-1}(W/f_0)]} \qquad (4)$$

In the diagram below, we have plotted the improvement D (expressed in decibels) versus the ratio W/f_0 for the two cases; when there is a transmission bandwidth constraint and when there is no such constraint:



Problem 6.15

In a PM system, the power spectral density of the noise at the phase discriminator output (in the absence of pre-emphasis and de-emphasis) is approximately constant. Therefore, the improvement in output signal-to-noise ratio obtained by using pre-emphasis in the transmitter and de-emphasis in the receiver of a PM system is given by

$$D = \frac{\int_0^W df}{\int_0^W |H_{de}(f)|^2 df}$$

With the transfer function $H_{de}(f)$ of the de-emphasis filter defined by

$$H_{de}(f) = \frac{1}{1 + (jf/f_0)},$$

we find that the corresponding value of D is

$$D = \frac{W}{\int_0^W \frac{df}{1 + (f/f_0)^2}}$$

$$= \frac{W/f_0}{\tan^{-1}(W/f_0)}$$

For the case when $W = 15$ kHz, $f_0 = 2.1$ kHz, we find that $D = 5$, or 7 dB. The corresponding value of the improvement ratio D for an FM system is equal to 13 dB Therefore, the improvement obtained by using pre-emphasis and de-emphasis in a PM system is smaller by an amount equal to 6 dB.

Problem 6.16

Problem 6.17

Chapter 7 Problems

Problem 7.1

Let $2W$ denote the bandwidth of a narrowband signal with carrier frequency f_c. The in-phase and quadrature components of this signal are both low-pass signals with a common bandwidth of W. According to the sampling theorem, there is no information loss if the in-phase and quadrature components are sampled at a rate higher than $2W$. For the problem at hand, we have

$f_c \;\; = 100$ kHz
$2W = 10$ kHz

Hence, $W = 5$ kHz, and the minimum rate at which it is permissible to sample the in-phase and quadrature components is 10 kHz.

From the sampling theorem, we also know that a physical waveform can be represented over the interval $-\infty < t < \infty$ by

$$g(t) = \sum_{n=-\infty}^{\infty} a_n \phi_n(t) \tag{1}$$

where $\{\phi_n(t)\}$ is a set of orthogonal functions defined as

$$\phi_n(t) = \frac{\sin\{\pi f_s(t - n/f_s)\}}{\pi f_s(t - n/f_s)}$$

where n is an integer and f_s is the sampling frequency. If $g(t)$ is a low-pass signal band-limited to W Hz, and $f_s \geq 2W$, then the coefficient a_n can be shown to equal $g(n/f_s)$. That is, for $f_s \geq 2W$, the orthogonal coefficients are simply the values of the waveform that are obtained when the waveform is sampled every $1/f_s$ second.

As already mentioned, the narrowband signal is two-dimensional, consisting of in-phase and quadrature components. In light of Eq. (1), we may represent them as follows, respectively:

$$g_I(t) = \sum_{n=-\infty}^{\infty} g_I(n/f_s)\phi_n(t)$$

$$g_Q(t) = \sum_{n=-\infty}^{\infty} g_Q(n/f_s)\phi_n(t)$$

Hence, given the in-phase samples $g_I\left(\frac{n}{f_s}\right)$ and quadrature samples $g_Q\left(\frac{n}{f_s}\right)$, we may reconstruct the narrowband signal $g(t)$ as follows:

$$\begin{aligned} g(t) &= g_I(t)\cos(2\pi f_c t) - g_Q(t)\sin(2\pi f_c t) \\ &= \sum_{n=-\infty}^{\infty}\left[g_I\left(\frac{n}{f_s}\right)\cos(2\pi f_c t) - g_Q\left(\frac{n}{f_s}\right)\sin(2\pi f_c t)\right]\phi_n(t) \end{aligned}$$

where $f_c = 100$ kHz and $f_s \geq 10$ kHz, and where the same set of orthonormal basis functions is used for reconstructing both the in-phase and quadrature components.

Problem 7.2

(a) Consider a periodic train $c(f)$ of rectangular pulses, each of duration T. The Fourier series expansion of $c(t)$ (assuming that a pulse of the train is centered on the origin) is given by

$$c(t) = \sum_{n=-\infty}^{\infty} f_s \operatorname{sinc}(nf_s T)\exp(j2\pi n f_s t)$$

where f_s is the repetition frequency, and the amplitude of a rectangular pulse is assumed to be $1/T$ (i.e., each pulse has unit area). The assumption that $f_s T >> 1$ means that the spectral lines (i.e., harmonics) of the periodic pulse train $c(t)$ are well separated from each other.

Multiplying a message signal $g(t)$ buy $c(t)$ yields

$$s(t) = c(t)g(t)$$

$$= \sum_{n=-\infty} f_s \operatorname{sinc}(nf_s T)g(t)\exp(j2\pi n f_s t) \qquad (1)$$

Taking the Fourier transform of both sides of Eq. (1) and using the frequency-shifting property of the Fourier transform:

$$S(f) = \sum_{n=-\infty}^{\infty} f_s \operatorname{sinc}(nf_s T)g(t)G(f - nf_s) \qquad (2)$$

where $G(f) = F[g(t)]$. Thus, the spectrum $S(f)$ consists of frequency-shifted replicas of the original spectrum $G(f)$, with the nth replica being scaled in amplitude by the factor $f_s\operatorname{sinc}(nf_sT)$.

(b) In accordance with the sampling theorem, let it be assumed that

- The signal $g(t)$ is band-limited with

  $$G(f) = 0 \text{ for } -W < f < W$$

- The sampling frequency f_s is defined by

  $$f_s > 2W$$

Then, the different frequency-shifted replicas of $G(f)$ involved in the construction of $S(f)$ will *not* overlap. Under the conditions described herein, the original spectrum $G(f)$, and therefore the signal $g(t)$, can be recovered exactly (except for a trivial amplitude scaling) by passing $s(t)$ through a low-pass filter of bandwidth W.

Problem 7.3

(a) $g(t) = \text{sinc}(200t)$

This sinc pulse corresponds to a bandwidth $W = 100$ Hz. Hence, the Nyquist rate is 200 Hz, and the Nyquist interval is 1/200 seconds.

(b) $g(t) = \text{sinc}^2(200t)$

This signal may be viewed as the product of the sinc pulse sinc($200t$) with itself. Since multiplication in the time domain corresponds to convolution in the frequency domain, we find that the signal $g(t)$ has a bandwidth equal to twice that of the sinc pulse sin($200t$), that is, 200 Hz. The Nyquist rate of $g(t)$ is therefore 400 Hz, and the Nyquist interval is 1/400 seconds.

(c) $g(t) = \text{sinc}(200t) + \text{sinc}^2(200t)$

The bandwidth of $g(t)$ is determined by the highest frequency component of sinc($200t$) or $\text{sinc}^2(200t)$, whichever one is the largest. With the bandwidth (i.e., highest frequency component of) the sinc pulse sinc($200t$) equal to 100 Hz and that of the squared sinc pulse $\text{sinc}^2(200t)$ equal to 200 Hz, it follows that the bandwidth of $g(t)$ is 200 Hz. Correspondingly, the Nyquist rate of $g(t)$ is 400 Hz, and its Nyquist interval is 1/400 seconds.

Problem 7.4

(a) The PAM wave is

$$s(t) = \sum_{n=-\infty}^{\infty} [1 + \mu m'(nT_s)]g(t - nT_s),$$

where $g(t)$ is the pulse shape, and $m'(t) = m(t)/A_m = \cos(2\pi f_m t)$. The PAM wave is equivalent to the convolution of the instantaneously sampled $[1 + \mu m'(t)]$ and the pulse shape $g(t)$:

$$s(t) = \left\{ \sum_{n-\infty}^{\infty} [1 + \mu m'(nT_s)]\delta(t - nT_s) \right\} H g(t)$$

$$= \left\{ 1 + \mu m'(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \right\} H g(t)$$

The spectrum of the PAM wave is,

$$S(f) = \left\{ [\delta(f) + \mu M'(f)] H \frac{1}{T_s} \sum_{m=-\infty}^{\infty} \delta\left(f - \frac{m}{T_s}\right) \right\} G(f)$$

$$= \frac{1}{T_s} G(f) \sum_{m=-\infty}^{\infty} \left[\delta\left(f - \frac{m}{T_s}\right) + \mu M'\left(f - \frac{m}{T_s}\right) \right]$$

For a rectangular pulse $g(t)$ of duration $T = 0.45s$, and with $AT = 1$, we have:

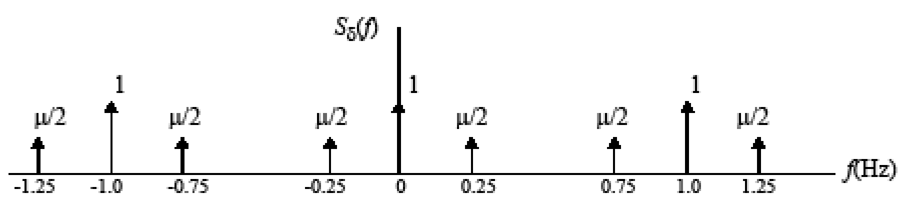
$$G(f) = AT \operatorname{sinc}(fT)$$

$$= \operatorname{sinc}(0.45f)$$

For $m'(t) = \cos(2\pi fmt)$, and with $f_m = 0.25$ Hz, we have:

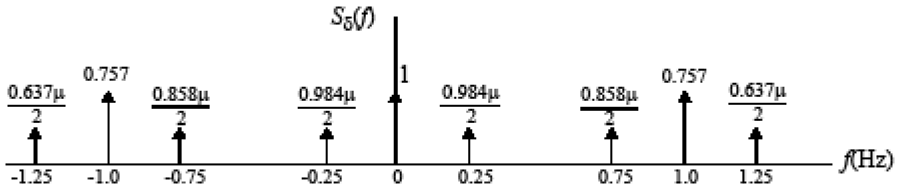
$$M'(f) = \frac{1}{2}[\delta(f - 0.25) + \delta(f + 0.25)]$$

For $T_s = 1s$, the ideally sampled spectrum is $S_\delta(f) = \sum_{m=-\infty}^{\infty} [\delta(f - m) + \mu M'(f - m)]$.

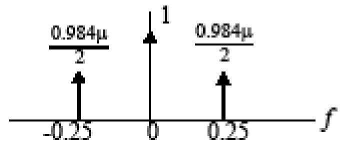


The actual sampled spectrum is:

$$S(f) = \sum_{m=-\infty}^{\infty} \operatorname{sinc}(0.45f)[\delta(f - m) + \mu M'(f - m)]$$



(b) The ideal reconstruction filter would retain the centre 3 delta functions of $S(f)$ or:



With no aperture effect, the two outer delta functions would have amplitude $\mu/2$. Aperture effect distorts the reconstructed signal by attenuating the high frequency portion of the message signal.

Problem 7.5

At $f = 1/2T_s$, which corresponds to the highest frequency component of the message signal for a sampling rate equal to the Nyquist rate, we find from Eq. (6.19) that the amplitude response

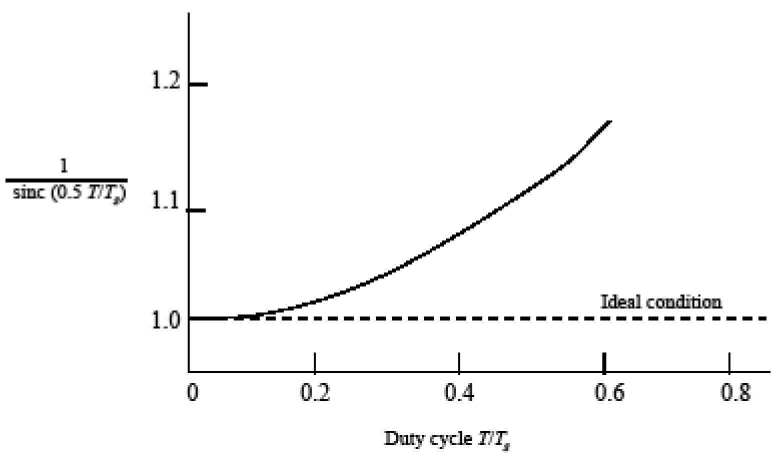


Figure 1

of the equalizer normalized to that at zero frequency is equal to

$$\frac{1}{\mathrm{sinc}(0.5T/T_s)} = \frac{(\pi/2)(T/T_s)}{\sin[(\pi/2)(T/T_s)]}$$

where the ratio T/T_s is equal to the duty cycle of the sampling pulses in Fig. 1, this result is plotted as a function of T/T_s. Ideally, it should be equal to one for all values of T/T_s. For a duty cycle of 10 percent, it is equal to 1.0041. It follows therefore that for duty cycles of less than 10 percent, the aperture effect becomes negligible, and the need for equalization may be omitted altogether.

Problem 7.6

Consider the full-load test tone $A\cos(2\pi f_m t)$. Denoting the kth sample amplitude of this signal by A_k, we find that the transmitted pulse is $A_k g(t)$, where $g(t)$ is defined by the spectrum:

$$G(f) = \begin{cases} \dfrac{1}{2B_T} & |f| < B_T \\ 0 & \text{otherwise} \end{cases}$$

The mean value of the transmitted signal power is

$$\begin{aligned} P &= E\left\{\lim_{L\to\infty} \frac{1}{2LT_s}\int_{-LT_s}^{LT_s}\left[\sum_{k=-L}^{L} A_k g(t)\right]^2 dt\right\} \\ &= E\left[\lim_{L\to\infty} \frac{1}{2LT_s}\int_{-LT_s}^{LT_s}\sum_{k=-L}^{L}\sum_{n=-L}^{L} A_k A_n g^2 dt\right] \\ &= \lim_{L\to\infty} \frac{1}{2LT_s}\sum_{k=-L}^{L}\sum_{n=-L}^{L} E[A_k A_n]\int_{-LT_s}^{LT_s} g^2(t)dt \end{aligned}$$

where T_s is the sampling period. However,

$$E[A_k A_n] = \begin{cases} \dfrac{A^2}{2}, & k = n \\ 0, & \text{otherwise} \end{cases}$$

Therefore,

$$P = \frac{A^2}{2T_s}\int_{-\infty}^{\infty} g^2(t)dt$$

Using Rayleigh's energy theorem, we may write

$$\begin{aligned} \int_{-\infty}^{\infty} g^2(t)dt &= \int_{-\infty}^{\infty} |G(f)|^2 df \\ &= \int_{-B_T}^{B_T}\left(\frac{1}{2B_T}\right)^2 df \\ &= \frac{1}{2B_T} \end{aligned}$$

Therefore

$$P = \frac{A^2}{4T_s B_T}$$

The average signal power at the receiver output is $A^2/2$. Hence, the output signal-to-noise ratio is given by

$$(\mathrm{SNR})_0 = \frac{A^2/2}{B_T N_0}$$

$$= \frac{A^2}{2B_T N_0}$$

$$= \frac{2T_s P}{N_0}$$

By choosing $B_T = 1/2T_s$, we get

$$(\text{SNR})_0 = \frac{P}{B_T N_0}$$

This shows that PAM and baseband signal transmission have the same signal-to-noise ratio for the same average transmitted power, with additive white Gaussian noise, and assuming the use of the minimum transmission bandwidth possible.

Problem 7.7

(a) The sampling interval is $T_s = 125$ μs. There are 24 channels and 1 sync pulse, so the time alloted to each channel is $T_c = T_s/25 = 5$ μs. The pulse duration is 1 μs, so the time between pulses is 4 μs.

(b) If sampled at the Nyquist rate, 6.8 kHz, then $T_s = 147$ μs, $T_c = 6.68$ μs, and the time between pulses is 5.68 μs.

Problem 7.8

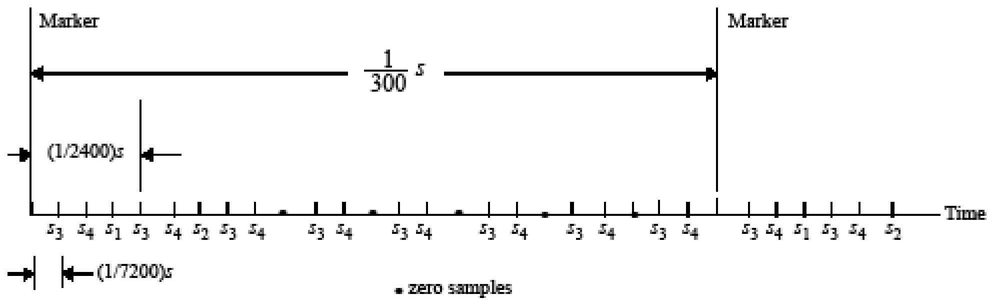
(a) The bandwidth required for each single sideband channel is 10 kHz. The total bandwidth for 12 channels is 120 kHz.

(b) The Nyquist rate for each signal is 20 kHz. For 12 TDM signals, the total data rate is 240 kHz. By using a sinc pulse whose smplitude varies in accordance with the modulation, and with zero crossings at multiples of (1/240) μs, we need a minimum bandwidth of 120 kHz.

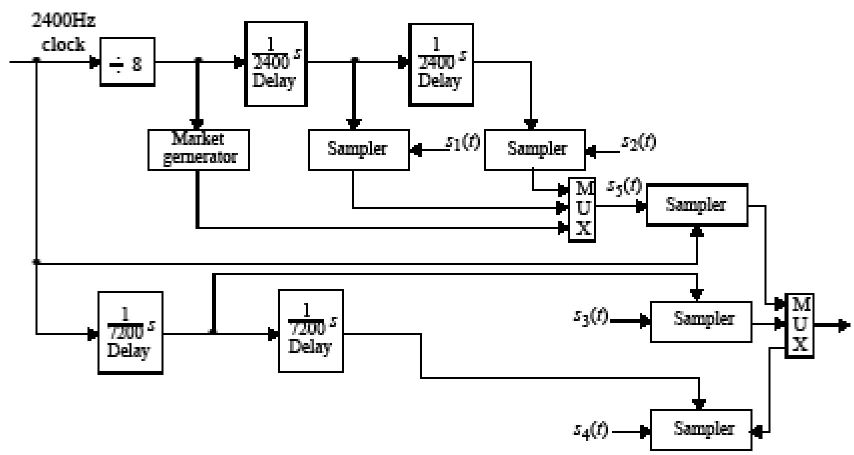
Problem 7.9

(a) The Nyquist rate for $s_1(t)$ and $s_2(t)$ is 160 Hz. Therefore $2400/2^R$ must be greater than 160, and the maximum R is 3.

(b) With $R = 3$, we may use the following signal format to multiplex the signals $s_1(t)$ and $s_2(t)$ into a new signal, and then multiplex $s_3(t)$ and $s_4(t)$ and $s_5(t)$ including markers for synchronization:



Based on this signal format, we may develop the following multiplexing system:

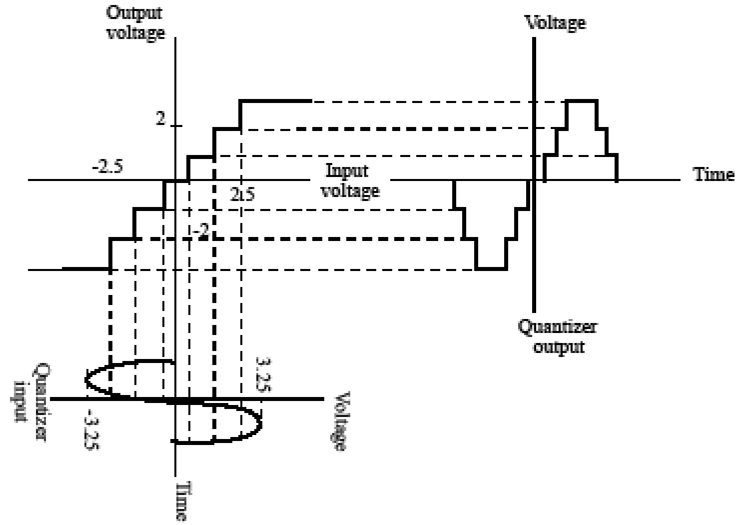


Problem 7.10

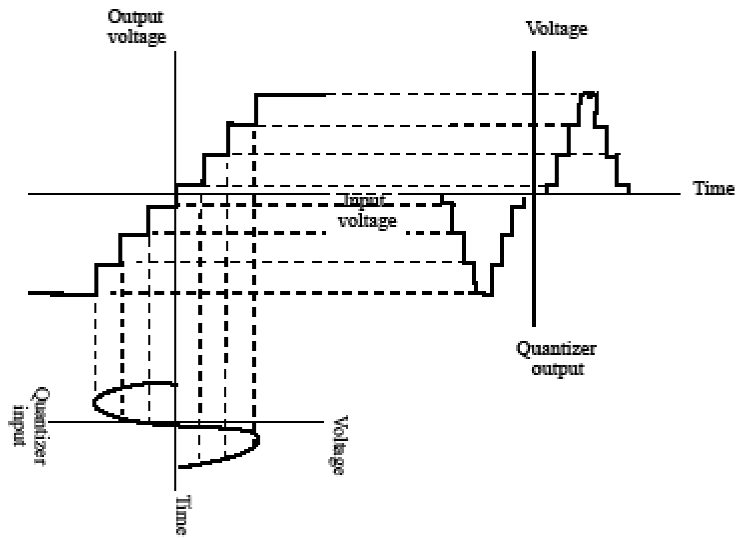
Problem 7.11

Problem 7.12

(a)

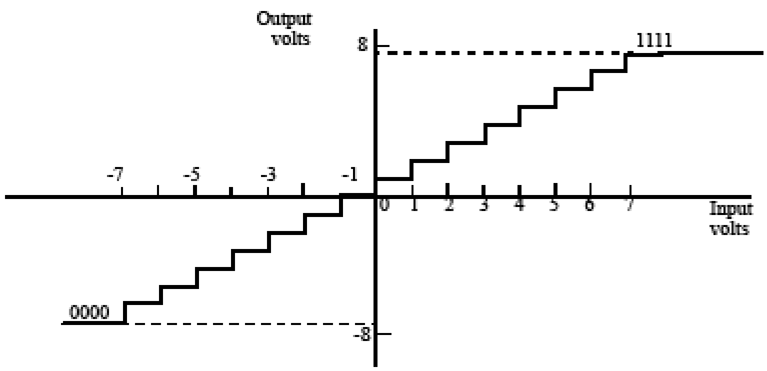


(b)



Problem 7.13

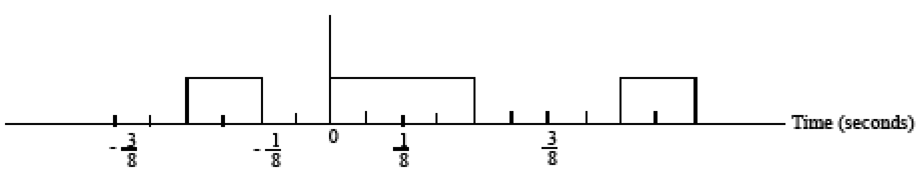
The quantizer has the following input-output curve:



At the sampling instants we have:

t	$m(t)$	code
-3/8	$-3\sqrt{2}$	0011
-1/8	$-3\sqrt{2}$	0011
+1/8	$3\sqrt{2}$	1100
+3/8	$3\sqrt{2}$	1100

And the coded waveform is (assuming on-off signaling):

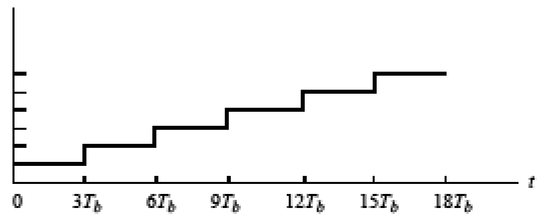


Problem 7.14

The transmitted code words are:

t/T_b	Code
1	001
2	010
3	011
4	100
5	101
6	110

The sampled analog signal is



Problem 7.15

Suppose that baseband signal $m(t)$ is modeled as the sample function of a Gaussian random process of zero mean, and that the amplitude range of $m(t)$ at the quantizer input extends from $-4A_{rms}$ to $4A_{rms}$. We find that samples of the signal $m(t)$ will fall outside the amplitudfe range $8A_{rms}$ with a probability of overload that is less than 1 in 10^4. If we further assume the use of a binary code with each code word having a length n, so that the number of quantizing levels is 2", we find that the resulting quantizer step size is

$$\delta = \frac{8A_{\text{rms}}}{2^R} \tag{1}$$

Substituting Eq. (1) to the formula for the output signal-to-quantization noise ration, we get

$$(\text{SNR})_o = \frac{3}{16}(2^{2R}) \tag{2}$$

Expressing the signal-to-noise ratio in decibels:

$$10\log_0(\text{SNR})_o = 6R - 7.2 \tag{3}$$

This formula states that each bit in the code word of a PCM system contributes 6 dB to the signal-to-noise ratio. It gives a good description of the noise performance of a PCM system, provided that the following conditions are satisfied:

1. The system operates with an average signal power above the error threshold, so that the effect of transmission noise is made negligible, and performance is thereby limited essentially by quantizing noise alone.
2. The quantizing error is uniformly distributed.
3. The quantization is fine enough (say $R > 6$) to prevent signal-correlated patterns in the quantizing error waveform.
4. The quantizer is aligned with the amplitude range from $-4A_{rms}$ to $4A_{rms}$.

In general, conditions (1) through (3) are true of toll quality voice signals. However, when demands on voice quality are not severe, we may use a coarse quantizer corresponding to $R \leq 6$. In such a case, degradation in system performance is reflected not only by a lower signal-to-noise ratio, but also by an undesirable presence of signal-dependent patterns in the waveform of quantizing error.

Problem 7.16

(a) Let the message bandwidth be W. Then, sampling the message signal at its Nyquist rate, and using an R-bit code to represent each sample of the message signal, we find that the bit duration is

$$T_b = \frac{T_s}{R} = \frac{1}{2WR}$$

The bit rate is

$$\frac{1}{T_b} = 2WR$$

The maximum value of message bandwidth is therefore

$$\begin{aligned} W_{\max} &= \frac{50 \times 10^6}{2 \times 7} \\ &= 3.57 \times 10^6 \text{ Hz} \end{aligned}$$

(b) The output signal-to-quantizing noise ratio is given by (see Example 2):

$$\begin{aligned} 10\log_{10}(\mathrm{SNR})_0 &= 1.8 + 6R \\ &= 1.8 + 6 \times 7 \\ &= 43.8 \text{ dB} \end{aligned}$$

Problem 7.17

Let a signal amplitude lying in the range

$$x_i - \frac{1}{2}\delta_i \le x \le x_i + \frac{1}{2}\delta_i$$

be represented by the quantizer amplitude x_i. The instantaneous square value of the error is $(x - x_i)^2$. Let the probability density function of the input signal be $f_X(x)$. If the step size δ_i is small in relation to the input signal excursion, then $f_X(x)$ varies little within the quantum step and may be approximated by $f_X(x_i)$. Then, the mean-square value of the error due to signals falling within this quantum is

$$\begin{aligned}
E[Q_i^2] &= \int_{x_i-\frac{1}{2}\delta_i}^{x_i+\frac{1}{2}\delta_i} (x-x_i)^2 f_X(x)\,dx \\
&= \int_{x_i-\frac{1}{2}\delta_i}^{x_i+\frac{1}{2}\delta_i} (x-x_i)^2 f_X(x_i)\,dx \\
&= f_X(x_i)\int_{x_i-\frac{1}{2}\delta_i}^{x_i+\frac{1}{2}\delta_i} (x-x_i)^2\,dx \\
&= f_X(x_i)\int_{-\frac{1}{2}\delta_i}^{\frac{1}{2}\delta_i} x^2\,dx \\
&= \frac{1}{12}\delta_i^3 f_X(x_i) \qquad (1)
\end{aligned}$$

The probability that the input signal amplitude lies within the ith interval is

$$p_i = \int_{x_i-\frac{1}{2}\delta_i}^{x_i+\frac{1}{2}\delta_i} f_X(x)\,dx \approx f_X(x_i)\int_{x_i-\frac{1}{2}\delta_i}^{x_i+\frac{1}{2}\delta_i} dx = f_X(x_i)\delta_i \qquad (2)$$

Therefore, eliminating $f_X(x_i)$ between Eqs. (1) and (2), we get

$$E[Q_i^2]\frac{1}{12}p_i\delta_i^2$$

The total mean-square value of the quantizing error is the sum of that contributed by each of the several quanta. hence,

$$\sum_i E[Q_i^2] = \frac{1}{12}\sum_i p_i\delta_i^2$$

Problem 7.18

(a) The probability p_1 of any binary symbol being inverted by transmission through the system is usually quite small, so that the probability of error after n regenerations in the system is very nearly equal to n p_1. For very large n, the probability of more than one inversion must be taken into account. Let p_n denote the probability that a binary symbol is in error after transmission through the complete system. Then, p_n is also the probability of an odd number of errors, since an even number of errors restores the original value. Counting zero as an even number, the probability of an even number of errors is 1-p_n. Hence.

$$\begin{aligned} p_{n+1} &= p_n(1-p_1)+(1-p_n)p_1 \\ &= (1-2p_1)p_n+p_1 \end{aligned}$$

This is a linear difference equation of the first order. Its solution is

$$p_n = \frac{1}{2}[1-(1-2p_1)^n]$$

(b) If p_1 is very small and n is not too large, then

$$(1-2p_1)^n \approx 1-2p_1 n$$

and

$$\begin{aligned} p_n &\approx \frac{1}{2}[1-(1-2p_1 n)] \\ &= p_1 n \end{aligned}$$

Problem 7.19

$$m(t) = A\tanh(\beta t)$$

To avoid slope overload, we require

$$\frac{\Delta}{T_s} \ge \max\left|\frac{dm(t)}{dt}\right| \qquad (1)$$

$$\frac{dm(t)}{dt} = A\beta\,\mathrm{sech}^2(\beta t) \qquad (2)$$

Hence, using Eq. (2) in (1):

$$\Delta \ge \max(A\beta\,\mathrm{sech}^2(\beta t)) \times T_s \qquad (3)$$

Since $\mathrm{sech}(\beta t) = \dfrac{1}{\cosh(\beta t)}$

$$= \frac{2}{e^{+\beta t}+e^{-\beta t}}$$

it follows that the maximum value of sech(βt) is 1, which occurs at time $t = 0$. Hence, from Eq. (3) we find that $\Delta \ge A\beta T_s$.

Problem 7.20

The modulating wave is

$$m(t) = A_m \cos(2\pi f_m t)$$

The slope of $m(t)$ is

$$\frac{dm(t)}{dt} = -2\pi f_m A_m \sin(2\pi f_m t)$$

The maximum slope of $m(t)$ is equal to $2\pi f_m A_m$.

The maximum average slope of the approximating signal $m_q(t)$ produced by the delta modulator is δ/T_s, where δ is the step size and T_s is the sampling period. The limiting value of A_m is therefore given by

$$2\pi f_m A_m > \frac{\delta}{T_s}$$

or

$$A_m > \frac{\delta}{2\pi f_m T_s}$$

Assuming a load of 1 ohm, the transmitted power is $A_m^2/2$. Therefore, the maximum power that may be transmitted without slope-overload distortion is equal to $\delta^2/8\pi^2 f_m^2 T_s^2$.

Problem 7.21

$f_s = 10 f_{\text{Nyquist}}$

$f_{\text{Nyquist}} = 6.8$ kHz

$f_s = 10 \times 6.8 \times 10^3 = 6.8 \times 10^4$ Hz

$$\frac{\Delta}{T_s} \geq \max\left|\frac{dm(t)}{dt}\right|$$

For the sinusoidal signal $m(t) = A_m \sin(2\pi f_m t)$, we have

$$\frac{dm(t)}{dt} = 2\pi f_m A_m \cos(2\pi f_m t)$$

Hence,

$$\left|\frac{dm(t)}{dt}\right|_{\max} = \left|2\pi f_m A_m\right|_{\max}$$

or, equivalently,

$$\frac{\Delta}{T_s} \geq \left|2\pi f_m A_m\right|_{\max}$$

Therefore,

$$\left|A_m\right|_{\max} = \frac{\Delta}{T_s \times 2\pi \times f_m}$$

$$= \frac{\Delta f_s}{2\pi f_m}$$

$$= \frac{0.1 \times 6.8 \times 10^4}{2\pi \times 10^3}$$

$$= 1.08\text{V}$$

Problem 7.22

The maximum slope of the signal $s(t) = A\sin(2\pi ft)$ is $2\pi fA$. Consequently, the maximum change during a sample period is approximately $2\pi AfTs$. To prevent slope overload, we require

$$\begin{aligned} 100mV &> 2\pi AfT_s \\ &= 2\pi A(1kHz)/(68kHz) \\ &= 0.092A \end{aligned}$$

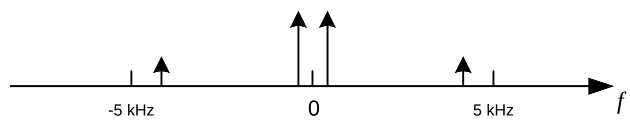
or A < 1.08 V.

Problem 7.23

(a) Theoretically, the sampled spectrum is given by

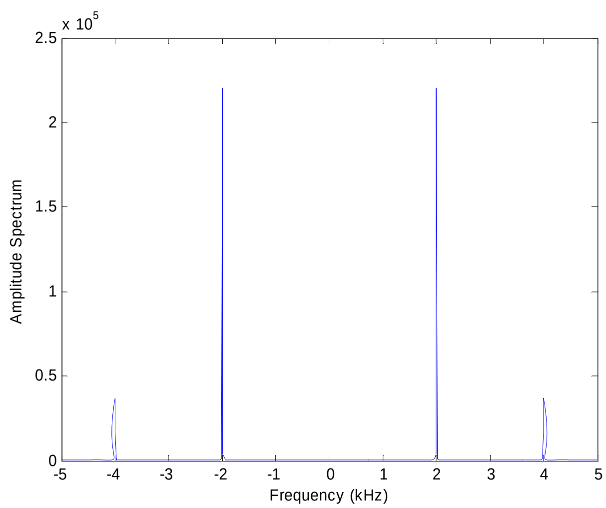
$$S_s(f) = \sum_{n=-\infty}^{\infty} H_s(f - nf_s)$$

where $H_s(f)$ is the spectrum of the signal $H(f)$ limited to $|f| \le f_s/2$. For this example, the sample spectrum should look as below.



(b)

The sampled spectrum is given by

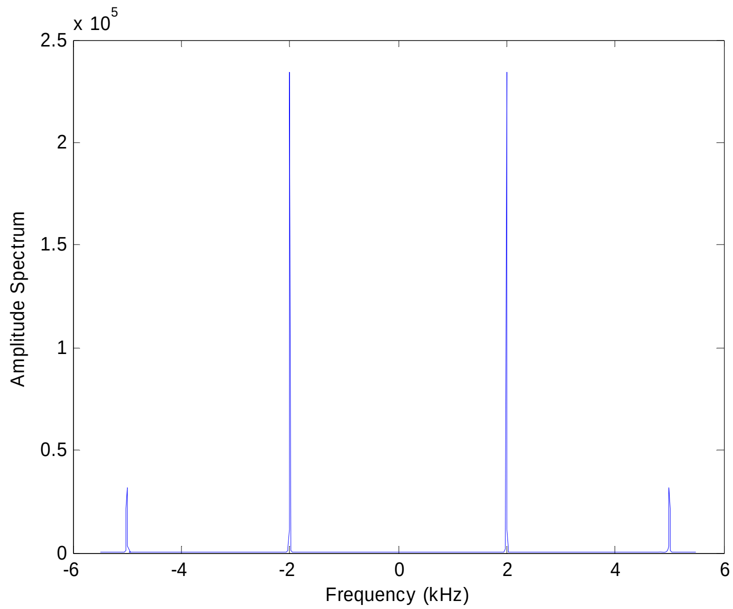


There are several features to comment on:

(i) The component at +4 kHz is due to aliasing of the -6 kHz sinusoid; and the component at -4kHz is due to aliasing of the +6 kHz sinusoid.

(ii) The lower frequency is at 2 kHz is six times larger than the one at 4 kHz. One would expect the power ratio to be 4:1, not 6:1. The difference is due to relationship between the FFTsize (period) and the sampling rate. (Try a sampling rate of 10.24 kHz and compare.)

(b) The spectrum with a 11 kHz sampling rate is shown below.



As expected the 2kHz component is unchanged in frequency, while the aliased component is shifted to reflect the new sampling rate.

Problem 7.23

(a) The expanding portion of the μ-law compander is given by

$$|m| = \frac{\exp\left[\log(1+\mu)|\upsilon|\right]-1}{\mu}$$
$$= \frac{(1+\mu)\exp\left[|\upsilon|\right]-1}{\mu}$$

(b)

(i) For the non-companded case, the rms quantization error is determined by step size. The step size is given by the maximum range over the number of quantization steps

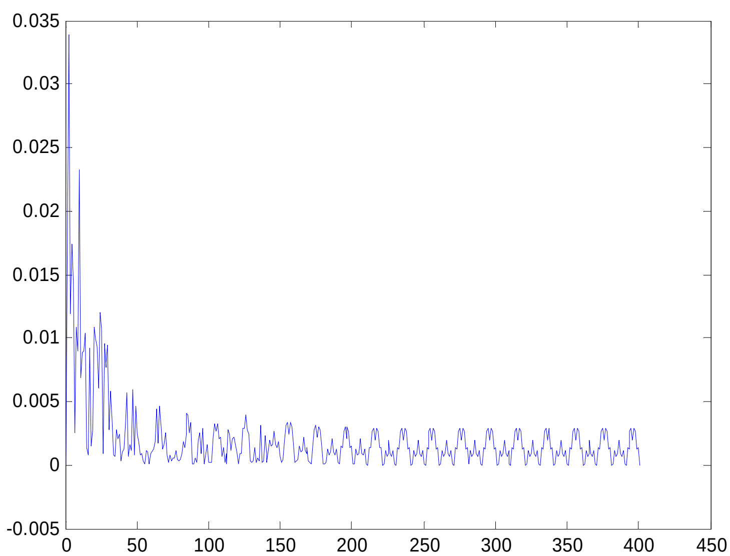
$$\Delta = \frac{2A}{2^Q}$$

For this signal the range is from +10 to -1, so A = 10 and with Q = 8, we have Δ = 0.078. From Eq. ( ) , the rms quantization error is then given by

$$\sigma_Q^2 = \frac{1}{3}m_{\max}^2 2^{-2R}$$
$$= \frac{1}{3}(10)^2 2^{-16}$$
$$= 0.0005086$$

and the rms error is σ_Q – 0.02255.

(ii) For a fair comparison, the signal must have similar amplitudes.
The rms error with companding is 0.0037 which is significantly less. The plot is shown below. Note that the error is always positive.



Rest TBD.

Problem 7.24

Problem 7.25

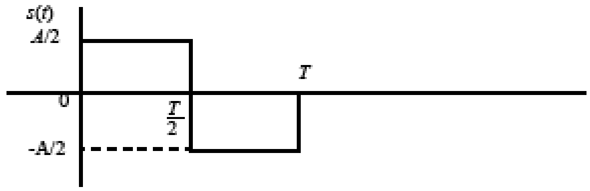
Chapter 8

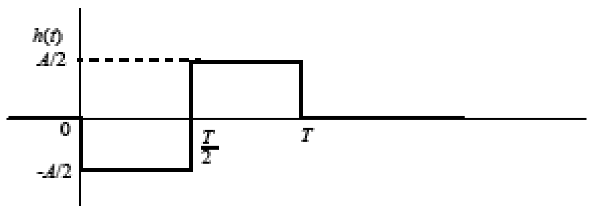
Problem 8.1

(a) The impulse response of the matched filter is

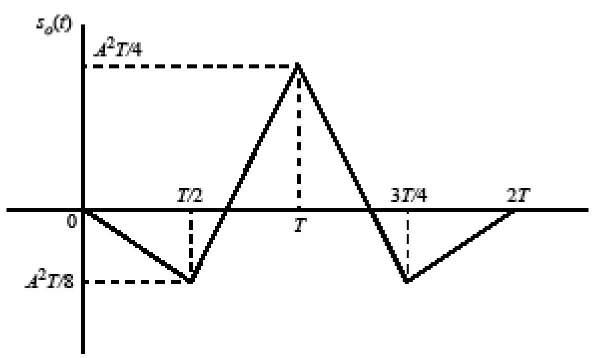
$$h(t) = s(T-t)$$

The $s(t)$ and $h(t)$ are shown below:





(b) The corresponding output of the matched filter is obtained by convolving $h(t)$ with $s(t)$. The result is shown below:



(c) The peak value of the filter output is equal to $A^2T/4$, occuring at $t = T$.

Problem 8.2

Problem 8.3

Problem 8.4

Ideal low-pass filter with variable bandwidth. The transfer function of the matched filter for a rectangular pulse of duration τ and amplitude A is given by

$$H_{opt}(f) = \text{sinc}(fT)\exp(-j\pi fT) \qquad (1)$$

The amplitude response $|H_{opt}(f)|$ of the matched filter is plotted in Fig. 1(a). We wish to approximate this amplitude response with an ideal low-pass filter of bandwidth B. The amplitude response of this approximating filter is shown in Fig. 1(b). The requirement is to determine the particular value of bandwidth B that will provide the best approximation to the matched filter.

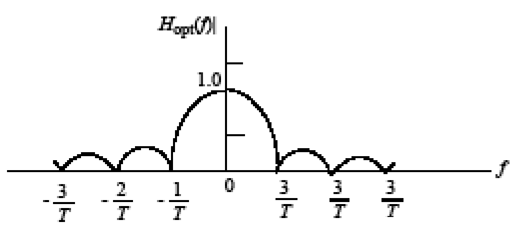
We recall that the maximum value of the output signal, produced by an ideal low-pass filter in response to the rectangular pulse occurs at $t = T/2$ for $BT \leq 1$. This maximum value, expressed in terms of the sine integral, is equal to $(2A/\pi)Si(\pi BT)$. The average noise power at the output of the ideal low-pass filter is equal to BN_0. The maximum output signal-to-noise ratio of the ideal low-pass filter is therefore

$$(\text{SNR})'_0 = \frac{(2A/\pi)^2 Si^2(\pi BT)}{BN_0} \qquad (2)$$

Thus, using Eqs. (1) and (2), and assuming that $AT = 1$, we get

$$\frac{(\text{SNR})'_0}{(\text{SNR})_0} = \frac{2}{\pi^2 BT} Si^2(\pi BT)$$

This ratio is plotted in Fig. 2 as a function of the time-bandwidth product BT. The peak value on this curve occurs for $BT = 0.685$, for which we find that the maximum signal-to-noise ratio of the ideal low-pass filter is 0.84 dB below that of the true matched filter. Therefore, the "best" value for the bandwidth of the ideal low-pass filter characteristic of Fig. 1(b) is $B = 0.685/T$.



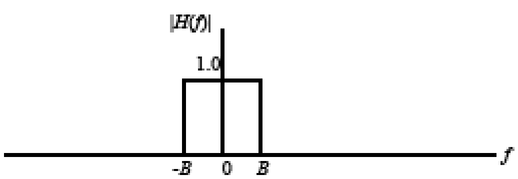


Figure 1

(b)

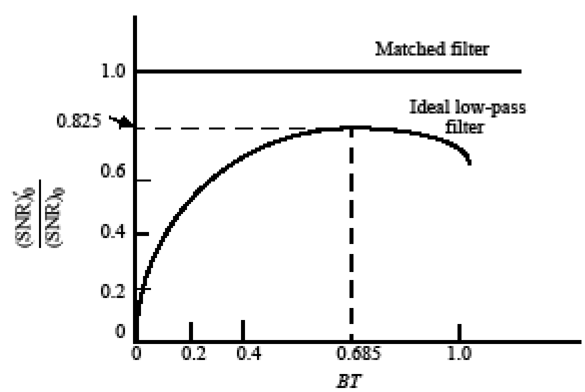


Figure 2

Problem 8.5

Problem 4.9

Consider the performance of a binary PCM system in the presence of channel noise; the receiver is depicted in Fig. 1. We do so by evaluating the average probability of error for such a system under the following assumptions:

1. The PCM system uses an on-off format, in which symbol 1 is represented by A volts and symbol 0 by zero volt.
2. The symbols 1 and 0 occur with equal probability.
3. The channel noise $w(t)$ is white and Gaussian with zero mean and power spectral density N0/2.

To determine the average probability of error, we consider the two possible kinds of error separately. We begin by considering the first kind of error that occurs when symbol 0 is sent and the receiver chooses symbol 1. In this case, the probability of error is just the probability that the correlator output in Fig. 1 will exceed the threshold λ owing to the presence of noise, so the transmitted symbol 0 is mistaken for symbol 1. Since the a priori probabilities of symbols 1 and 0 are equal, to have $p_0 = p_1$. Correspondingly, the expression for the threshold λ simplifies as follows:

$$\lambda = \frac{A^2 T_b}{2} \tag{1}$$

where T_b is the bit duration, and $A^2 T_b$ is the signal energy consumed in the transmission of symbol 1. Let y denote the correlator output:

$$y = \int_0^{T_b} s(t)x(t)dt \tag{2}$$

Under hypothesis H_0, corresponding to the transmission of symbol 0, the received signal $x(t)$ equals the channel noise $w(t)$. Under this hypothesis we may therefore describe the correlator output as

$$H_0 : y = A\int_0^{T_b} w(t)dt \tag{3}$$

Since the white noise $w(t)$ has zero mean, the correlator output under hypothesis H_0 also has zero mean. In such a situation, we speak of a *conditional mean*, which (for the situation at hand) we describe by writing

$$\mu_0 = E[Y|H_0] = E\left[\int_0^{T_b} W(t)dt\right] = 0 \tag{4}$$

where the random variable Y represents the correlatoroutput with y as its sample value and $W(t)$ is a white-noise process with $w(t)$ as its sample function. The subscript 0 in the conditional mean μ_0 refers to the condition that hypothesis H_0 is true. Correspondingly, let σ_0^2 denote the *conditional variance* of the correlator output, given that hypothesis H_0 is true. We may therefore write

$$\sigma_0^2 = E[Y^2|H_0]$$
$$= E\left[\int_0^{T_b}\int_0^{T_b} W(t_1)W(t_2)(dt_1)dt_2\right] \quad (5)$$

The double integration in Eq. (5) accounts for the squaring of the correlator output. Interchanging the order of integration and expectation in Eq. (5), we may write

$$\sigma_0^2 = \int_0^{T_b}\int_0^{T_b} E[W(t_1)W(t_2)]dt_1 dt_2$$
$$= \int_0^{T_b}\int_0^{T_b} R_w(t_1 - t_2)dt_1 dt_2 \quad (6)$$

The parameter ($R_w(t_1 - t_2)$ is the ***ensemble-averaged autocorrelation function*** of the white-noise process $W(t)$. From random process theory, it is recognized that the autocorrelation function and power spectral density of a random process form a Fourier transform pair. Since the white-noise process $W(t)$ is assumed to have a constant power spectral density of $N_0/2$, it follows that the autocorrelation function of such a process consists of a delta function weighted by $N_0/2$. Specifically, we may write

$$R_w(t_1 - t_2) = \frac{N_0}{2}\delta(\tau - t_1 + t_2) \quad (7)$$

Substituting Eq. (7) in (6), and using the property that the total area under the Dirac delta function $\delta(\tau - t_1 + t_2)$ is unity, we get

$$\sigma_0^2 = \frac{N_0 T_b A^2}{2} \quad (8)$$

The statistical characterization of the correlator output is computed by noting that it is Gaussian distributed, since the white noise at the correlator input is itself Gaussian (by assumption). In summary, we may state that under hypothesis H_0 the correlator output is a Gaussian random variable with zero mean and variance $N_0 T_b A^2/2$, as shown by

$$f_0(y) = \frac{1}{\sqrt{\pi N_0 T_b} A}\exp\left(-\frac{y^2}{N_0 T_b A^2}\right) \quad (9)$$

where the subscript in $f_0(y)$ signifies the condition that symbol 0 was sent.

Figure 2(a) shows the bell-shaped curve for the probability density function of the correlator output, given that symbol 0 was transmitted. The probability of the receiver deciding in favor of symbol 1 is given by the area shown shaded in Fig. 2(a). The part of the y-axis covered by this area corresponds to the condition that the correlator output y is in excess of the threshold l defined by Eq. (1). Let P_{e0} denote the *conditional probability of error, given that symbol 0 was sent*. Hence, we may write

$$p_{10} = \int_{\lambda}^{\infty} f_0(y)dy$$

$$= \frac{1}{\sqrt{\pi N_0 T_b} a}\int_{A^2 T_b/2}^{\infty} \exp\left(-\frac{y^2}{N_0 T_b A^2}\right) dy \tag{10}$$

Define

$$z = \frac{y}{\sqrt{N_0 T_b} A} \tag{11}$$

We may then rewrite Eq. (10) in terms of the new variable z as

$$p_{10} = \frac{1}{\sqrt{\pi}}\int_{\sqrt{A^2 T_b/2N_0}}^{\infty} \exp(-z^2)dz \tag{12}$$

which may be reformulated in terms of *complementary error function*

$$\mathrm{erfc}(u) = \frac{2}{\sqrt{\pi}}\int_{u}^{\infty} \exp(-z^2)dz \tag{13}$$

Accordingly, we may redefine the conditional probability of error P_{e0} as

$$p_{10} = \frac{1}{2}\mathrm{erfc}\left(\sqrt{\frac{A^2 T_b}{4N_0}}\right) \tag{14}$$

Consider next the second kind of error that occurs when symbol 1 is sent and the receiver chooses symbol 0. Under this condition, corresponding to hypothesis H_1, the correlator input consists of a rectangular pulse of amplitude A and duration T_b plus the channel noise $w(t)$. We may thus apply Eq. (2) to write

$$H_1 : y = A\int_0^{T_b} [A + w(t)]dt \tag{15}$$

The fixed quantity A in the integrand of Eq. (15) serves to shift the correlator output from a mean value of zero volt under hypothesis H_0 to a mean value of A^2T_b under hypothesis H_1. However, the conditional variance of the correlator output under hyupothesis H_1 has the same value as that under hypothesis H_0. Moreover, the correlator output is Gaussian distributed as before. In summary, the correlator output under hypothesis H_1 is a Gaussian random variable with mean A^2T_b and variance $N_0T_b^2/2$, as depicted in Fig. 2(b), which corresponds to those values of the correlator output less than the threshold λ set at $A^2T_b/2$. From the symmetric nature of the Gaussian density function, it is clear that

$$p_{01} - p_{10} \tag{16}$$

Note that this statement is only true when the a priori probabilities of binary symbols 0 and 1 are equal; this assumption was made in calculating the threshold λ.

To determine the average probability of error of the PCM receiver, we note that the two possible kinds of error just considered are mutually exclusive events. Thus, with the a priori probability of transmitting a 0 equal to p_0, and the a priori probability of transmitting a 1 equal to p_1, we find that the *average probability of error*, P_e, is given by

$$P_e = p_0 p_{10} + p_1 p_{01} \tag{17}$$

Since $p_{01} = p_{10}$ and $p_0 + p_1 = 1$, Eq. (17) simplifies as

$$P_e = p_{10} = p_{01}$$

or

$$P_e = \frac{1}{2}\operatorname{erfc}\left(\frac{1}{2}\sqrt{\frac{A^2 T_b}{N_0}}\right) \tag{18}$$

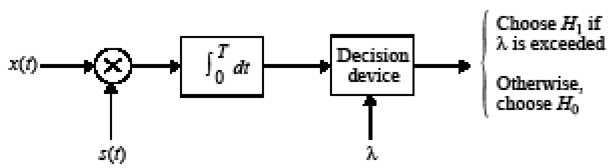


Figure 1

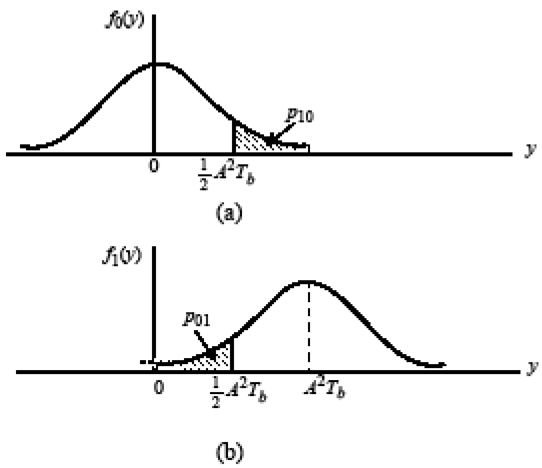


Figure 2

Problem 8.6

In a binary PCM system, with NRZ signaling, the average probability of error is

$$P_e = \frac{1}{2}\operatorname{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right)$$

The signal energy per bit is

$$E_b = A^2 T_b$$

where A is the pulse amplitude and T_b is the bit (pulse) duration. If the signaling rate is doubled, the bit duration T_b is reduced by half. Correspondingly, E_b is reduced by half.

Let $u = \sqrt{E_b/N_0}$. We may then set

$$P_e = 10^{-6} = \frac{1}{2}\operatorname{erfc}(u)$$

Solving for u, we get

$$u = 3.3$$

When the signaling rate is doubled, the new value of P_e is

$$\begin{aligned} P'_e &= \frac{1}{2}\operatorname{erfc}\left(\frac{u}{\sqrt{2}}\right) \\ &= \frac{1}{2}\operatorname{erfc}(2.33) \\ &= 10^{-3} \end{aligned}$$

Problem 8.7

(a) The average probability of error is

$$P_e = \frac{1}{2}\text{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right)$$

where $E_b = A^2T_b$. We may rewrite this formula as

$$P_e = \frac{1}{2}\text{erfc}\left(\sqrt{\frac{A}{\sigma}}\right) \tag{1}$$

where A is the pulse amplitude at $\sigma = \sqrt{N_0T_b}$. We may view σ^2 as playing the role of noise variance at the decision device input. Let

$$u = \sqrt{\frac{E_b}{N_0}} = \frac{A}{\sigma}$$

We are given that

$$\sigma^2 = 10^{-2}\ \text{volts}^2, \qquad \sigma = 0.1\ \text{volt}$$

$$P_e = 10^{-8}$$

Since P_e is quite small, we may approximate it as follows:

$$\text{erfc}(u) \approx \frac{\exp(-u^2)}{\sqrt{\pi}u}$$

We may thus rewrite Eq. 1 as (with $P_e = 10^{-8}$)

$$\frac{\exp(-u^2)}{2}\sqrt{\pi}u = 10^{-8}$$

Solving this equation for u, we get

$u = 3.97$

The corresponding value of the pulse amplitude is

$A = \sigma u = 0.1 \text{ x } 3/97$

$= 0.397$ volts

(b) Let σ_i^2 denote the combined variance due to noise and interference; that is

$$\sigma_T^2 = \sigma^2 + \sigma_i^2$$

where σ^2 is due to noise and σ_i^2 is due to the interference. The new value of the average probability of error is 10^{-6}. Hence

$$10^{-6} = \frac{1}{2}\text{erfc}\left(\frac{A}{\sigma_T}\right)$$

$$= \frac{1}{2}\text{erfc}(u_T) \tag{2}$$

where

$$u_T = \frac{A}{\sigma_T}$$

Equation (2) may be approximated as (with $P_e = 10^{-6}$)

$$\frac{\exp(-u_T^2)}{2\sqrt{\pi}u_T} \approx 10^{-6}$$

Solving for u_T, we get

$$u_T = 3.37$$

The corresponding value of σ_T^2 is

$$\sigma_T^2 = \left(\frac{A}{u_T}\right)^2 = \left(\frac{0.397}{3.37}\right)^2 = 0.0138 \text{ volts}^2$$

The variance of the interference is therefore

$$\begin{aligned}\sigma_i^2 &= \sigma_T^2 - \sigma^2 \\ &= 0.0138 - 0.01 \\ &= 0.0038 \text{ volts}^2\end{aligned}$$

Problem 8.8

(i) $p_0 > p_1$

The transmitted symbol is more likely to be 0. Hence, the average probability of symbol error is smaller when a 0 is transmitted than when a 1 is transmitted. In such a situation, the threshold λ in Figs. 4.5(a) and (b) in the textbook is moved to the right.

(ii) $p_1 > p_0$

The transmitted symbol is more likely to be 1. Hence, the average probability of symbol error is smaller when a 1 is transmitted than when a 0 is transmitted. In this second situation, the threshold λ in Figs. 4.5(a) and (b) in the textbook is moved to the left.

Problem 8.9

Problem 8.10

Since $P(f)$ is an even real function, its inverse Fourier transform equals

$$p(t) = 2\int_0^{\infty} P(f)\cos(2\pi ft)df \tag{1}$$

The $P(f)$ is itself defined by Eq. (7.60) which is reproduced here in the form

$$P(f) = \begin{cases} \dfrac{1}{2W} & 0 < |f| f_1 \\ \dfrac{1}{4W}\left\{1 + \cos\left[\dfrac{\pi(|f| - f_1)}{2W - 2f_1}\right]\right\}, & f_1 < f < 2W - f_1 \\ 0 & |f| > 2W - f_1 \end{cases} \tag{2}$$

Hence, using Eq. (2) in (1):

$$p(t) = \frac{1}{W}\int_0^{f_1}\cos(2\pi ft)df + \frac{1}{2B}\int_{f_1}^{2W\text{-}f_1}\left[1 + \cos\left(\frac{\pi(f-f_1)}{2W\alpha}\right)\right]\cos(2\pi ft)df$$

$$= \left[\frac{\sin(2\pi ft)}{2\pi Wt}\right] + \left[\frac{\sin(2\pi ft)}{4\pi Wt}\right]_{f_1}^{2W-f_1}$$

$$+ \frac{1}{4}W\left[\frac{\sin\left(2\pi ft + \dfrac{\pi(f-f_1)}{2W\alpha}\right)}{2\pi t + \pi/2W\alpha}\right]_{f_1}^{2W-f_1} + \frac{1}{4W}\left[\frac{\sin\left(2\pi ft - \dfrac{\pi(f-f_1)}{2W\alpha}\right)}{2\pi t + \pi/2W\alpha}\right]_{f_1}^{2W-f_1}$$

$$= \frac{\sin(2\pi f_1 t)}{4\pi Wt} + \frac{\sin[2\pi t(2W-f_1)]}{4\pi Wt}$$

$$-\frac{1}{4W}\ \frac{\sin(2\pi f_1 t) + \sin[2\pi t(2W-f_1)]}{2\pi t - \pi/2W\alpha} + \frac{\sin(2\pi f_1 t) + \sin[2\pi t(2W-f_1 t)]}{2\pi t - \pi/2W\alpha}$$

$$= \frac{1}{W}[\sin(2\pi f_1 t) + \sin[2\pi t(2W-f_1)]]\left[\frac{1}{4Wt} - \frac{\pi t}{2(\pi t)^2 - (\pi/2W\alpha)^2}\right]$$

$$= \frac{1}{W}[\sin(2\pi Wt)\cos(2\pi\alpha W)]\left[\frac{-\pi/(2W\alpha)^2}{4\pi t[(2\pi t)^2 - \pi/(2W\alpha)^2]}\right]$$

$$= \mathrm{sinc}(2Wt)\cos(2\pi\alpha Wt)\left[\frac{1}{1 - 16\alpha^2 W^2 t^2}\right]$$

Problem 8.11

The minimum bandwidth, B_T, is equal to $1/2T$, where T is the pulse duration. For 64 quantization levels, $\log_2 64 = 6$ bits are required.

Problem 8.12

The effect of a linear phase response in the channel is simply to introduce a constant delay τ into the pulse $p(t)$. The delay τ is defined as $-1/(2\pi)$ times the slope of the phase response; see Eq. 2.144

.

Problem 8.13

The bandwidth B of a raised cosine pulse spectrum is $2W - f_1$, where $W = 1/2T_b$ and $f_1 = W(1 - \alpha)$. Thus $B = W(1 + \alpha)$. For a data rate of 56 kilobits per second, $W = 28$ kHz.

(a) For $\alpha = 0.25$,
$B = 28 \text{ kHz} \times 1.25$
$= 35 \text{ kHz}$

(b) $B = 28 \text{ kHz} \times 1.5$
$= 42 \text{ kHz}$

(c) $B = 49 \text{ kHz}$

(d) $B = 56 \text{ kHz}$

Problem 8.14

The use of eight amplitude levels ensures that 3 bits can be transmitted per pulse. The symbol period can be increased by a factor of 3. All four bandwidths in Problem 7.12 will be reduced to 1/3 of their binary PAM values.

Problem 8.15

(a) For a unity rolloff, raised cosine pulse spectrum, the bandwidth B equals $1/T$, where T is the pulse length. Therefore, T in this case is 1/12 kHz. Quarternary PAM ensures 2 bits per pulse, so the rate of information is

$$\frac{2 \text{ bits}}{T} = 24 \text{ kilobits per second}$$

(b) For 128 quantizing levels, 7 bits are required to transmit an amplitude. the additional bit for synchronization makes each code word 8 bits. The signal is transmitted at 24 kilobits/s, so it must be sampled at

$$\frac{24 \text{ bits/s}}{8 \text{ bits/sample}} = 3 \text{ kHz}$$

The maximum possible value for the signal's highest frequency component is 1.5 kHz, in order to avoid aliasing.

Problem 8.16

The raised cosing pulse bandwidth $B = 2W - f_1$, where $W = 1/2T_b$. For this channel, $B = 75$ kHz. For the given bit duration, $W = 50$ kHz. Then,

$$\begin{aligned} f_1 &= 2W - B \\ &= 25 \text{ kHz} \end{aligned}$$

$$\begin{aligned} \alpha &= 1 - f_1/B_T \\ &= 0.5 \end{aligned}$$

Problem 8.17

Problem 8.18

Problem 8.19

Problem 8.20

(a) The output symbols of the modulo-2 adder are independent because:

1. the input sequence to the adder has independent symbols, and therefore
2. knowing the previous value of the adder does not improve prediction of the present value, i.e.,

   $f(y_n|y_{n-1}) = f(y_n)$,

   where y_n is the value of the adder ouytput at time nT_b. The adder output sequence is another on-off binary wave with independent symbols. Such a wave has the power spectral density (from Problem 4.10),

$$S_Y(f) = \frac{A^2}{4}\delta(f) + \frac{A^2 T_b}{4}\mathrm{sinc}^2(fT_b) \quad .$$

The correlative coder has the transfer function

$H(f) = 1 - \exp(-j2\pi fT_b)$

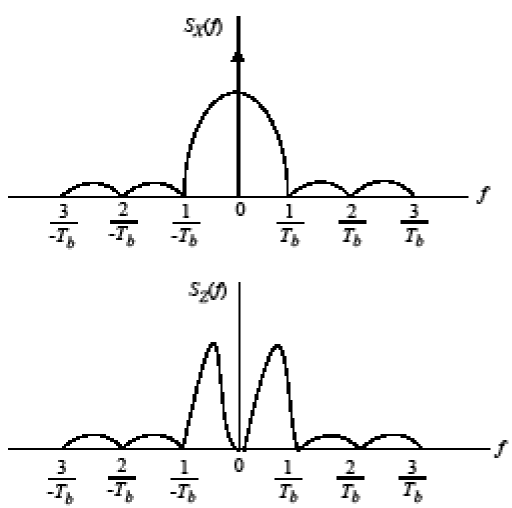
Hence, the output wave has the power spectral density

$$\begin{aligned}
S_Z(f) &= |H(f)|^2 S_Y(f) \\
&= [1-\exp(-j2\pi fT_b)][1-\exp(j2\pi fT_b)]S_Y(f) \\
&= [2-2\cos(2\pi fT_b)]S_Y(f) \\
&= 4\sin^2(\pi fT_b)S_Y(f) \\
&= 4\sin^2(\pi fT_b)\left[\frac{A^2}{4}\delta(f)\,\mathrm{sinc}^2(fT_b)\right] \\
&= A^2 T_b \sin^2(\pi fT_b)\,\mathrm{sinc}^2(fT_b)
\end{aligned}$$

In the last line we have used the fact that

$\sin(\pi fT_b) = 0$ at $f = 0$.

(b)



Note that the bipolar wave has no dc component.

(Note: the power spectral density of a bipolar signal derived in part (a) assumes the use of a pulse of full duration T_b.

Problem 8.21

(a) Polar Signaling ($M = 2$)

In this case we have

$$m(t) = \sum_n A_n \operatorname{sinc}\left(\frac{t}{T} - n\right)$$

where $A_n = \pm A/2$. Digits 0 and 1 are thus represented by $-A/2$ and $+A/2$, respectively. The Fourier transform of $m(t)$ is

$$M(f) = \sum_n A_n F\left[\operatorname{sinc}\left(\frac{t}{T} - n\right)\right]$$
$$= T\operatorname{rect}(fT)\sum_n A_n \exp(-j2\pi nfT)$$

Therefore, $m(t)$ is passed through the ideal low-pass filter with no distortion.

The noise appearing at the low-pass filter output has a variance given by

$$\sigma^2 = \frac{N_0}{2T}$$

Suppose we transmit digit 1. Then, at the sampling instant, we obtain a random variable at the input of the decision device, defined by

$$X = \frac{A}{2} + N$$

where N denotes the contribution due to noise. The decision level is 0 volts. If $X > 0$, the decision device chooses symbol 1, which is a correct decision. If $X < 0$, it chooses symbol 0, which is in error. The probability of making an error is

$$P(X < 0) = \int_{-\infty}^{0} f_X(x)dx$$

The expected value of X is $A/2$, and its variance is σ^2. Hence,

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{\left(x - \frac{A}{2}\right)^2}{2\sigma^2}\right]$$

$$P(X < 0) = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{0}\exp\left(-\frac{\left(x - \frac{A}{2}\right)^2}{2\sigma^2}\right)dx$$
$$= \frac{1}{2}\operatorname{erfc}\left(\frac{A}{2\sqrt{2}\sigma}\right)$$

Similarly, if we transmit symbol 0, an error is made when $X > 0$, and the probability of this error is

$$P(X > 0) = \frac{1}{2}\operatorname{erfc}\left(\frac{A}{2\sqrt{2}\sigma}\right)$$

Since the symbols 1 and 0 are equally probable, we find that the average probability of error is

$$P_e = \frac{1}{2}P(X < 0|\text{transmit } 1) + \frac{1}{2}P(X > 0|\text{transmit } 0)$$
$$= \frac{1}{2}\operatorname{erfc}\left(\frac{A}{2\sqrt{2}\sigma}\right)$$

(b) <u>Polar ternary signaling</u>

In this case we have

$$m(t) = \sum_n A_n \operatorname{sinc}\left(\frac{t}{T} - n\right)$$

where
$A_n = 0, \pm A$.

The 3 digits are defined as follows

Digit	Level
0	-A
1	0
2	+A

Suppose we transmit digit 2, which, at the input of the decision device, yields the random variable
$X = A + N$

The probability density function of X is

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(x-A)^2}{2\sigma^2}\right)$$

The decision levels are set at -A/2 and A/2 volts. Hence, the probability of choosing digit 1 is

$$P\left(-\frac{A}{2} < X < \frac{A}{2}\right) = \int_{-A/2}^{A/2} \frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{(x-A)^2}{2\sigma^2}\right]dx$$

$$= \frac{1}{2}\left[\operatorname{erfc}\left(\frac{A}{2\sqrt{2}\sigma}\right) - \operatorname{erfc}\left(\frac{3A}{2\sqrt{2}\sigma}\right)\right]$$

Next, the probability of choosing digit 0 is

$$P\left(X < \ -\frac{A}{2}\right) = \frac{1}{2}\operatorname{erfc}\left(\frac{3A}{2\sqrt{2}\sigma}\right)$$

If we transmit digit 1, the random variable at the input of the decision device is
$X = N$

The probability density function of X is therefore

$$f_X(x) = \frac{1}{\sqrt{2}\sigma}\exp\left(-\frac{x^2}{2\sigma^2}\right)$$

The probability of choosing digit 2 is

$$P\left(X < \ -\frac{A}{2}\right) = \frac{1}{2}\operatorname{erfc}\left(\frac{A}{2\sqrt{2}\sigma}\right)$$

Next, suppose we transmit digit 0. Then, the random variable at the input of the decision device is
$X = -A + N$

The probability density function of X is therefore

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{(x+A)^2}{2\sigma^2}\right]$$

The probability of choosing digit 1 is

$$P\left(-\frac{A}{2} < X < \frac{A}{2}\right) = \frac{1}{2}\left[\mathrm{erfc}\left(\frac{A}{2\sqrt{2}\sigma}\right) - \mathrm{erfc}\left(\frac{3A}{2\sqrt{2}\sigma}\right)\right]$$

The probability of choosing digit 2 is

$$P\left(X > \frac{A}{2}\right) = \frac{1}{2}\mathrm{erfc}\left(\frac{3A}{2\sqrt{2}\sigma}\right)$$

Assuming that digits 0, 1, and 2 are equally probable, the average probability of error is

$$\begin{aligned} P_e = {} & \frac{1}{3}\left[\frac{1}{2}\mathrm{erfc}\left(\frac{A}{2\sqrt{2}\sigma}\right) - \frac{1}{2}\mathrm{erfc}\left(\frac{3A}{2\sqrt{2}\sigma}\right)\right] + \frac{1}{3}\cdot\frac{1}{2}\mathrm{erfc}\left(\frac{3A}{2\sqrt{2}\sigma}\right) \\ & + \frac{1}{3}\cdot\frac{1}{2}\left[\mathrm{erfc}\left(\frac{A}{2\sqrt{2}\sigma}\right)\right] + \frac{1}{3}\cdot\frac{1}{2}\left[\mathrm{erfc}\left(\frac{A}{2\sqrt{2}\sigma}\right)\right] \\ & + \frac{1}{3}\cdot\frac{1}{2}\left[\mathrm{erfc}\left(\frac{A}{2\sqrt{2}\sigma}\right) - \mathrm{erfc}\left(\frac{3A}{2\sqrt{2}\sigma}\right)\right] + \frac{1}{3}\cdot\frac{1}{2}\mathrm{erfc}\left(\frac{3A}{2\sqrt{2}\sigma}\right) \\ = {} & \frac{2}{3}\mathrm{erfc}\left(\frac{A}{2\sqrt{2}\sigma}\right) \end{aligned}$$

(c) Polar quaternary signaling

In this case, we have

$$A_n = \pm\frac{A}{2}, \pm\frac{3A}{2}$$

and the 4 digits are represented as follows:

Digit	Level
0	$-\frac{3A}{2}$
1	$-\frac{A}{2}$
2	$+\frac{A}{2}$
3	$+\frac{3A}{2}$

Suppose we transmit digit 3, which, at the input of the decision device, yields the random variable:
$X = \frac{3A}{2} + N$.

The decision levels are 0, $\pm A$. The probability of choosing digit 2 is

$$P(0 < X < A) = \frac{1}{\sqrt{2\pi}\sigma}\int_0^A \exp\left[-\frac{\left(x-\frac{3A}{2}\right)^2}{2\sigma^2}\right]dx$$

$$= \frac{1}{2}\left[\operatorname{erfc}\left(\frac{A}{2\sqrt{2}\sigma}\right) - \operatorname{erfc}\left(\frac{3A}{2\sqrt{2}\sigma}\right)\right]$$

The probability of choosing digit 1 is

$$P(-A < X < 0) = \frac{1}{\sqrt{2\pi}\sigma}\int_{-A}^0 \exp\left[-\frac{\left(x-\frac{3A}{2}\right)^2}{2\sigma^2}\right]dx$$

$$= \frac{1}{2}\left[\operatorname{erfc}\left(\frac{3A}{2\sqrt{2}\sigma}\right) - \operatorname{erfc}\left(\frac{5A}{2\sqrt{2}\sigma}\right)\right]$$

The probability of choosing digit 0 is

$$P(X < -A) = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{-A} \exp\left[-\frac{\left(x-\frac{3A}{2}\right)^2}{2\sigma^2}\right]dx$$

$$= \frac{1}{2}\operatorname{erfc}\left(\frac{5A}{2\sqrt{2}\sigma}\right).$$

Suppose next we transmit digit 2, obtaining

$X = \frac{A}{2} + N$.

The probability of choosing digit 3 is

$$P(X > A) = \frac{1}{\sqrt{2\pi}\sigma}\int_A^{\infty} \exp\left[-\frac{\left(x-\frac{A}{2}\right)^2}{2\sigma^2}\right]dx$$

$$= \frac{1}{2}\operatorname{erfc}\left(\frac{A}{2\sqrt{2}\sigma}\right).$$

The probability of choosing digit 1 is

$$P(-A < X < 0) = \frac{1}{\sqrt{2\pi}\sigma}\int_{-A}^0 \exp\left[-\frac{\left(x-\frac{A}{2}\right)^2}{2\sigma^2}\right]dx$$

$$= \frac{1}{2}\left[\operatorname{erfc}\left(\frac{A}{2\sqrt{2}\sigma}\right) - \operatorname{erfc}\left(\frac{3A}{2\sqrt{2}\sigma}\right)\right]$$

The probability of choosing digit 0 is

$$P(X<-A) = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{-A} \exp\left[-\frac{\left(x-\frac{A}{2}\right)^2}{2\sigma^2}\right]dx$$

$$= \frac{1}{2}\text{erfc}\left(\frac{3A}{2\sqrt{2}\sigma}\right).$$

Suppose next we transmit digit 1, obtaining

$$X = -\frac{A}{2}+N$$

The probability of choosing digit 0 is

$$P(X<-A) = \frac{1}{2}\text{erfc}\left(\frac{A}{2\sqrt{2}\sigma}\right)$$

The probability of choosing digit 2 is

$$P(0<X<A) = \frac{1}{2}\left[\text{erfc}\left(\frac{A}{2\sqrt{2}\sigma}\right)-\text{erfc}\left(\frac{3A}{2\sqrt{2}\sigma}\right)\right]$$

The probability of choosing digit 3 is

$$P(X>A) = \frac{1}{2}\text{erfc}\left(\frac{3A}{2\sqrt{2}\sigma}\right).$$

Finally, suppose we transmit digit 0, obtaining

$$X = -\frac{3A}{2}+N$$

The probability fo choosing digit 1 is

$$P(-A<X<0) = \frac{1}{2}\left[\text{erfc}\left(\frac{A}{2\sqrt{2}\sigma}\right)-\text{erfc}\left(\frac{3A}{2\sqrt{2}\sigma}\right)\right]$$

The probability of choosing digit 2 is

$$P(0<X<A) = \frac{1}{2}\left[\text{erfc}\left(\frac{3A}{2\sqrt{2}\sigma}\right)-\text{erfc}\left(\frac{5A}{2\sqrt{2}\sigma}\right)\right]$$

The probability of choosing digit 3 is

$$P(X>A) = \frac{1}{2}\text{erfc}\left(\frac{5A}{2\sqrt{2}\sigma}\right)$$

Since all 4 digits are equally probable, with a probability of occurence equal to 1/4, we find that the average probability of error is

$$P_e = \frac{1}{4}\cdot 2\cdot\frac{1}{2}\left\{\operatorname{erfc}\left(\frac{A}{2\sqrt{2}\sigma}\right) - \operatorname{erfc}\left(\frac{3A}{2\sqrt{2}\sigma}\right)\right.$$

$$+\operatorname{erfc}\left(\frac{3A}{2\sqrt{2}\sigma}\right) - \operatorname{erfc}\left(\frac{5A}{2\sqrt{2}\sigma}\right)$$

$$+\operatorname{erfc}\left(\frac{5A}{2\sqrt{2}\sigma}\right)$$

$$+\operatorname{erfc}\left(\frac{A}{2\sqrt{2}\sigma}\right)$$

$$+\operatorname{erfc}\left(\frac{A}{2\sqrt{2}\sigma}\right) - \operatorname{erfc}\left(\frac{3A}{2\sqrt{2}\sigma}\right)'$$

$$\left.+\operatorname{erfc}\left(\frac{3A}{2\sqrt{2}\sigma}\right)\right\}$$

$$= \frac{3}{4}\operatorname{erfc}\left(\frac{A}{2\sqrt{2}\sigma}\right).$$

Problem 8.22

The average probability of error is (from the solution to Problem 7.23)

$$P_e = \left(1 - \frac{1}{M}\right)\operatorname{erfc}\left(\frac{A}{2\sqrt{2}\sigma}\right) \tag{1}$$

The received signal-to-noise ratio is

$$(\mathrm{SNR})_\mathrm{R} = \frac{A^2(M^2-1)}{12\sigma^2}$$

That is,

$$\frac{A}{\sigma} = \sqrt{\frac{12(\mathrm{SNR})_\mathrm{R}}{M^2-1}} \tag{2}$$

Substituting Eq. (2) in (1), we get

$$P_e = \left(1 - \frac{1}{M}\right)\operatorname{erfc}\sqrt{\frac{3(\mathrm{SNR})_\mathrm{R}}{2(M^2-1)}}$$

With $P_e = 10^{-6}$, we may thus write

$$10^{-6} = \left(1 - \frac{1}{M}\right)\operatorname{erfc}(u) \tag{3}$$

where

$$u^2 = \frac{3(\mathrm{SNR})_\mathrm{R}}{2(M^2-1)}$$

For a specified value of M, we may solve Eq. (3) for the corresponding value of u. We may thus construct the following table:

M	u
2	3.37
4	3.42
8	3.45
16	3.46

We thus find that to a first degree of approximation, the minimum value of received signal-to-noise ratio required for $P_e < 10^{-6}$ is given by

$$\frac{3(\text{SNR})_{\text{R, min}}}{2(M^2-1)} \approx (3.42)^2$$

That is, $(\text{SNR})_{\text{R, min}} \approx 7.8(M^2-1)$

Problem 8.23

(a) The channel output is

$$x(t) = a_1 s(t - t_{01}) + a_2 s(t - t_{02})$$

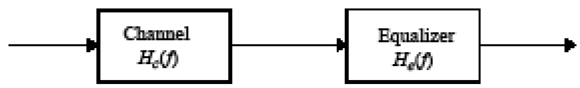
Taking the Fourier transform of both sides:

$$X(f) = [a_1 \exp(-j2\pi f t_{01}) + a_2 \exp{-j2(\pi f t_{02})}]S(f)$$

The transfer function of the channel is

$$\begin{aligned} H_c(f) &= \frac{X(f)}{X(f)} \\ &= a_1 \exp(-j2\pi f t_{01}) + a_2 \exp(-j2\pi f t_{02}) \end{aligned}$$

(b)



Ideally, the equalizer should be designed so that

$$H_c(f) H_e(f) = K_0 \exp(-j2\pi f t_0)$$

where K_0 is a constant gain and t_0 is the transmission delay. The transfer function of the equalizer is

$$\begin{aligned} H_e(f) &= w_0 + w_1 \exp(-j2\pi f T) + w_2 \exp(-j4\pi f T) \\ &= w_0 \left[1 + \frac{w_1}{w_0} \exp(-j2\pi f T) + \frac{w_2}{w_0} \exp(-j4\pi f T) \right] \end{aligned} \qquad (1)$$

Therefore

$$\begin{aligned} H_e(f) &= \frac{K_0 \exp(-j2\pi f t_0)}{H_c(f)} \\ &= \frac{K_0 \exp(-j2\pi f t_0)}{a_1 \exp(-j2\pi f t_{01}) + a_2 \exp(-j2\pi f t_{02})} \\ &= \frac{(K_0/a_1) \exp[-j2\pi f(t_0 - t_{01})]}{1 + \dfrac{a_2}{a_1} \exp[-j2\pi f(t_{02} - t_{01})]} \end{aligned}$$

Since $a_2 << a_1$, we may approximate $H_e(f)$ as follows

$$\begin{aligned} H_e(f) = \frac{K_0}{a_1} \exp[-j2\pi f(t_0 - t_{01})] &\left\{ 1 - \frac{a_2}{a_1} \exp[-j2\pi f(t_{02} - t_{01})] \right. \\ &\left. + \left(\frac{a_2}{a_1}\right)^2 \exp[-j4\pi f(t_{02} - t_{01})] \right\} \end{aligned} \qquad (2)$$

Comparing Eqs. (1) and (2), we deduce that

$$\frac{K_0}{a_1} \approx w_0$$

$$t_0 - t_{01} \approx 0$$

$$-\frac{a_2}{a_1} \approx \frac{w_1}{w_0}$$

$$\left(\frac{a_2}{a_1}\right)^2 \approx \frac{w_2}{w_0}$$

$$T \approx t_{02} - t_{01}$$

Choosing $K_0 = a_1$, we find that the tap weights of the equalizer are as follows

$$w_0 = 1$$

$$w_1 = -\frac{a_2}{a_1}$$

$$w_2 = \left(\frac{a_2}{a_1}\right)^2$$

Problem 8.24

The Fourier transform of the tapped-delay-line equalizer output is defined by

$$Y_{\text{out}}(f) = H(f)X_{\text{in}}(f) \tag{1}$$

where $H(f)$ is the equalizer's transfer function and $X_{\text{in}}(f)$ is the Fourier transform of the input signal. The input signal consists of a uniform sequence of samples, denoted by $\{x(nT)\}$. We may therefore write (see Eq. (6.2):

$$X_{\text{in}}(f) = \frac{1}{T}\sum_k X\left(f-\frac{k}{T}\right) \tag{2}$$

where T is the sampling period and $s(t)$ is the signal from which the sequence of samples is derived. For perfect equalization, we require that

$$Y_{\text{out}}(f) = 1 \qquad \text{for all } f.$$

From Eqs. (1) and (2) we therefore find that

$$H(f) = \frac{T}{\sum_k X(f-k/T)} \tag{3}$$

Let the impulse response (sequence) of the equalizer be denoted by $\{w_n\}$. Assuming an infinite number of taps, we have

$$H(f) = \sum_{n=-\infty}^{\infty} w_n \exp(j2\pi fT)$$

We now immediately see that $H(f)$ is in the form of a complex Fourier series with real coefficients defined by the tap weights of the equalizer. The tap-weights are themselves defined by

$$w_n = \frac{1}{T}\int_{-1/2T}^{1/2T} H(f)\exp(-j2\pi fT)\ , \qquad n = 0, +1,\ +2,\ldots$$

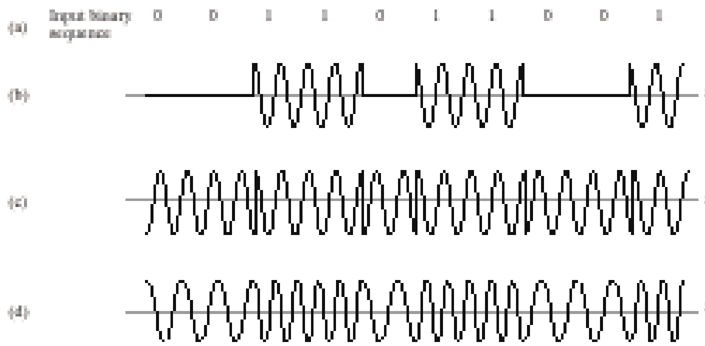
The transfer function $H(f)$ is itself defined in terms of the input signal by Eq. (3). Accordingly, a tapped-delay-line equalizer of infinite length can approximate any function in the frequency interval (-1/2T, 1/2T).

Problem 8.25
Problem 8.26
Problem 8.29

Chapter 9

Problem 9.1

The three waveforms are shown below for the sequence 0011011001. (b) is ASK, (c) is PSK; and (d) is FSK.



Problem 9.2

The bandpass signal is given by

$$s(t) = g(t)\cos\left(2\pi f_c t\right)$$

The corresponding amplitude spectrum, using the multiplication theorem for Fourier transforms, is given by

$$\begin{aligned}|S(f)| &= G(f) * \left[\delta(f - f_c) + \delta(f + f_c)\right] \\ &= G(f - f_c) + G(f + f_c)\end{aligned}$$

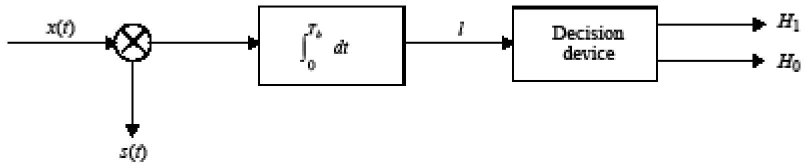
For a triangular spectrum *G*(*f*), the corresponding sketch is shown below.

Problem 9.3

To be done

Problem 9.4

(a) <u>ASK with coherent reception</u>



Denoting the presence of symbol 1 or symbol 0 by hypothesis H_1 or H_0, respectively, we may write

H_1: $x(t) = s(t) + w(t)$

H_0: $x(t) = w(t)$

where $s(t) = A_c \cos(2\pi f_c t)$, with $A_c = \sqrt{2E_b/T_b}$. Therefore,

$$l = \int_0^{T_b} x(t)s(t)dt$$

If $l > E_b/2$, the receiver decides in favor of symbol 1. If $l < E_b/2$, it decides in favor of symbol 0.

The conditional probability density functions of the random variable L, whose value is denoted by l, are defined by

$$f_{L|0}(l|0) = \frac{1}{\sqrt{\pi N_0 E_b}} \exp\left(-\frac{l^2}{N_0 E_b}\right)$$

$$f_{L|1}(l|1) = \frac{1}{\sqrt{\pi N_0 E_b}} \exp\left[-\frac{(l-E_b)^2}{N_0 E_b}\right]$$

The average probability of error is therefore,

$$P_e = P_o \int_{E_b/2}^{\infty} f_{L|0}(l|0)l + dp_1 \int_{-\infty}^{E_b/2} f_{L|1}(l|1)dl$$

$$= \frac{1}{2}\int_{E_b/2}^{\infty} \frac{1}{\sqrt{\pi N_0 E_b}} \exp\left(-\frac{l^2}{N_0 E_b}\right) dl + \frac{1}{2}\int_{\infty}^{E_b/2} \frac{1}{\sqrt{\pi N_0 E_b}} \exp\left[-\frac{(l-E_b)^2}{N_0 E_b}\right] dl$$

$$= \frac{1}{\sqrt{\pi N_0 E_b}} \int_{E_b/2}^{\infty} \exp\left(-\frac{l^2}{N_0 E_b}\right) dl$$

$$= \frac{1}{2}\mathrm{erfc}\left(\frac{1}{2}\sqrt{E_b/N_0}\right)$$

(b) ASK with noncoherent reception

In this case, the signal s(t) is defined by

$$S(t) = A_c \cos(2\pi f_c t + \theta)$$

where $A_c = \sqrt{2(E_b/T_b)}$, and

$$f_\theta(\theta) = \begin{cases} \dfrac{1}{2\pi}, & 0 \le \theta \le 2\pi \\ 0, & \text{otherwise} \end{cases}$$

For the case when symbol 0 is transmitted, that is, under hypothesis H_0, we find that the random variable L, at the input of the decision device, is Rayleigh-distributed:

$$f_{L|0}(l|0) = \frac{4l}{N_0 T_b}\exp\left(-\frac{2l^2}{N_0 T_b}\right)$$

For the case when symbol 1 is transmitted, that is, under hypothesis H_1, we find that the random variable L is Rician-distributed:

$$f_{L|1}(l|1) = \frac{4l}{N_0 T_b}\exp\left(-\frac{l^2 + A_c^2 T_b^2/4}{N_0 T_b/2}\right) I_0\left(\frac{2lA_c}{N_0}\right)$$

where $I_0 2l(A_c/N_0)$ is the modified Bessel function of the first kind of zero order.

Before we can obtain a solution for the error performance of the receiver, we have to determine a value for the threshold. Since symbols 1 and 0 occur with equal probability, the minimum probability of error criterion yields:

$$\exp\left(-\frac{A_c^2 T_b}{2N_0}\right) I_0\left(\frac{2lA_c}{N_0}\right) \underset{H_0}{\overset{H_1}{\gtrless}} 1 \qquad (1)$$

For large values of E_b/N_0, we may approximate $I_0(2lA_c/N_0)$ as follows:

$$I_0\left(\frac{2lA_c}{N_0}\right) \approx \frac{\exp(2lA_c/N_0)}{\sqrt{4\pi A_c/N_0}}$$

Using this approximation, we may rewrite Eq. (1) as follows:

$$\exp\left[\frac{A_c[4 - A_c T_b]}{2N_0}\right] \underset{H_0}{\overset{H_1}{\gtrless}} \sqrt{\frac{4\pi l A_c}{N_0}}$$

Taking the logarithm of both sides of this relation, we get

$$l \underset{H_0}{\overset{H_1}{\gtrless}} \frac{A_c T_b}{4} + \frac{1}{2}\sqrt{\frac{\pi l N_0}{A_c}}$$

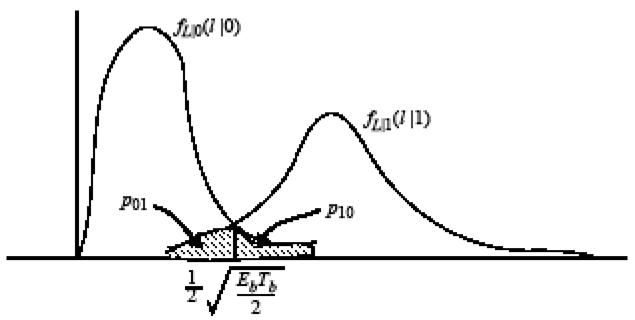
Neglecting the second term on the right hand side of this relation, and using the fact that

$$E_b = \frac{A_c^2 T_b}{2}$$

we may write

$$l \underset{H_0}{\overset{H_1}{\gtrless}} \frac{1}{2}\sqrt{\frac{E_b/T_b}{2}}$$

The threshold $\frac{1}{2}\sqrt{\frac{E_b/T_b}{2}}$ is at the point corresponding to the crossover between the two probability density functions, as illustrated below.



The average probability of error is therefore

$$P_e = p_0 P_{10} + p_1 P_{01}$$

where

$$P_{10} = \int_{\infty}^{\sqrt{E_b/T_b}/2\sqrt{2}} f_{L|0}(l|0)\,dl$$

$$= \int_{\sqrt{E_b/T_b}/2\sqrt{2}}^{\infty} \frac{4l}{N_0 T_b} \exp\left(-\frac{2l^2}{N_0 T_b}\right) dl$$

$$= \left[-\exp\left(-\frac{2l^2}{N_0 T_b}\right)\right]_{\sqrt{E_b/T_b}/2\sqrt{2}}^{\infty}$$

$$= \exp\left(-\frac{E_b}{4N_0}\right)$$

$$\begin{aligned}
P_{01} &= \int_0^{\sqrt{E_b/T_b}/2\sqrt{2}} f_{L|1}(l|1)dl \\
&= \int_0^{\sqrt{E_b/T_b}/2\sqrt{2}} \frac{4l}{N_0T_b}\exp\left(-\frac{l^2 + A_c^2T_b^2/4}{(N_0T_b)/2}\right)I_0\left(\frac{2lA_c}{N_0}\right)dl \\
&\approx \int_0^{\sqrt{E_b/T_b}/2\sqrt{2}} \frac{4l}{N_0T_b}\exp\left(-\frac{l^2 + A_c^2T_b^2/4}{N_0T_b/2}\right)\cdot\frac{\exp(2lA_c/N_0)}{\sqrt{4\pi lA_c/N_0}}dl \\
&= \int_0^{\sqrt{E_b/T_b}/2\sqrt{2}} \sqrt{\frac{2l}{A_cT_b}}\sqrt{\frac{2}{\pi N_0T_b}}\exp\left[-\frac{(l - A_cT_b/2)^2}{N_0T_b/2}\right]dl \qquad (2)
\end{aligned}$$

The integrand in Eq. (2) is the product of $\sqrt{2l/A_cT_b}$ and the probability density function of a Gaussian random variable of mean $A_cT_b/2$ and variance $N_0T_b/4$. For high values of E_b/N_0, the standard deviation $\sqrt{N_0T_b/4}$ is much less than the threshold $\sqrt{E_b/T_b}/2\sqrt{2}$ is quite small, that is, $P_{01} \approx 0$. Then, we may approximate the average probability of error as

$$\begin{aligned}
P_e &\approx p_oP_{10} \\
&= \frac{1}{2}rxp\left(-\frac{E_b}{4N_0}\right)
\end{aligned}$$

where it is assumed that symbols 0 and 1 occur with equal probability.

Problem 9.5

The transmitted binary PSK signal is defined by

$$s(t) = \begin{cases} \sqrt{E_b}\phi(t), & 0 \le t \le T_b, \quad \text{symbol } 1 \\ -\sqrt{E_b}\phi(t), & 0 \le t \le T_b, \quad \text{symbol } 0 \end{cases}$$

where the basis function $\phi(t)$ is defined by

$$\phi(t) = \sqrt{\frac{2}{T_b}}\cos(2\pi f_c t)$$

The locally generated basis function in the receiver is

$$\begin{aligned}\phi_{\text{rec}}(t) &= \sqrt{\frac{2}{T_b}}\cos(2\pi f_c t + \varphi) \\ &= \sqrt{\frac{2}{T_b}}[\cos(2\pi f_c t)\cos\varphi - \sin(2\pi f_c t)\sin\varphi]\end{aligned}$$

where φ is the phase error. The correlator output is given by

$$y = \int_0^{T_b} x(t)\varphi_{\text{rec}}(t)dt$$

where

$$x(t) = s_k(t) + w(t), \qquad k = 1, 2$$

Assuming that f_c is an integer multiple of $1/T_b$, and recognizing that $\sin(2\pi f_c t)$ is orthogonal to $\cos(2\pi f_c t)$ over the interval $0 \le t \le T_b$, we get

$$y = \pm\sqrt{E_b}\cos\varphi + W$$

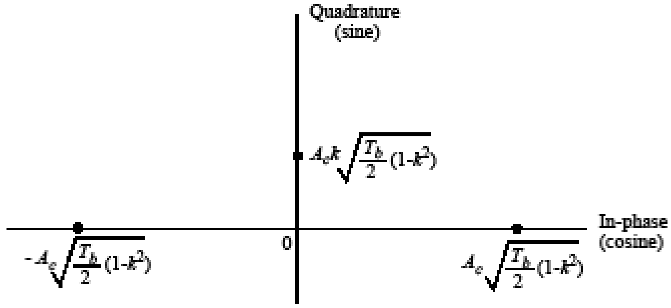
when the plus sign corresponds to symbol 1 and the minus sign corresponds to symbol 0, and W is a zero-mean Gaussian variable of variance $N_0/2$. Accordingly, the average probability of error of the binary PSK system with phase error φ is given by

$$P_e = \frac{1}{2}\text{erfc}\left(\sqrt{\frac{E_b\cos\varphi}{N_0}}\right)$$

When $\varphi = 0$, this formula reduces to that for the standard PSK system equipped with perfect phase recovery. At the other extreme, when $\varphi = \pm 90°$, P_e attains its worst value of unity.

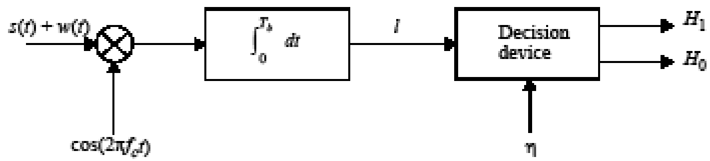
Problem 9.6

(a) The signal-space diagram of the scheme described in this problem is two-dimensional, as shown by



This signal-space diagram differs from that of the conventional PSK signaling scheme in that it is two-dimensional, with a new signal point on the quadrature axis at $A_c k\sqrt{T_b/2}$. If k is reduced to zero, the above diagram reduces to the same form as that shown in Fig. 8.14.

(b)



The signal at the decision device input is

$$l = \pm\frac{A_c}{2}\sqrt{1-k^2}T_b + \int_0^{T_b} w(t)\cos(2\pi f_c t)dt \qquad (1)$$

Therefore, following a procedure similar to that used for evaluating the average probability of error for a conventional PSK system, we find that for the system dfefined by Eq. (1) the average probability of error is

$$P_e = \frac{1}{2}\mathrm{erfc}\left(\sqrt{E_b(1-k^2)/N_0}\right)$$

where $E_b = \frac{1}{2}A_c^2 T_b$.

**The problem here is solved as "erfc" here and in the old edition, but listed in the textbook question as "Q(x)".

(c) For the case when $P_e = 10^{-4}$ and $k^2 = 0.1$, we get

$$10^{-4} = \frac{1}{2}\text{erfc}(u)$$

where $u^2 = \dfrac{0.9E_b}{N_0}$

Using the approximation

$$\text{erfc}(u) \approx \frac{\exp(-u^2)}{\sqrt{\pi}u}$$

we obtain

$$\exp(-u^2) - 2\sqrt{\pi} \times 10^{-4}u = 0$$

The solution to this equation is $u = 2.64$. The corresponding value of E_b/N_0 is

$$\frac{E_b}{N_0} = \frac{(2.64)^2}{0.9} = 7.74$$

Expressed in decibels, this value corresponds to 8.9 dB.

(d) For a conventional PSK system, we have

$$P_e = \frac{1}{2}\text{erfc}(\sqrt{E_b/N_0})$$

In this case, we find that

$$\frac{E_b}{N_0} = (2.64)^2 = 6.92$$

Problem 9.7

The bit duration is

$$T_b = \frac{1}{2.5 \times 10^6 \text{ Hz}} = 0.4\mu\text{s}$$

The signal energy per bit is

$$E_b = \frac{1}{2}A_c^2 T_b$$
$$= \frac{1}{2}(10^{-6}) \times 0.4 \times 10^{-6} = 2 \times 10^{-19}\text{joules}$$

(a) Coherent Binary FSK

The average probability of error is

$$P_e = \frac{1}{2}\text{erfc}(\sqrt{E_b/2N_0})$$
$$= \frac{1}{2}\text{erfc}\left(\sqrt{2 \times 10^{-19}/4 \times 10^{-20}}\right)$$
$$= \frac{1}{2}\text{erfc}(\sqrt{5})$$

Using the approximation

$$\text{erfc}(u) \approx \frac{\exp(-u^2)}{\sqrt{\pi}u}$$

we obtain the result

$$P_e = \frac{1}{2}\ \frac{\exp(-5)}{\sqrt{5\pi}} = 0.85 \times 10^{-3}$$

(b) MSK

$$P_e = \text{erfc}(\sqrt{E_b/N_0})$$
$$= \text{erfc}(\sqrt{10})$$
$$\approx \frac{\exp(-10)}{\sqrt{10\pi}}$$
$$= 0.81 \times 10^{-5}$$

(c) Noncoherent Binary FSK

$$P_e = \frac{1}{2}\exp\left(-\frac{E_b}{2N_0}\right)$$
$$= \frac{1}{2}\exp(-5)$$
$$= 3.37 \times 10^{-3}$$

Problem 9.8

(a) The correlation coefficient of the signals $s_0(t)$ and $s_1(t)$ is

$$\rho = \frac{\int_0^{T_b} s_0(t)s_1(t)dt}{\left[\int_0^{T_b} s_0^2(t)dt\right]^{1/2}\left[\int_0^{T_b} s_1^2(t)dt\right]^{1/2}}$$

$$= \frac{A_c^2\int_0^{T_b}\cos\left[2\pi\left(f_c+\frac{1}{2}\Delta f\right)t\right]\cos\left[2\pi\left(f_c - \frac{1}{2}\Delta f\right)t\right]}{\left[\frac{1}{2}A_c^2T_b\right]^{1/2}\left[\frac{1}{2}A_c^2T_b\right]^{1/2}}$$

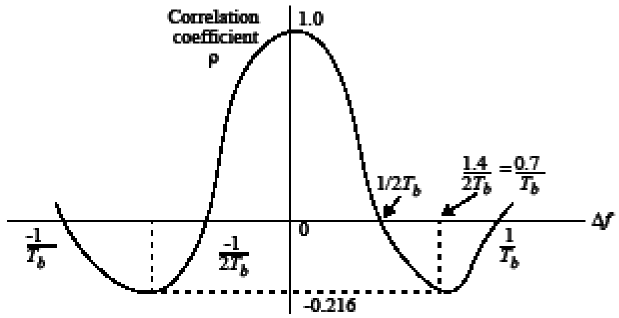
$$= \frac{1}{T_b}\int_0^{T_b}[\cos(2\pi\Delta ft)+\cos(4\pi f_ct)]dt$$

$$= \frac{1}{2\pi T_b}\left[\frac{\sin(2\pi\Delta fT_b)}{\Delta f}+\frac{\sin(4\pi f_cT_b)}{2f_c}\right] \qquad (1)$$

Since $f_c >> \Delta f$, then we may ignore the second term in Eq. (1), obtaining

$$\rho \approx \frac{\sin(2\pi\Delta fT_b)}{2\pi T_b\Delta f} = \mathrm{sinc}(2\Delta fT_b)$$

(b) The dependence of ρ on Δf is as shown in Fig. 1.



$s_0(t)$ and $s_1(t)$ are orthogonal when $\rho = 0$. Therefore, the minimum value of Δf for which they are orthogonal is $1/2T_b$.

(c) The average probability of error is given by

$$E_b = \frac{1}{2}\text{erfc}(\sqrt{E_b(1-\rho)/2N_0})$$

The most negative value of ρ is -0.216, occuring at $\Delta f = 0.7/T_b$. The minimum value of P_e is therefore

$$P_{e,\,\min} = \frac{1}{2}\text{erfc}(\sqrt{0.608E_b/N_0})$$

(d) For a coherent binary PSK system, the average probability of error is

$$P_e = \frac{1}{2}\text{erfc}(\sqrt{E_b/N_0})$$

Therefore, the E_b/N_0 of this coherent binary FSK system must be increased by the factor 1/0.608 = 1.645 (or 2.16 dB) so as to realize the same average probability of error as a coherent binary PSK system.

Problem 9.9

(a) Since the two oscillators used to represent symbols 1 and 0 are independent, we may view the resulting binary FSK wave as the sum of two on-off keying (00K) signals. One 00K signal operates with the oscillator of frequency f_1. The second 00K signal operates with the oscillator of f_2.

The power spectral density of a random binary wave $X_1(t)$, in which symbol 1 is represented by A volts and symbol 0 by zero volts, it is given by (see Problem 4.10)

$$S_{X_1}(f) = \frac{A^2}{4}\delta(f) + \frac{A^2T_b}{4}\text{sinc}^2(fT_b)$$

where T_b is the bit duration. When this binary wave is multiplied by a sinusoidal wave of unit amplitude and frequency $f_c + \Delta f/2$, we get the first 00K signal with

$$A = \sqrt{2E_b/T_b}$$

The power spectral density of this 00K signal equals

$$S_1(f) = \frac{1}{4}\left[S_{X_1}\left(f-f_c-\frac{\Delta f}{2}\right) + S_{X_1}\left(f+f_c+\frac{\Delta f}{2}\right)\right]$$

The power spectral density of the random binary wave $X_2(t) = \overline{X_1(t)}$, in which symbol 1 is represented by zero volts and symbol 0 by A volts, is given by

$$S_{X_2}(f) = S_{X_1}(f)$$

When $X_2(t)$ is multiplied by the second sinusoidal wave of unit amplitude and frequency $f_c - \Delta f/2$, we get the second 00K signal whose power spectral density equals

$$S_2(f) = \frac{1}{4}\left[S_{X_2}\left(f - f_c - \frac{\Delta f}{2}\right) + S_{X_2}\left(f + f_c + \frac{\Delta f}{2}\right)\right]$$

The power spectral density of the FSK signal equals:

$$S_{\text{FSK}}(f) = S_1(f) + S_2(f)$$

$$= \frac{E_b}{8T_b}\left[\delta\left(f - f_c - \frac{\Delta f}{2}\right) + \delta\left(f + f_c + \frac{\Delta f}{2}\right) + \delta\left(f - f_c + \frac{\Delta f}{2}\right) + \delta\left(f + f_c - \frac{\Delta f}{2}\right)\right]$$

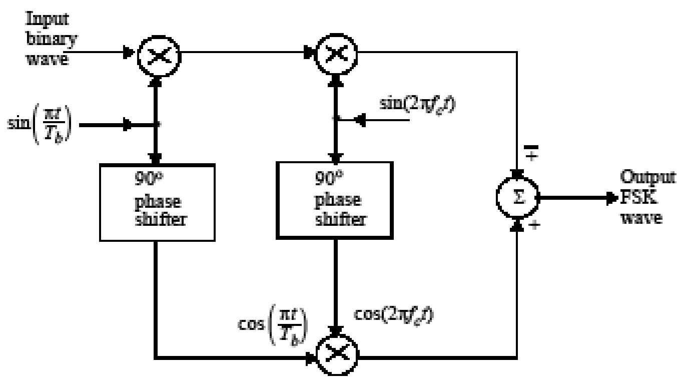
$$+ \frac{E_b}{8}\left\{\operatorname{sinc}^2\left[T_b\left(f - f_c - \frac{\Delta f}{2}\right)\right] + \operatorname{sinc}^2\left[T_b\left(f + f_c + \frac{\Delta f}{2}\right)\right]\right.$$

$$\left. + \operatorname{sinc}^2\left[T_b\left(f - f_c + \frac{\Delta f}{2}\right)\right] + \operatorname{sinc}^2\left[T_b\left(f + f_c - \frac{\Delta f}{2}\right)\right]\right\}$$

This result shows that the power spectrum of this binary FSK wave contains delta functions at $f = f_c \pm \Delta f/2$.

(b) At high values of x, the function sinc(x) falls off as $1/x$. Hence, at high frequencies, S_{FSK} falls off as $1/f^2$.

Problem 9.10



Problem 9.11

(a) For coherent binary PSK,

$$P_e = \frac{1}{2}\text{erfc}\left(\frac{E_b}{N_0}\right).$$

For P_e to equal 10^{-4}, $\sqrt{E_b/N_0} = 2.64$. This yields $E_b/N_0 = 7.0$. Hence, $E_b = 3.5 \text{ x } 10^{-10}$. The required average carrier power is 0.35 mW.

(b) For DPSK,

$$P_e = \frac{1}{2}\text{erfc}\left(-\frac{E_b}{N_0}\right).$$

For P_e to equal 10^{-4}, we have $E_b/N_0 = 8.5$. Hence $E_b = 4.3 \text{ x } 10^{-10}$. The required average power is 0.43 mW.

Problem 9.12

(a) For a coherent PSK system, the average probability of error is

$$P_e = \frac{1}{2}\text{erfc}[\sqrt{(E_b/N_0)_1}]$$

$$\approx \frac{1}{2}\ \frac{\exp[-(E_b/N_0)_1]}{\sqrt{\pi}\sqrt{(E_b/N_0)_1}} \qquad (1)$$

For a DPSK system, we have

$$P_e = \frac{1}{2}\exp[-(E_b/N_0)_2] \qquad (2)$$

Let

$$\left(\frac{E_b}{N_0}\right)_2 = \left(\frac{E_b}{N_0}\right)_1 + \delta$$

Then, we may use Eqs. (1) and (2) to obtain

$$\sqrt{\pi}\sqrt{(E_b/N_0)_1} = \exp\delta$$

We are given that

$$\left(\frac{E_b}{N_0}\right)_1 = 7.2$$

Hence,

$$\delta = \ln[\sqrt{7.2\pi}]$$
$$= 1.56$$

Therefore,

$$10\log_{10}\left(\frac{E_b}{N_0}\right)_1 = 10\log_{10}7.2 = 8.5\text{dB}$$

$$10\log_{10}\left(\frac{E_b}{N_0}\right)_2 = 10\log_{10}(7.2 + 1.56)$$
$$= 9.42\text{dB}$$

The separation between the two (E_b/N_0) ratios is therefore 9.42 - 8.57 = 0.85 dB.

(b) For a coherent PSK system, we have

$$P_e = \frac{1}{2}\operatorname{erfc}[\sqrt{(E_b/N_0)_1}]$$

$$\approx \frac{1}{2}\frac{\exp[-(E_b/N_0)_1]}{\sqrt{\pi}\sqrt{(E_b/N_0)_1}}$$

For a QPSK system, we have

$$P_e = \operatorname{erfc}[\sqrt{(E_b/N_0)_2}]$$

$$\approx \frac{\exp[-(E_b/N_0)_2]}{\sqrt{\pi}\sqrt{(E_b/N_0)_2}}$$

Here again, let

$$\left(\frac{E_b}{N_0}\right)_2 = \left(\frac{E_b}{N_0}\right)_1 + \delta$$

Then we may use Eqs. (3) and (4) to obtain

$$\frac{1}{2} = \frac{\exp(-\delta)}{\sqrt{1+\delta/(E_b/N_0)_1}}$$

Taking logarithms of both sides:

$$-\ln 2 = -\delta - 0.5\ln[1+\delta/(E_b/N_0)_1]$$

$$\approx -\delta - 0.5\frac{\delta}{(E_b/N_0)_1}$$

Solving for δ:

$$\delta \approx \frac{\ln 2}{1+0.5/(E_b/N_0)_1}$$

$$= 0.65$$

Therefore,

$$10\log_{10}\left(\frac{E_b}{N_0}\right)_1 = 10\log_{10}(7.2) = 8.57\text{dB}$$

$$10\log_{10}\left(\frac{E_b}{N_0}\right)_2 = 10\log_{10}(7.2+65)$$

$$= 8.95\text{dB}$$

The separation between the two (E_b/N_0) ratios is 8.95 - 8.57 = 0.38 dB.

(c) For a coherent binary FSK system, we have

$$P_e = \frac{1}{2}\mathrm{erfc}[\sqrt{(E_b/N_0)_1}]$$

$$= \frac{1}{2}\ \frac{\exp\left(-\frac{1}{2}\left(\frac{E_b}{N_0}\right)_1\right)}{\sqrt{\pi}\sqrt{(E_b/2N_0)_1}} \tag{6}$$

For a noncoherent binary FSK system, we have

$$P_e = \frac{1}{2}\exp\left(-\frac{1}{2}\left(\frac{E_b}{N_0}\right)_2\right) \tag{7}$$

Hence,

$$\sqrt{\frac{\pi}{2}\left(\frac{E_b}{N_0}\right)_1} = \exp\left(\frac{\delta}{2}\right) \tag{8}$$

We are given that $(E_b/N_0) = 13.5$. Therefore,

$$\delta = \ln\left(\frac{13.5\pi}{2}\right)$$

$$= 3.055$$

We thus find that

$$10\log_{10}\left(\frac{E_b}{N_0}\right)_1 = 10\log_{10}(13.5)$$

$$= 11.3\mathrm{dB}$$

$$10\log_{10}\left(\frac{E_b}{N_0}\right)_2 = 10\log_{10}(13.5 + 3.055)$$

$$= 12.2\mathrm{dB}$$

Hence, the separation between the two (E_b/N_0) ratios is 12.1 - 11.3 = 0.9 dB.

(d) For a coherent binary FSK system, we have

$$P_e = \frac{1}{2}\mathrm{erfc}[\sqrt{(E_b/2N_0)_1}]$$

$$= \frac{1}{2}\ \frac{\exp\left(-\frac{1}{2}\left(\frac{E_b}{N_0}\right)_1\right)}{\sqrt{\pi}\sqrt{(E_b/2N_0)_1}} \qquad (9)$$

For a MSK system, we have

$$P_e = \frac{1}{2}\mathrm{erfc}[\sqrt{(E_b/2N_0)_2}]$$

$$\approx \frac{\exp\left(-\frac{1}{2}\left(\frac{E_b}{N_0}\right)_2\right)}{\sqrt{\pi}\sqrt{(E_b/2N_0)_2}} \qquad (10)$$

Hence, using Eqs. (9) and (10), we have

$$\ln 2 - \frac{1}{2}\ln\left[1 + \frac{\delta}{(E_b/N_0)_1}\right] \approx \frac{1}{2}\delta \qquad (11)$$

Noting that

$$\frac{\delta}{(E_b/N_0)_1} << 1$$

We may approximate Eq. (11) to obtain

$$\ln 2 - \frac{1}{2}\left[1 + \frac{\delta}{(E_b/N_0)_1}\right] \approx \frac{1}{2}\delta \qquad (12)$$

Solving for δ, we obtain

$$\delta = \frac{1\ln 2}{1 + \frac{1}{(E_b/N_0)_1}}$$

$$= \frac{2 \times 0.693}{1 + \frac{1}{13.5}}$$

$$= 1.29$$

We thus find that

$$10\log_{10}\left(\frac{E_b}{N_0}\right)_1 = 10\log_{10}(13.5) = 10 \times 1.13 = 11.3\mathrm{dB}$$

$$10\log_{10}\left(\frac{E_b}{N_0}\right)_2 = 10\log_{10}(13.5 + 1.29) = 11.7\mathrm{dB}$$

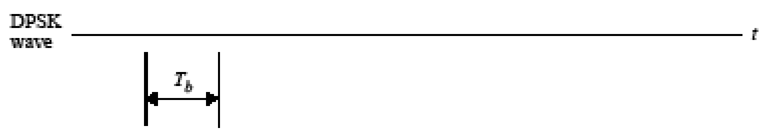
Therefore, the separation between the two (E_b/N_0) ratios is 11.7 - 11.3 = 0.4 dB.

Problem 9.13
Problem 9.14

(a)

b_k		1	1	0	0	1	0	0	0	1	0
$d_{k\text{-}1}$		1	1	1	0	1	1	0	1	0	0
d_k	1	1	1	0	1	1	0	1	0	0	1
Transmitted phase	0	0	0	π	0	0	π	0	π	π	0

The waveform of the DPSK signal is thus as follows:



(b) Let x_I = output of the integrator in the in-phase channel

x_Q = output of the integrator in the quadrature channel

x_I' = one-bit delayed version of x_I

x_Q' = one-bit delayed version of x_Q

l_I = in-phase channel output
$= x_I x_I'$

l_Q = quadrature channel output
$= x_Q x_Q'$

$y = l_I + l_Q$

Transmitted phase (radians)	**0**	**0**	**0**	**π**	**0**	**0**	**π**	**0**	**π**	**π**	**0**
Polarity of x_I	+	+	+	-	+	+	-	+	-	-	+
Polarity of x_I'		+	+	+	-	+	+	-	+	-	-
Polarity of l_I		+	+	-	-	+	-	-	-	+	-
Polarity of x_Q	-	-	-	+	-	-	+	-	+	+	-
Polarity of x_Q'		-	-	-	+	-	-	+	-	+	+
Polarity of l_Q		+	+	-	-	+	-	-	-	+	-
Polarity of y		+	+	-	-	+	-	-	-	+	-
Reconstructed data stream		**1**	**1**	**0**	**0**	**1**	**0**	**0**	**0**	**1**	**0**

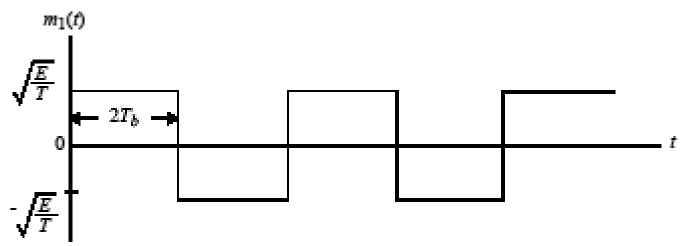
Problem 9.15

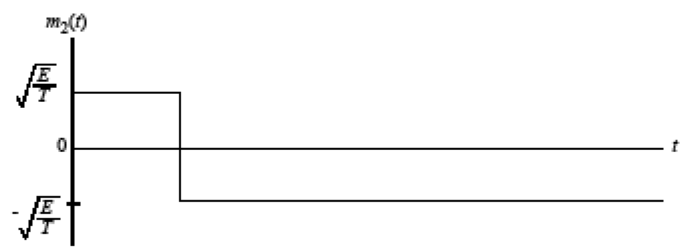
(a) The QPSK wave can be expressed as

$$s(t) = m_1(t)\cos(2\pi f_c t) + m_2(t)\sin(2\pi f_c t)$$

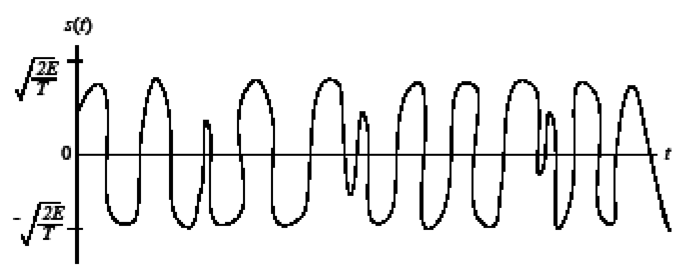
Dividing the binary wave into dibits and finding $m_1(t)$ and $m_2(t)$ for each dibit:

dibit	11	00	10	00	10
$m_1(t)$	$\sqrt{E/T}$	$-\sqrt{E/T}$	$\sqrt{E/T}$	$-\sqrt{E/T}$	$\sqrt{E/T}$
$m_2(t)$	$\sqrt{E/T}$	$-\sqrt{E/T}$	$-\sqrt{E/T}$	$-\sqrt{E/T}$	$-\sqrt{E/T}$





(b)



Problem 9.16

Let P_{eI} = average probability of symbol error into the in-phase channel

P_{eQ} = average probability of symbol error into the quadrature channel

Since the individual outputs of the in-phase and quadrature channels are statistically independent, the overall average probability of correct reception is

$$\begin{aligned} P_c &= (1-P_{eI})(1-P_{eQ}) \\ &= 1-P_{eI}-P_{eQ}+P_{eI}P_{eQ} \end{aligned}$$

The overall average probability of error is therefore

$$\begin{aligned} P_e &= 1-P_c \\ &= P_{eI}+P_{eQ}-P_{eI}P_{eQ} \end{aligned}$$

Problem 9.17

For coherent MSK, the probability of error is

$$P_e = \text{erfc}(\sqrt{E_b/N_0}).$$

While for noncoherent MSK, (i.e., noncoherent binary FSK)

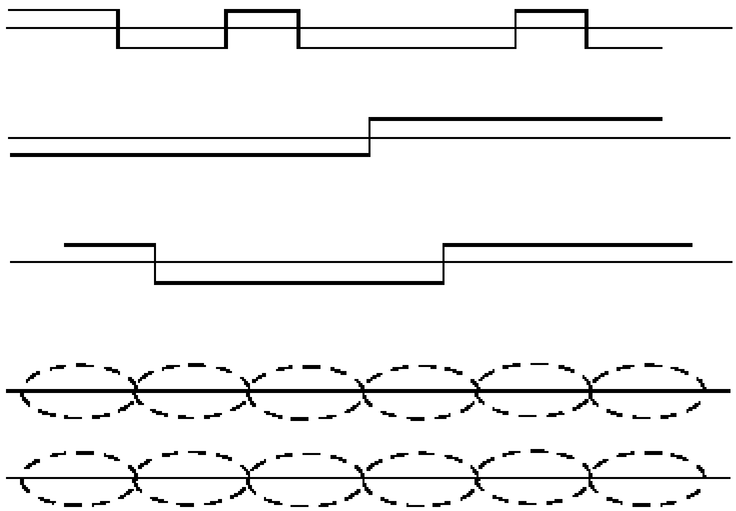
$$P_e = \frac{1}{2}\exp\left(-\frac{E_b}{2N_0}\right).$$

To maintain $P_e = 10^{-5}$ for coherent MSK, $\frac{E_b}{N_0} = 9.8$. To maintain the same probability of symbol error for noncoherent MSK,

$\frac{E_b}{N_0} = 21.6$, which is an increase of 3.4 dB.

Problem 9.18

(a)



(b)

Problem 9.19

The important point to note here, in comparison to the results plotted in Fig. 1 is that the error performance of the coherent QPSK is slightly degraded with respect to that of coherent PSK and coherent MSK. Otherwise, the observations made in Section 9.5 still hold here.

Problem 9.20

Let

$$x(t) = A_c \cos(2\pi f_c t + \theta)$$
$$= A_c \cos(2\pi f_c t)\cos\theta - A_c \sin(2\pi f_c t)\sin\theta$$

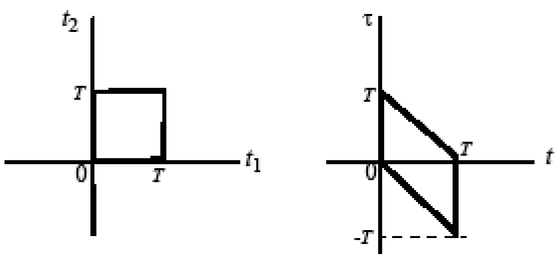
The output of the square-law envelope detector in Fig. P8.2, sampled at time $t = T$, is given by

$$y(T) = \left[\int_0^T x(t)\cos(2\pi f_c t)dt\right]^2 + \left[\int_0^T x(t)\sin(2\pi f_c t)dt\right]^2$$

This may be written as

$$y(T) = \int_0^T\int_0^T x(t_1)x(t_2)[\cos(2\pi f_c t_1)\cos(2\pi f_c t_2) + \sin(2\pi f_c t_1)\sin(2\pi f_c t_2)]dt_1 dt_2 \qquad (1)$$

Put $t_1 = t$, and $t_2 = t + \tau$. This transformation is illustrated below:



Then, we may rewrite Eq. (1) as follows

$$y(T) = \int_0^T\int_{-t}^{T-t} x(t)x(t+\tau)[\cos(2\pi f_c t)\cos(2\pi f_c t + 2\pi f_c \tau)$$
$$+ \sin(2\pi f_c t)\sin(2\pi f_c t + 2\pi f_c \tau)]dt d\tau \qquad (2)$$

However,

$$\cos(2\pi f_c t)\cos(2\pi f_c t + 2\pi f_c \tau) + \sin(2\pi f_c t)\sin(2\pi f_c t + 2\pi f_c \tau) = \cos(2\pi f_c \tau)$$

Therefore, we may simplify Eq. (2) as follows

$$y(T) = \int_0^T\int_{-t}^{T-t} x(t)x(t+\tau)\cos(2\pi f_c \tau)d\tau dt$$
$$= 2\int_0^T\int_0^{T-t} x(t)x(t+\tau)\cos(2\pi f_c \tau)d\tau dt\ , \qquad 0 \le \tau \le T \qquad (3)$$

Define

$$R_X(\tau) = \int_0^{T-t} x(t)x(t+\tau)dt \qquad 0 \le \tau \le T$$

Then, we may rewrite Eq. (3) in terms of $R_X(\tau)$ as follows

$$\begin{aligned} y(T) &= 2\int_0^T R_X(\tau)\cos(2\pi f_c \tau)\,d\tau \\ &= 2S_X(f_c) \end{aligned} \tag{4}$$

where

$$S_X(f) = \int_0^T R_X(\tau)\cos(2\pi f_c \tau)\,d\tau$$

Equation (4) is the desired result.

Problem 9.21

Problem 9.22

The average power for any modulation scheme is

$$P = \frac{E_b}{T_b}.$$

This can be demonstrated for the three types given by integrating their power spectral densities from $-\infty$ to ∞,

$$\begin{aligned} P &= \int_{-\infty}^{\infty} S(f)df \\ &= \frac{1}{4}\int_{-\infty}^{\infty} [S_B(f-f_c) + S_B(f+f_c)]df \\ &= \frac{1}{2}\int_{-\infty}^{\infty} S_B(f)df \end{aligned}$$

The baseband power spectral densities for each of the modulation techniques are:

	PSK	QPSK	MSK
$S_B(f)$	$2E_b \mathrm{sinc}^2(fT_b)$	$4E_b \mathrm{sinc}^2(2fT_b)$	$\frac{32E_b}{\pi^2}\left[\frac{\cos(2\pi f T_b)}{16f^2T_b^2 - 1}\right]^2$

Since $\int_{-\infty}^{\infty} a \ \mathrm{sinc}^2(ax)dx = 1$, $P = \frac{E_b}{T_b}$ is easily derived for PSK and QPSK. For MSK we have

$$\begin{aligned} P &= \frac{16E_b}{\pi^2}\int_{-\infty}^{\infty}\left[\frac{\cos(2\pi f T_b)}{16f^2T_b^2 - 1}\right]df \\ &= \frac{16E_b}{\pi^2 T_b}\int_{-\infty}^{\infty}\frac{\cos^2(2\pi x)}{16x^2 - 1}dx \\ &= \frac{8E_b}{\pi^2 T_b}\int_{-\infty}^{\infty}\frac{1 + \cos(4\pi x)}{16x^2\left(x^2 - \frac{1}{16}\right)}dx \end{aligned}$$

$$= \frac{E_b}{16\pi^2 T_b}\int_{-\infty}^{\infty} \frac{\cos 0 + \cos(4\pi x)}{\left(x^2 - \frac{1}{16}\right)^2} dx$$

From integral tables

$$\int_0^x \frac{\cos(ax)dx}{(b^2 - x^2)^2} = \frac{\pi}{4b^3}[\sin(ab) - ab\cos(ab)]$$

For $a = 0$, the integral is 0.

For a - 4π, $b = 1/4$, we have

$$P = \frac{E_b}{16\pi^2 T_b}\int_{-\infty}^{\infty} \frac{\cos(ax)}{(b^2 - x^2)^2} dx = \frac{E_b}{T_b}$$

For the three schemes, the values of $S(f_c)$ are as follows:

	PSK	QPSK	MSK
$S(f_c)$	$\frac{E_b}{2}$	E_b	$\frac{8E_b}{\pi^2}$

Hence, the noise equivalent bandwidth for each technique is as follows:

	PSK	QPSK	MSK
Bandwidth	$\frac{1}{T_b}$	$\frac{1}{2T_b}$	$\frac{0.62}{T_b}$

Problem 9.23

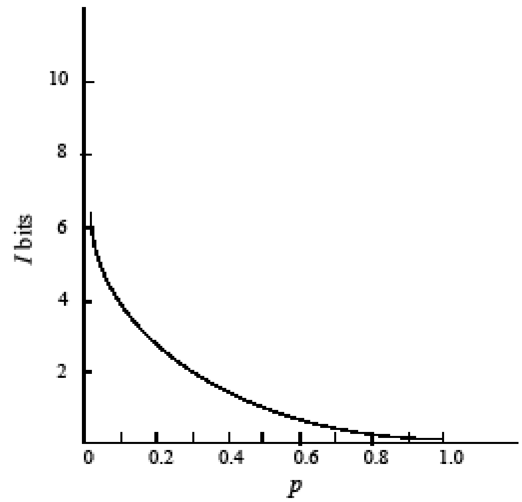
Chapter 10 Problems

Problem 10.1

Amount of information gained by the occurrence of an event of probability p is

$$I = \log_2\left(\frac{1}{p}\right) \text{bits}$$

I varies with p as shown below:



Problem 10.2

Let the event $S = s_k$ denote the emission of symbol s_k by the source. Hence,

$$I(s_k) = \log_2\left(\frac{1}{p}\right) \text{bits}$$

s_k	s_0	s_1	s_2	s_3
p_k	0.4	0.3	0.2	0.1
$I(s_k)$ bits	1.322	1.737	2.322	3.322

Problem 10.3

Entropy of the source is

$$\begin{aligned} H(S) &= p_0 \log_2\left(\frac{1}{p_0}\right) + p_1 \log_2\left(\frac{1}{p_1}\right) + p_2 \log_2\left(\frac{1}{p_2}\right) + p_3 \log_2\left(\frac{1}{p_3}\right) \\ &= \frac{1}{3}\log_2(3) + \frac{1}{6}\log_2(6) + \frac{1}{4}\log_2(4) + \frac{1}{4}\log_2(4) \\ &= 0.528 + 0.431 + 0.5 + 0.5 \\ &= 1.959 \text{ bits} \end{aligned}$$

Problem 10.4

Let X denote the number showing on a single roll of a dice. With a dice having six faces, we note that p_X is 1/6. Hence, the entropy of X is

$$\begin{aligned} H(X) &= p_X \log_2\left(\frac{1}{p_X}\right) \\ &= \frac{1}{6}\log_2(6) = 0.431 \text{ bits} \end{aligned}$$

Problem 10.5

The entropy of the quantizer output is

$$H = -\sum_{k=1}^{4} P(X_k)\log_2 P(X_k)$$

where X_k denotes a representation level of the quantizer. Since the quantizer input is Gaussian with zero mean, and a Gaussian density is symmetric about its mean, we find that

$$P(X_1) = P(X_4)$$
$$P(X_2) = P(X_3)$$

The representation level $X_1 = 1.5$ corresponds to a quantizer input $+1 \le Y < \infty$. Hence,

$$\begin{aligned} P(X_1) &= \int_1^\infty \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{y^2}{2}\right)dy \\ &= \frac{1}{2} - \frac{1}{2}\mathrm{erf}\left(\frac{4}{\sqrt{2}}\right) \\ &= 0.1611 \end{aligned}$$

The representation level $X_2 = 0.5$ corresponds to the quantizer input $0 \le Y < 1$. Hence,

$$\begin{aligned} P(X_2) &= \int_0^1 \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{y^2}{2}\right)dy \\ &= \frac{1}{2}\mathrm{erf}\left(\frac{4}{\sqrt{2}}\right) \\ &= 0.3389 \end{aligned}$$

Accordingly, the entropy of the quantizer output is

$$\begin{aligned} H &= -2\left[0.1611\log_2\left(\frac{1}{0.1611}\right) + 0.3389\log_2(0.3389)\right] \\ &= 1.91 \text{ bits} \end{aligned}$$

Problem 10.6

(a) For a discrete memoryless source:

$$P(\sigma_i) = P(s_{i_1})P(s_{i_1})\dots P(s_{i_n})$$

Noting that $M = K^n$, we may therefore write

$$\begin{aligned}\sum_{i=0}^{M-1} P(\sigma_i) &= \sum_{i=0}^{M-1} P(s_{i_1})P(s_{i_2})\dots P(s_{i_n}) \\ &= \sum_{i_1=0}^{K-1}\sum_{i_2=0}^{K-1}\cdots\sum_{i_n=0}^{K-1} P(s_{i_1})P(s_{i_2})\dots P(s_{i_n}) \\ &= \sum_{i_1=0}^{K-1} P(s_{i_1})\sum_{i_2=0}^{K-1} P(s_{i_2})\dots\sum_{i_n=0}^{K-1} P(s_{i_n}) \\ &= 1\end{aligned}$$

(b) For $k = 1,2,\dots,n$, we have

$$\sum_{i=0}^{M-1} P(\sigma_i)\log_2\left(\frac{1}{p_{i_k}}\right) = \sum_{i=0}^{M-1} P(s_{i_1})P(s_{i_2})\dots P(s_{i_n})\log_2\left(\frac{1}{p_{i_k}}\right)$$

For $k = 1$, say, we may thus write

$$\begin{aligned}\sum_{i=0}^{M-1} P(\sigma_i)\log_2\left(\frac{1}{p_{i_1}}\right) &= \sum_{i_n=0}^{K-1} P(s_{i_1})\log_2\left(\frac{1}{p_{i_k}}\right)\sum_{i=0}^{K-1} P(s_{i_2})\dots\sum_{i=0}^{K-1} P(s_{i_n}) \\ &= \sum_{i=0}^{K-1} P(s_{i_1})\log_2\left(\frac{1}{p_{i_1}}\right) \\ &= H(S)\end{aligned}$$

Clearly, this result holds not only for $k = 1$, but also $k = 2,\dots,n$.

(c)

$$\begin{aligned}H(S^n) &= \sum_{i=0}^{M-1} P(\sigma_i)\log 2\frac{1}{P(\sigma_i)} \\ &= \sum_{i=0}^{M-1} P(\sigma_i)\log_2\frac{1}{P(s_{i_1})P(s_{i_2})\dots P(s_{i_n})} \\ &= \sum_{i=0}^{M-1} P(\sigma_i)\log_2\frac{1}{P(s_{i_1})} + P(\sigma_i)\log_2\frac{1}{P(s_{i_2})} \\ &\quad + \dots + \sum_{i=0}^{M-1} P(\sigma_i)\log_2\frac{1}{P(s_{i_n})}\end{aligned}$$

Using the result of part (b), we thus get

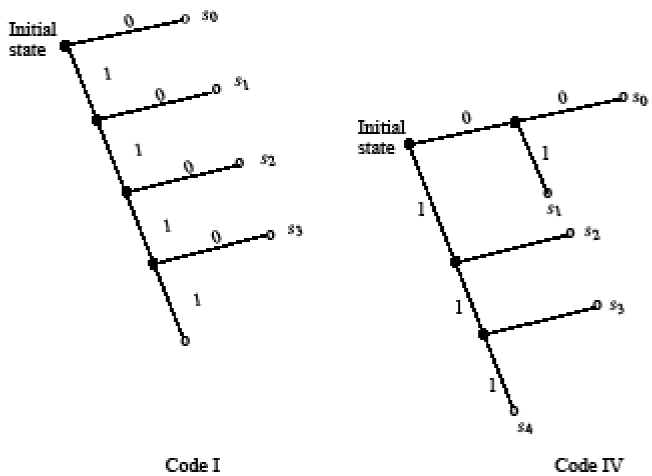
$$\begin{aligned}H(S^n) &= H(S) + H(S) + \dots + H(S) \\ &= nH(S)\end{aligned}$$

Problem 10.7

a)

A prefix code is defined as a code in which no code word is the prefix of any other code word. By inspection, we see therefore that codes I and IV are prefix codes, whereas codes II and III are not.

To draw the decision tree for a prefix code, we simply begin from some starting node, and extend branches forward until each symbol of the code is represented. We thus have:

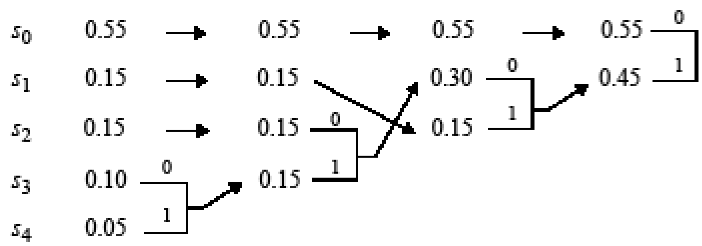


b)To be done

Problem 10.8

We may construct two different Huffman codes by choosing to place a combined symbol as *low* or as *high* as possible when its probability is equal to that of another symbol.

We begin with the Huffman code generated by placing a combined symbol as low as possible:



The source code is therefore

s_0	0
s_1	1 1
s_2	1 0 0
s_3	1 0 1 0
s_4	1 0 1 1

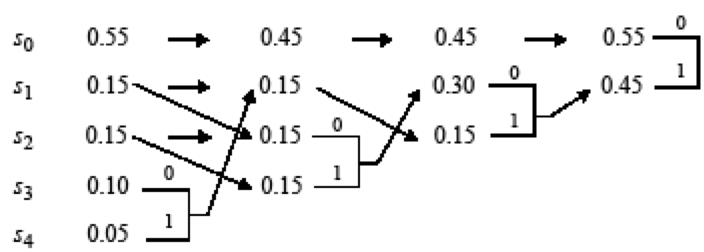
The average code-word length is therefore

$$
\begin{aligned}
\bar{L} &= \sum_{k=0}^{4} p_k l_k \\
&= 0.55(1) + 0.15(2) + 0.15(3) + 0.1(4) + 0.05(4) \\
&= 1.9
\end{aligned}
$$

The variance of $\bar{L}$ is

$$
\begin{aligned}
\sigma^2 &= \sum_{k=0}^{4} p_k (l_k - \bar{L})^2 \\
&= 0.55(-0.9)^2 + 0.15(0.1)^2 + 0.15(0.55)(1.1)^2 + 0.1(2.1)^2 + 0.05(2.1)^2 \\
&= 1.29
\end{aligned}
$$

Next placing a combined symbol as high as possible, we obtain the second Huffman code:



Correspondingly, the Huffman code is

s_0	0
s_1	1 0 0
s_2	1 0 1
s_3	1 1 0
s_4	1 1 1

The average code-word length is

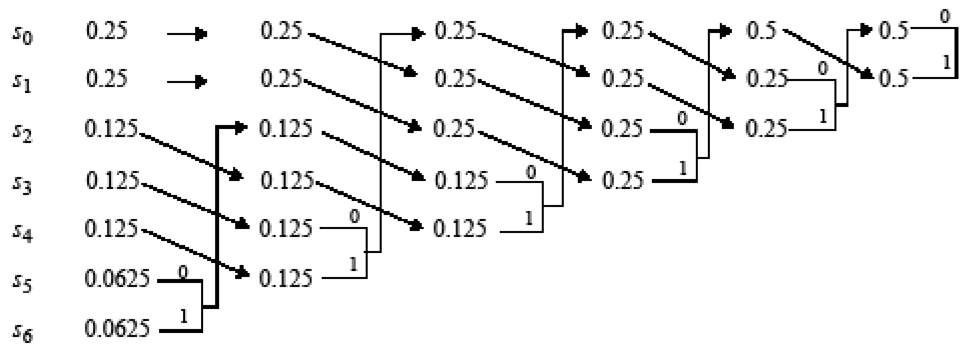
$$\bar{L} = 0.55(1)^2 + (0.15 + 0.15 + 0.1 + 0.05)(3)$$
$$= 1.9$$

The variance of $\bar{L}$ is

$$\sigma^2 = 0.55(-0.9)^2 + 0.15 + 0.15 + 0.1 + 0.05(1.1)^2$$
$$= 0.99$$

The two Huffman codes described herein have the same average code-word length but different variances.

Problem 10.9



The Huffman code is therefore

s_0	1 0
s_1	1 1
s_2	0 0 1
s_3	0 1 0
s_4	0 1 1
s_5	0 0 0 0
s_6	0 0 0 1

The average code-word length is

$$\begin{aligned}\bar{L} &= \sum_{k=0}^{6} p_k l_k \\ &= 0.25(2)(2) + 0.125(3)(3) + 0.0625(4)(2) \\ &= 2.625\end{aligned}$$

The entropy of the source is

$$\begin{aligned}H(S) &= \sum_{k=0}^{6} p_k \log_2\left(\frac{1}{p_k}\right) \\ &= 0.25(2)\log_2\left(\frac{1}{0.25}\right) + 0.125(3)\log_2\left(\frac{1}{0.125}\right) \\ &\quad + 0.0625(2)\log_2\left(\frac{1}{0.0625}\right) \\ &= 2.625\end{aligned}$$

The efficiency of the code is therefore

$$\eta = \frac{H(S)}{\bar{L}} = \frac{2.625}{2.625} = 1$$

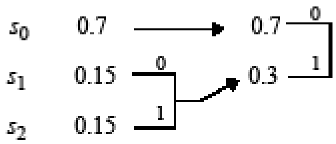
We could have shown that the efficiency of the code is 100% by inspection since

$$\eta = \frac{\sum_{k=0}^{6} p_k \log_2(1/p_k)}{\sum_{k=0}^{6} p_k l_k}$$

where $l_k = \log_2(1/p_k)$.

Problem 10.10

(a)



The Huffman code is therefore

s_0	0
s_1	1 0
s_2	1 1

The average code-word length is

$$\begin{aligned} \bar{L} &= 0.7(1) + 0.15(2) + 0.12(2) \\ &= 1.3 \end{aligned}$$

(b) For the extended source we have

Symbol	s_0s_0	s_0s_1	s_0s_2	s_1s_0	s_2s_0	s_1s_1	s_1s_2	s_2s_1	s_2s_2
Probability	0.49	0.105	0.105	0.105	0.105	0.0225	0.0225	0.0225	0.0225

Applying the Huffman algorithm to the extended source, we obtain the following source code:

s_0s_0	1
s_0s_1	0 0 1
s_0s_2	0 1 0
s_1s_0	0 1 1
s_2s_0	0 0 0 0
s_1s_1	0 0 0 1 0 0
s_1s_2	0 0 0 1 0 1
s_2s_1	0 0 0 1 1 0
s_2s_2	0 0 0 1 1 1

The corresponding value of the average code-word length is

$$\begin{aligned} \bar{L}_2 &= 0.49(1) + 0.105(3)(3) + 0.105(4) + 0.0225(4)(4) \\ &= 2.395 \text{ bits/extended symbol} \end{aligned}$$

$$\frac{\bar{L}_2}{2} = 1.1975 \text{ bits/symbol}$$

(c) The original source has entropy

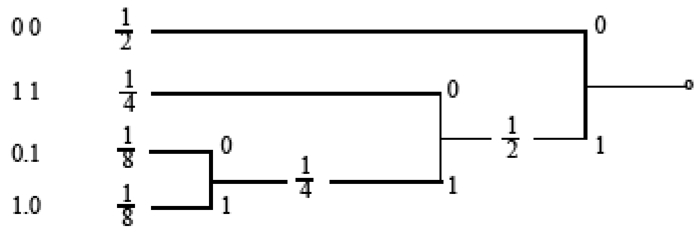
$$\begin{aligned} H(S) &= 0.7\log_2\left(\frac{1}{0.7}\right) + 0.15(2)\log_2\left(\frac{1}{0.15}\right) \\ &= 1.18 \end{aligned}$$

According to Eq. (10.28),

$$H(S) \le \frac{\bar{L}_n}{n} \le H(S) + \frac{1}{n}$$

This is a condition which the extended code satisfies.

Problem 10.11



Computer code	Probability	Huffman code
0 0	$\frac{1}{2}$	0
1 1	$\frac{1}{4}$	1 0
0 1	$\frac{1}{8}$	1 1 0
1 0	$\frac{1}{8}$	1 1 1

The number of bits used for the instrutions based on the computer code, in a probabilistic sense, is equal to

$$2\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{8}\right) = 2 \text{ bits}$$

On the other hand, the number of bits used for instr4uctions based on the Huffman code, is equal to

$$1\times\frac{1}{2}+2\times\frac{1}{4}+3\times\frac{1}{8}+3\times\frac{1}{8} = \frac{7}{4}$$

The percentage reduction in the number of bits used for instruction, realized by adopting the Huffman code, is therefore

$$100\times\frac{1/4}{2} = 12.5 \text{ percent}$$

Problem 10.12

<u>Initial step</u>
Subsequences stored: 0
Data to be parsed: 1 1 1 0 1 0 0 1 1 0 0 0 1 0 1 1 0 1 0 0 ...

<u>Step 1</u>
Subsequences stored: 0, 1, 11
Data to be parsed: 1 0 1 0 0 1 1 0 0 0 1 0 1 1 0 1 0 0 ..

<u>Step 2</u>
Subsequences stored: 0, 1, 11, 10
Data to be parsed: 1 0 0 1 1 0 0 0 1 0 1 1 0 1 0 0 ....

<u>Step 3</u>
Subsequences stored: 0, 1, 11, 10, 100
Data to be parsed: 1 1 0 0 0 1 0 1 1 0 1 0 0 ...

<u>Step 4</u>
Subsequences stored: 0, 1, 11, 10, 100, 110
Data to be parsed: 0 0 1 0 1 1 0 1 0 0 ...

<u>Step 5</u>
Subsequences stored: 0, 1, 11, 10, 100, 110, 00
Data to be parsed: 1 0 1 1 0 1 0 0 ....

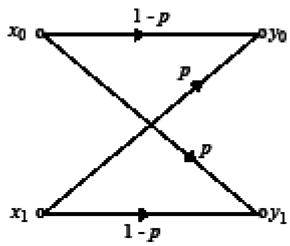
<u>Step 6</u>
Subsequences stored: 0, 1, 11, 10, 100, 110, 00, 101
Data to be parsed: 1 0 1 0 0 ...

<u>Step 7</u>
Subsequences stored: 0, 1, 11, 10, 100, 110, 00, 101, 1010
Data to be parsed: 0 ....

Now that we have done as far as we can go with data parsing for the given sequence, we write

Numerical positions	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
Subsequences	**0,**	**1,**	**11,**	**10,**	**100,**	**110,**	**00,**	**101,**	**1010**
Numerical representations			**22,**	**21,**	**41,**	**31,**	**11,**	**42,**	**81**
Binary encoded blocks			**0101,**	**0100,**	**0100,**	**0110,**	**0010,**	**1001,**	**10000**

Problem 10.13



$$p(x_0) = p(x_1) = \frac{1}{2}$$

$$\begin{aligned} p(y_0) &= (1-p)p(x_0) + p\ \ p(x_1) \\ &= (1-p)\left(\frac{1}{2}\right) + p\left(\frac{1}{2}\right) \\ &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} p(y_1) &= p\ \ p(x_0) + (1-p)p(x_1) \\ &= p\left(\frac{1}{2}\right) + (1-p)\left(\frac{1}{2}\right) \\ &= \frac{1}{2} \end{aligned}$$

Problem 10.14

$$\begin{aligned} p(x_0) &= \frac{1}{4} \\ p(x_1) &= \frac{3}{4} \end{aligned}$$

$$\begin{aligned} p(y_0) &= (1-p)\left(\frac{1}{4}\right) + p\left(\frac{3}{4}\right) \\ &= \frac{1}{4} + \frac{p}{2} \end{aligned}$$

$$\begin{aligned} p(y_i) &= p\left(\frac{1}{4}\right) + (1-p)\left(\frac{3}{4}\right) \\ &= \frac{3}{4} + \frac{p}{2} \end{aligned}$$

Problem 10.15

(a) When each symbol is repeated three times, we have

Messages	Unused signals	Channel outputs
000	001	000
	010	001
	011	010
	100	100
	101	101
	110	110
111		111

We note the following:

1. The probability that no errors occur in the transmission of three 0s or three 1s is $(1 - p)^3$.
2. The probability of just one error occurring is $3p(1 - p)^2$.
3. The probability of two errors occurring is $3p^2(1 - p)$.
4. The probability of receiving all three bits in error is p^3.

With the decision-making based on a majority vote, it is clear that contributions 3 and 4 lead to the probability of error

$$P_3 = 3p^2(1-p)+p^3$$

(b) When each symbol is transmitted five times, we have

Messages	Unused signals	Channel outputs
00000		00000
	00001	00001
	00010	00010
	00011	00011
	⋮	⋮
	11110	11110
11111		11111

The probability of zero, one, two, three, four, or five bit errors in transmission is as follows, respectively:

$(1 - p)^5$
$5p(1 - p)^4$
$10p^2(1 - p)^3$
$10p^3(1 - p)^2$
$5p^4(1 - p)$
p^5

The last three contributions constitute the probability of error

$$P_e = p^5+5p^4(1-p)+10p^3(1-p)^2$$

(c) For the general case of $n = 2m + 1$, we note that the decision-making process (based on a majority vote) makes an error when $m + 1$ bits or more out of the n bits of a message are received in error. The probability of i message bits being received in error is

$$\binom{n}{i} p^i (1-p)^{n-i}$$

Hence, the probability of error is (in general)

$$P_e = \sum_{i=m+1}^{n} \binom{n}{i} p_i (1-p)^{n-i}$$

The results derived in parts (a) and (b) for $m = 1$ and $m = 2$ are special cases of this general formula.

Problem 10.16

(a) Channel bandwidth $B = 3.4$ kHz
Received signal-to-noise ratio SNR = $10^3 \cong 30$ dB

Hence, the channel capacity is

$$\begin{aligned} C &= B\log_2(1 + \text{SNR}) \\ &= 3.4 \times 10^3 \log_2(1 + 10^3) \\ &= 33.9 \times 10^3 \text{ bits/second} \end{aligned}$$

(b) $4800 = 3.4 \times 10^3 \log_2(1 + \text{SNR})$

Solving for the unknown SNR, we get
SNR = $1.66 \cong 2.2$ dB

Problem 10.17

With 10 distinct brightness levels with equal probability, the information in each level is $\log_2 10$ bits. With each picture frame containing 3×10^5 elements, the information content of each picture frame is $3 \times 10^5 \log_2 10$ bits. Thus, a rate of information transmission of 30 frames per second corresponds to

$$30 \times 3 \times 10^5 \log_2 10 = 9 \times 10^6 \log_2 10 \text{ bits/second}$$

That is, the channel capacity is

$$C = 9 \times 10^6 \log_2 10 \text{ bits/second}$$

From the information capacity theorem:

$$C = B\log_2(1 + \text{SNR})$$

With a signal-to-noise ratio SNR = 10^3 = 30 dB, the channel bandwidth is therefore

$$\begin{aligned} B &= \frac{C}{\log_2(1 + \text{SNR})} \\ &= \frac{9 \times 10^6 \log_2 10}{\log_2 1001} \\ &= 3 \times 10^3 \text{Hz} \end{aligned}$$

Problem 10.18

Message Sequence	Single-paritypcheck code
0 0 0	0 0 0 0
0 0 1	0 0 1 1
0 1 0	0 1 0 1
0 1 1	0 1 1 0
1 0 0	1 0 0 1
1 0 1	1 0 1 0
1 1 0	1 1 0 0
1 1 1	1 1 1 1

Problem 10.19

For the (4,1) repetition code, the parity check matrix is

$$\mathbf{H} = \left[\begin{array}{ccc:c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

For a (7,4) Hamming code, we have

$$\mathbf{H} = \left[\begin{array}{ccc:cccc} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{array}\right]$$

For the Hamming code, the parity check matrix $\mathbf{H}$ is more structured than that for the repetition code. Indeed, the matrix $\mathbf{H}$ for the Hamming code includes that for the repetition code as a submatrix.

Problem 10.20

The generator matrix for the (7,4) Hamming code is

$$\mathbf{G} = \left[\begin{array}{ccc|cccc} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

The parity-check matrix is

$$\mathbf{H} = \left[\begin{array}{ccc|cccc} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{array}\right]$$

Hence,

$$\mathbf{HG}^T = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \text{mod-2}$$

Problem 10.21

(a) Viewing the matrix

$$\mathbf{H} = \left[\begin{array}{ccc:cccc} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{array}\right]$$

as a generator matrix, we may define the code vector **c** in terms of the message vector **m** as

$$\mathbf{c} = \mathbf{m}\,\mathbf{H}$$

The message word length is

$$n - k = 7\text{-}4 = 3$$

Hence, we may construct the following table

Message word	Code word	Hamming weight
0 0 0	0 0 0 0 0 0 0	0
0 0 1	0 0 1 0 1 1 1	4
0 1 0	0 1 0 1 1 1 0	4
0 1 1	0 1 1 1 0 0 1	4
1 0 0	1 0 0 1 0 1 1	4
1 0 1	1 0 1 1 1 0 0	4
1 1 0	1 1 0 0 1 0 1	4
1 1 1	1 1 1 0 0 1 0	5

(b) The minimum value of the Hamming weight defines the Hamming distance of the dual code as

$$d_{\min} = 4$$

Problem 10.22

(a) For a (5,1) repetition code:

$$\mathbf{G} = \begin{bmatrix} 1\ 1\ 1\ 1\ 1 & \vdots & 1 \end{bmatrix}$$

$$\mathbf{H} = \begin{bmatrix} 1 & 0 & 0 & 0 & \vdots & 1 \\ 0 & 1 & 0 & 0 & \vdots & 1 \\ 0 & 0 & 1 & 0 & \vdots & 1 \\ 0 & 0 & 0 & 1 & \vdots & 1 \end{bmatrix}$$

$$\mathbf{H}^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

The syndrome is

$$\mathbf{s} = \mathbf{e}\,\mathbf{H}^T$$

where $\mathbf{e}$ is the error pattern. For a single error, we thus have

Error pattern	Syndrome
0 0 0 0 1	1 1 1 1
0 0 0 1 0	0 0 0 1
0 0 1 0 0	0 0 1 0
0 1 0 0 0	0 1 0 0
1 0 0 0 0	1 0 0 0

(b) For two errors in the received word, we have

Error pattern	Syndrome
0 0 0 1 1	1 1 1 0
0 0 1 0 1	1 1 0 1
0 1 0 0 1	1 0 1 1
1 0 0 0 1	0 1 1 1
0 0 1 1 0	0 0 1 1
0 1 0 1 0	0 1 0 1
1 0 0 1 0	1 0 0 1
0 1 1 0 0	0 1 1 0
1 0 1 0 0	1 0 1 0
1 1 0 0 0	1 1 0 0

We note that the syndromes for all single-error and double-error patterns are distinct. This is intuitively satisfying since a (5,1) repetition code is capable of correcting up to two errors in the received vector

$$\mathbf{y} = \mathbf{e} + \mathbf{c}$$

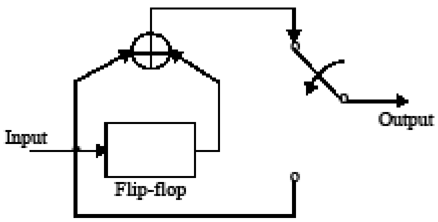
Problem 10.23

The encoder is realized by inspection:

$g^{(1)} = (1,0,1)$
$g^{(2)} = (1,1,0)$
$g^{(3)} = (1,1,1)$

For the Hamming code, the parity check matrix **H** is more structured than that for the repetition code. Indeed, the matrix **H** for the Hamming code includes that for the repetition code as a submatrix.

Problem 10.24



Using this encoder, we may construct the following table by inspection:

Message	1	0	1	1	1	1	...
Output	11	10	11	01	01	01	...

Original message

The code is in fact symstematic.

Problem 10.25

The generator polynomials are

$$g^{(1)}(X) = 1 + X + X^2 + X^3$$
$$g^{(2)}(X) = 1 + X + X^3$$

The message polynomial is

$$m(X) = 1 + X^2 + X^3 + X^4 + \dots$$

Hence,

$$\begin{aligned} c^{(1)}(X) &= g^{(1)}(X)m(X) \\ &= 1 + X + X^3 + X^4 + X^5 + \dots \end{aligned}$$

$$\begin{aligned} c^{(2)}(X) &= g^{(2)}(X)m(X) \\ &= 1 + X + X^2 + X^3 + X^6 + X^7 + \dots \end{aligned}$$

Hence,

$$\{c^{(1)}\} = 1, 1, 0, 1, 1, 1, \dots$$
$$\{c^{(2)}\} = 1, 1, 1, 1, 0, 0, \dots$$

The encoder output is therefore 11, 11, 01, 11, 10, 10.

Problem 10.26

The encoder of Fig. 10.25 (b) has three generator sequences for each of the two input paths; they are as follows (from top to bottom)

$$g_1^{(1)} = (1, 1), \quad g_1^{(2)} = (1, 0), \quad g_1^{(3)} = (1, 1)$$
$$g_2^{(1)} = (0, 1), \quad g_2^{(2)} = (1, 1), \quad g_2^{(3)} = (0, 0)$$

Hence

$$g_1^{(1)}(X) = 1 + X, \quad g_1^{(2)}(X) = 1, \quad g_1^{(3)}(X) = 1 + X$$
$$g_2^{(1)}(X) = X, \quad g_2^{(2)}(X) = 1 + X, \quad g_2^{(3)}(X) = 0$$

The incoming message sequence 10111... enters the encoder two bits at a time; hence

$$m^{(1)} = 1 \;\; 1 \ldots$$
$$m^{(2)} = 0 \;\; 1 \ldots$$

The message polynomials are therefore

$$m_1(X) = 1 + X + \ldots$$
$$m_2(X) = X + \ldots$$

Hence, the output polynomials are

$$\begin{aligned} c^{(1)}(X) &= g_1^{(1)}(X)m_1(X) + g_2^{(1)}(X)m_2(X) \\ &= (1 + X)(1 + X + \ldots) + X(X + \ldots) \\ &= 1 + \ldots \end{aligned}$$

$$\begin{aligned} c^{(2)}(X) &= g_1^{(2)}(X)m_1(X) + g_2^{(2)}(X)m_2(X) \\ &= (1)(1 + X + \ldots) + (1 + X)(X + \ldots) \\ &= 1 + X^2 + \ldots \end{aligned}$$

$$\begin{aligned} c^{(3)}(X) &= g_1^{(3)}(X)m_1(X) + g_2^{(3)}(X)m_2(X) \\ &= (1 + X)(1 + X + \ldots) + (0)(X + \ldots) \\ &= 1 + X^2 + \ldots \end{aligned}$$

The output sequences are correspondingly as follows:

$$c^{(1)} = 1,0, \ldots$$
$$c^{(2)} = 1,0, \ldots$$
$$c^{(3)} = 1,0, \ldots$$

The encoder output is therefore (1,1,1), (0,0,0), ...

Problem 10.27

(a) Coding gain for binary symmetric channel is

$$G_a = 10\log_2\left(\frac{10 \times 1/2}{2}\right)$$
$$= 10\log_{10}2.5$$
$$= 4 \text{ dB}$$

(b) Coding gain for additive white Gaussian noise channel is

$$G_a = 10\log_{10}\left(10 \times \frac{1}{2}\right)$$
$$= 10\log_{10}5$$
$$= 7 \text{ dB}$$

Problem 10.28

Let the code rate of turbo code be R. We can write

$$\left(\frac{1}{R} - 1\right) = \left(\frac{1}{r_c^{(1)}} - 1\right) + \left(\frac{1}{r_c^{(2)}} - 1\right)$$

$$\frac{1}{R} = \left(\frac{1}{r_c^{(1)}}\right) + \left(\frac{1}{r_c^{(2)}} - 1\right)$$

$$= \left(\frac{q_1}{p} + \frac{q_2}{p} - 1\right)$$

$$= \frac{q_1 + q_2 - p}{p}$$

Hence

$$R = p/(q_1 + q_2 - p)$$

Problem 10.29

Figure 1 is a reproduction of the 8-state RSC encoder of Figure 10.34 used as encoder 1 and encoder 2 in the turbo encoder of Fig. 10.25 of the textbook. For an input sequence consisting of symbol 1 followed by an infinite number of symbols 0, the outputs of the RSC encoders will contain an infinite number of ones as shown in Table 1.

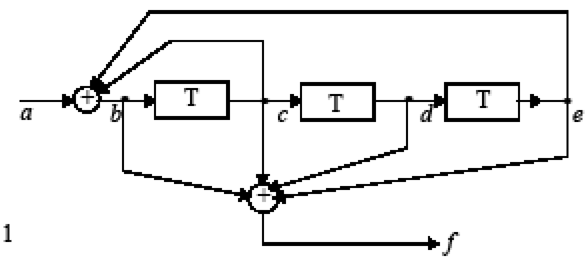


Fig. 1

$b = a \oplus c \oplus e$

$f = b \oplus c \oplus d \oplus e$

Initial conditions: $c = d = e = 0$ {empty}

(Input)	Intermediate inputs				(output)
a	b	c	d	e	f
1	1	0	0	0	1
0	1	1	0	0	0
0	1	1	1	0	1
0	0	1	1	1	1
0	1	0	1	1	1
0	0	1	0	1	0
0	0	0	1	0	1
0	1	0	0	1	0
0	1	1	0	0	0

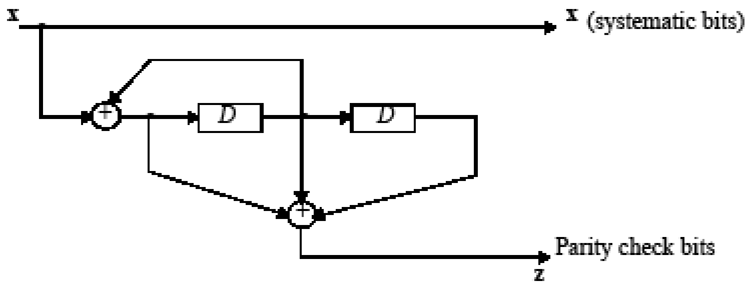
The output is 1011101001110100111...

Therefore, an all zero sequence with a single bit error (1) will cause an infinite number of channel errors.

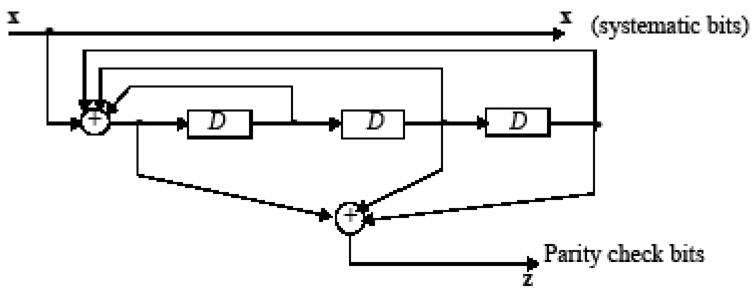
[Note: The all zero input sequence produces an all zero output sequence.]

Problem 10.30

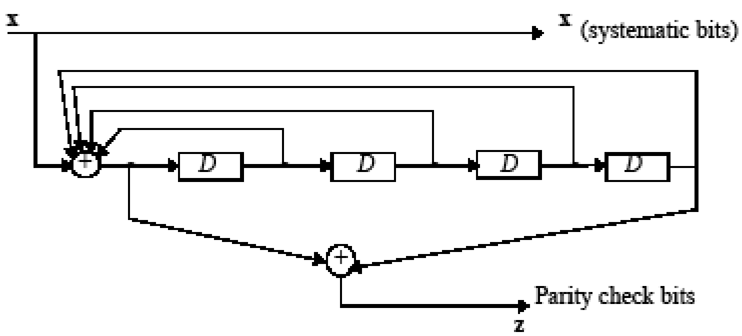
(a) 4-state encoder



8-state encoder



16-state encoder



(b) 4-state encoder

$$\mathbf{g}(D) = \left[1, \frac{1+D+D^2}{1+D^2}\right]$$

By definition, we have

$$\left(\frac{B(D)}{M(D)}\right) = \frac{1+D+D^2}{1+D^2}$$

where $B(D)$ denotes the transform of the parity sequence $\{b_i\}$ and $M(D)$ denotes the transform of the message sequence $\{m_i\}$. Hence,

$$(1+D^2)B(D) = (1+D+D^2)M(D)$$

The parity-check equation is given by

$$(m_i + m_{i-1} + m_{i-2}) + (b_i + b_{i-2}) = 0$$

where the addition is modulo-2.

Similarly for the 8-state encoder, we find that the parity-check equation is

$$m_i + m_{i-2} + m_{i-3} + b_i + b_{i-1} + b_{i-2} + b_{i-3} = 0$$

For the 16-state encoder, the parity-check equation is

$$m_i + m_{i-4} + b_i + b_{i-1} + b_{i-2} + b_{i-3} + b_{i-4} = 0$$