بسم الله الرحمن الرحيم

تلخيص مادة أشارات وأنظمة

الفصل الدراسي الاول 2021\2020

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Signals & Systems

Lecture #1
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Chapter 1

Introduction

* physical systems in real life can be modeled by mathematical equations.

Example :- Circuit

* Physical signals can be modeled as mathematical functions.

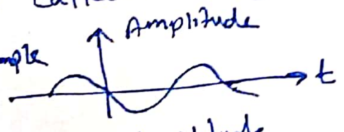
Example :- voltage that applied for any circuit.

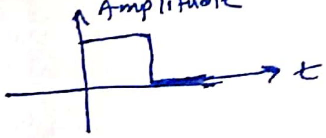
* A signal might change based on different parameters, i.e., time, freq, distance, ..., etc.

* We choose this independent parameter to be time

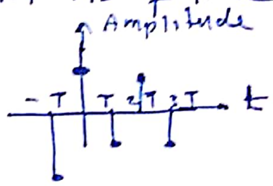
Time signals

* Continuous-time signals
  - signal defined for all values of time
  - is called analoge signal.
  - Example





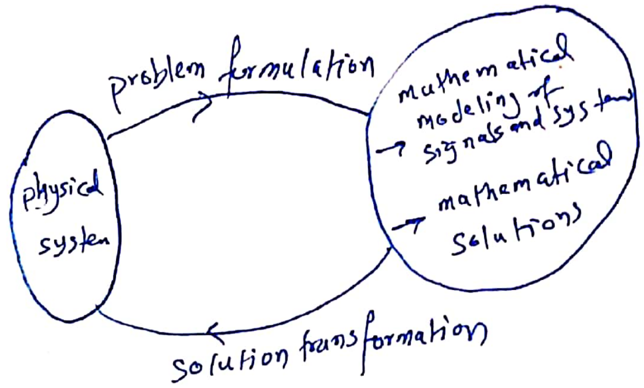
discrete-time signals
- defined at only certain instants of time
-



* In this course, we will learn Continous-time signals

* Discrete-time signals will be studied in Digital signal processing (DSP)

• Mathematical analysis of physical systems can be represented as following :-



* In this course, we will learn how to deal with mathematical signals and systems.

End of chapter 1.

2/2/2020

Chapter (2) :- Continous time signals and system.

(2:1) :- transformations of signals.

time transformations — amplitude transformation

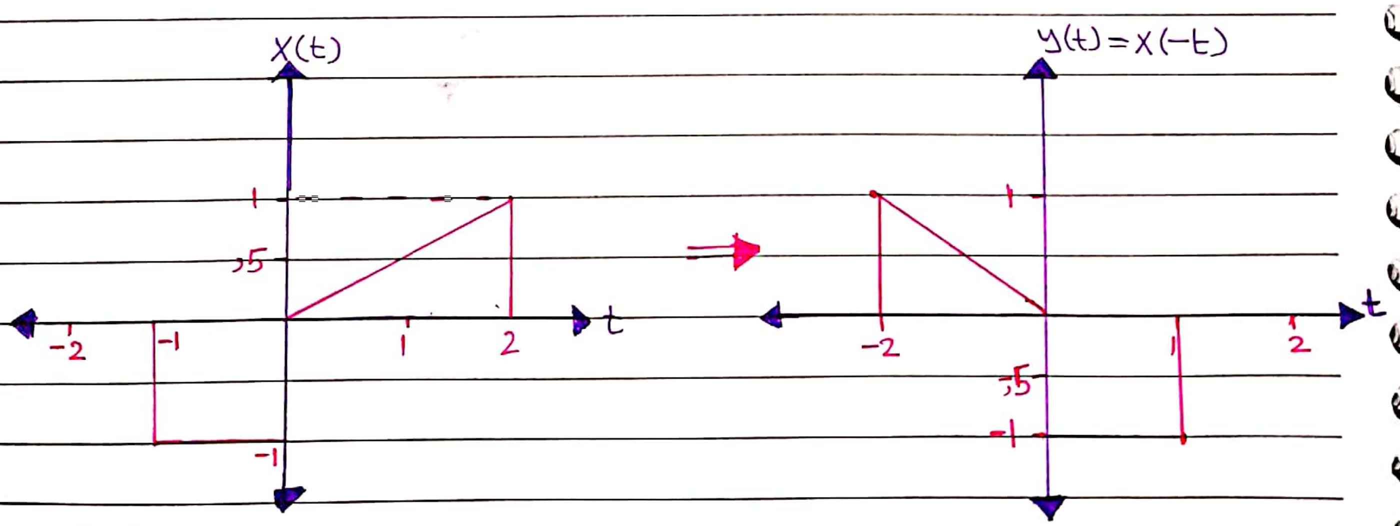
time reversal — timescaling — time-shift

[1] Time transformations :-

a) time reversal, given a signals $x(t)$, $y(t) = x(-t)$, $y(t)$ is the image mirror of the original signals $x(t)$.

b) time scaling, $y(t) = x(at)$, $|a| > 1 \rightarrow$ time compression (sped-up)

$|a| < 1 \rightarrow$ time stretched (slow-down)



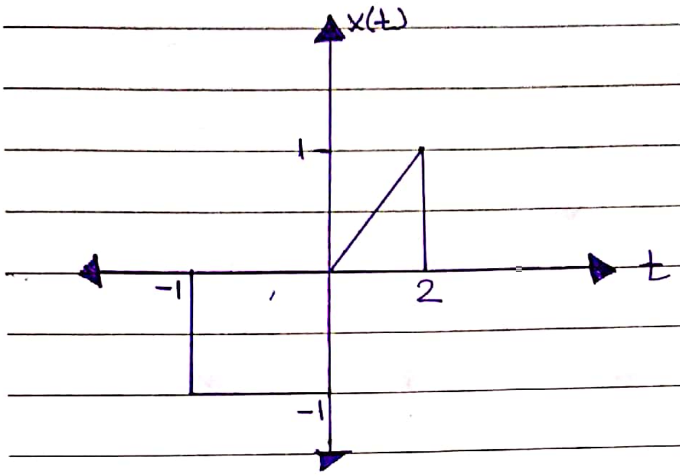
$y(t) = x(-t)$

$y(2) = x(-2) = 0$

$y(1) = x(-1) = -1$

$y(-2) = x(2) = 1$

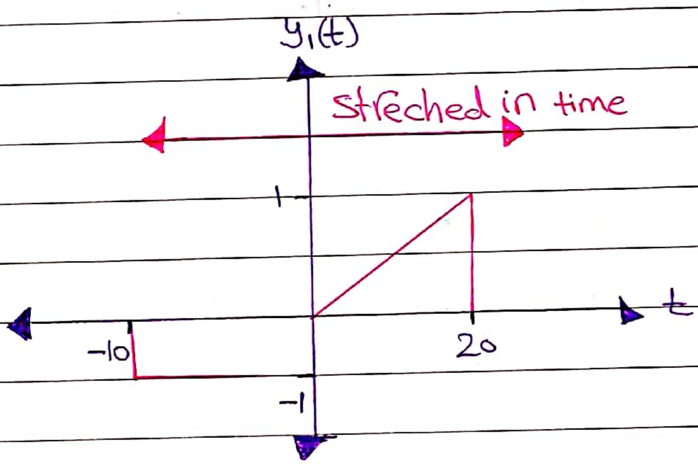
example :- For the given $x(t)$, Plot $y_1(t) = x(.1\,t)$ & $y_2(t) = x(2t)$.



$y_1(-10) = x(-1) = -1$

$y_1(0) = x(0) = 0$

$y_1(20) = x(2) = 1$

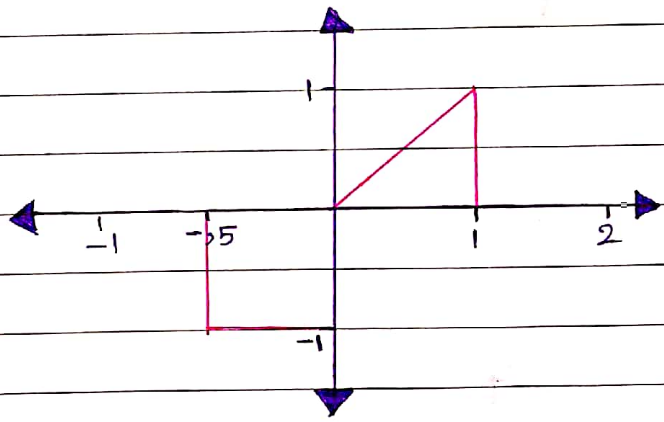


$y_2(t) = x(2t)$

$y_2(-.5) = x(-1) = -1$

$y_2(0) = x(0) = 0$

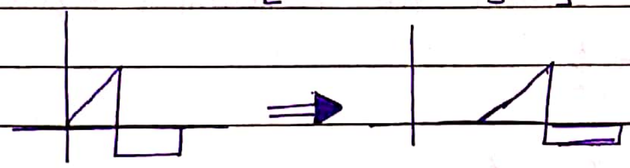
$y_2(1) = x(2) = 1$



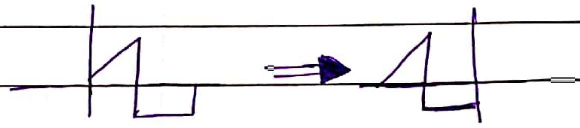
c) time shift.

$y(t) = x(t - t_0)$

t_0 : Positive [shift to right] → [delay in time]



t_0 : Negative [Shift to left] → [advanced]



example :- plot $y_3(t) = x(t-3)$.

$y_3(t_0) = x(t_0 - 3)$

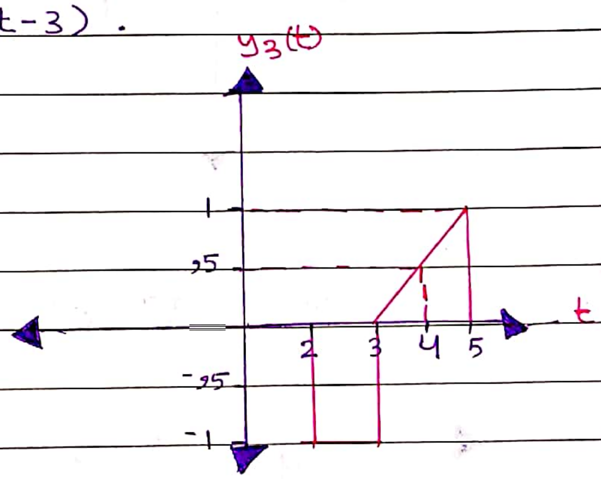
$y(-1) = x(-4) = 0$

$y(1) = x(-3) = 0$

$y(2) = x(-1) = -1$

$y(3) = x(0) = 0$

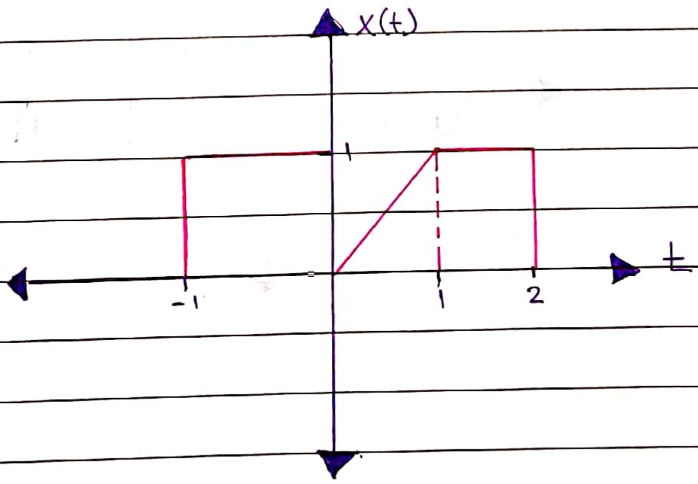
$y(4) = x(1) = .5$



try $y_4(t) = x(t+3)$.

In general :- $y(t) = x(at+b)$.

example :- given $x(t)$ as show below, Plot $y(t) = x(1 - \frac{t}{2})$.

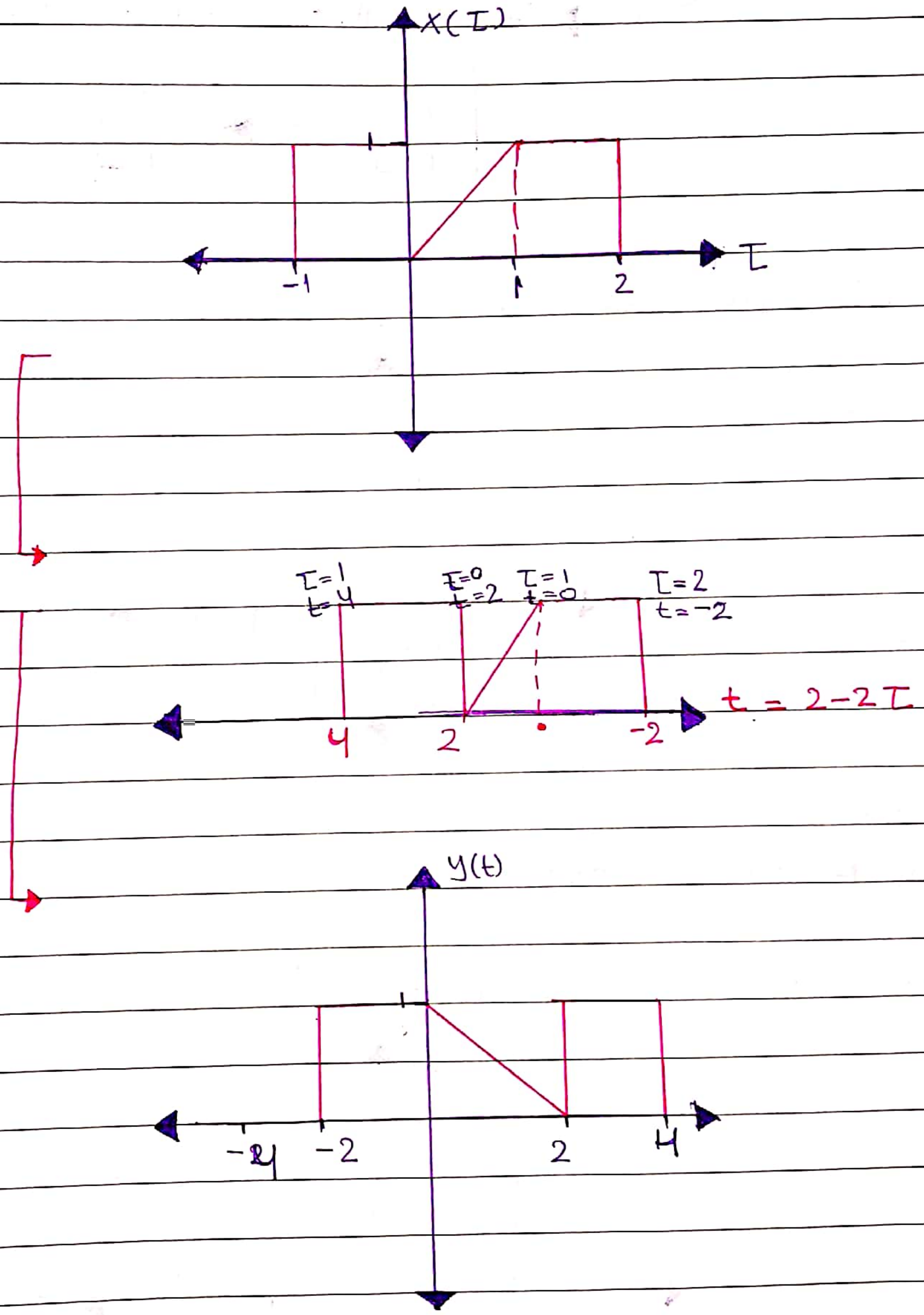


① replace τ insted of t in the original signal.

② write down the following equation.

$$\tau = 1 - \frac{t}{2}$$

$$\frac{t}{2} = 1 - \tau \rightarrow t = 2 - 2\tau .$$



2) Amplitude transformation:

In general :- $y(t) = \underset{\pm ve}{A}\,x(t) + \underset{shift}{B}$

if A is (−) → Amplitude Reversal.

if A > 1 → Amplitude increasing.

4/2/2020

(2.1):- Signals transformations.

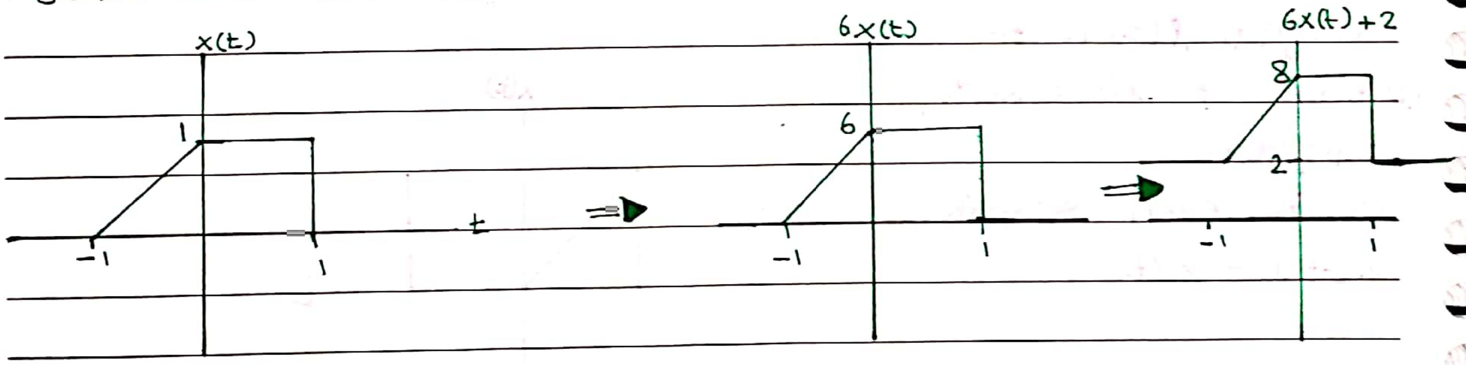
- time transformations.
- amplitude transformations.

$y(t) = A\,x(t) + B$

$y(t) = A\,x(at + Bt_0) + B$.

plot :-

$y(t) = 6\,x(t) + 2$.



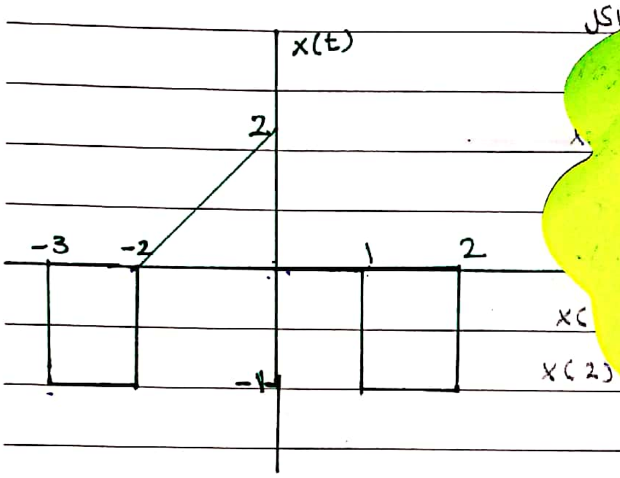
* time transformation + Amplitude . transformation.

$y(t) = Ax(at+B) + B$

↙ amplitude transformation.

↙ time transformation.

find $y(t) = -2x(-2t+3)-2$?



Section (2.2):- Charctorstics of signals.

1. even and odd signals.
2. Poriodic signals.

* even and odd signals :-

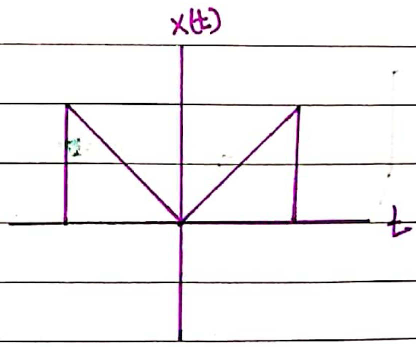
asignal $x(t)$ is :-

(1) even if $x(t) = x(-t)$.

example :-

$x(t)$ is even signals

$x(-t) = x(t)$

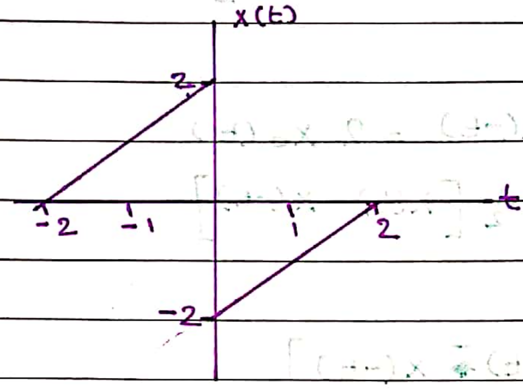


المثال هو الرسمة والجواب ↑

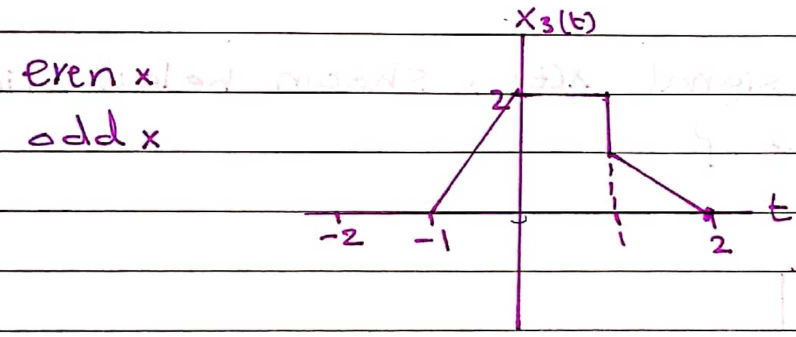
ملاحظة :- odd = 0
part

(2) odd if $x(t) = -x(\bar{t})$

example:- $x_2(t)$ is odd signal, $x_2(-t) = -x(\bar{t})$.



(3) niether odd or even.



* IN general, any signal $x(t)$ can be written as a summation of even and odd part.

$x(t) = \underbrace{x_e(t)}_{\text{even part}} + \underbrace{x_o(t)}_{\text{odd part}}$ ---- ①

$x(-t) = x_e(-t) + x_o(-t)$ .... ②

even:- $x_e(-t) = x_e(t)$.

odd:- $x_o(-t) = -x_o(t)$.

$x(-t) = x_e(t) - x_o(t)$ .... ③

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①+③ :- $x(t) + x(-t) = 2x_e(t)$

even part $x_e(t) = \frac{1}{2}\left[x(t) + x(-t)\right]$

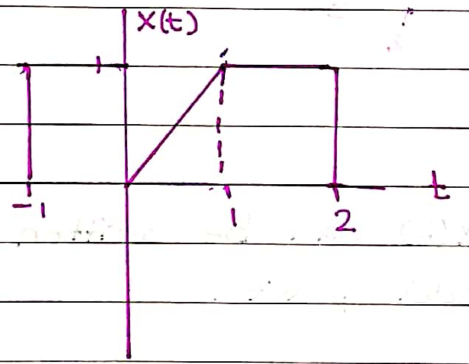
①−③ :- $x(t) - x(-t) = 2x_o(t)$

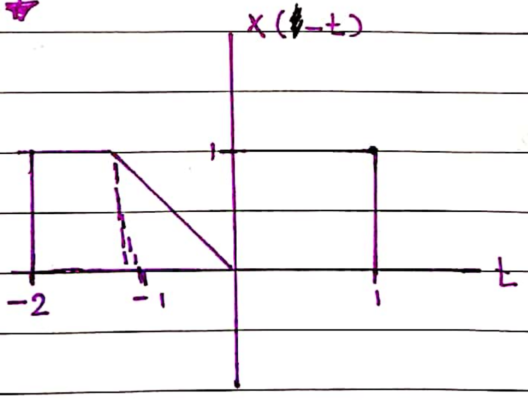
odd part $x_o(t) = \frac{1}{2}\left[x(t) - x(-t)\right]$

✓ $x_o(t) = \frac{1}{2}\left[x(t) - x(-t)\right]$

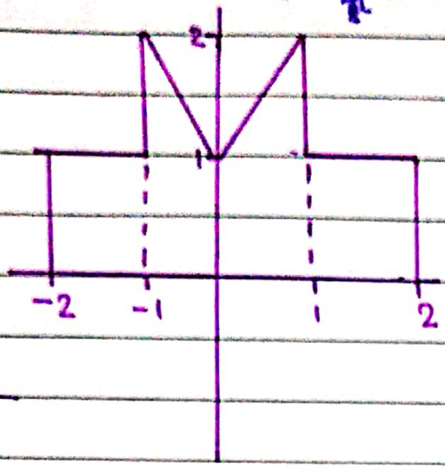
✓ $x_e(t) = \frac{1}{2}\left[x(t) + x(-t)\right]$

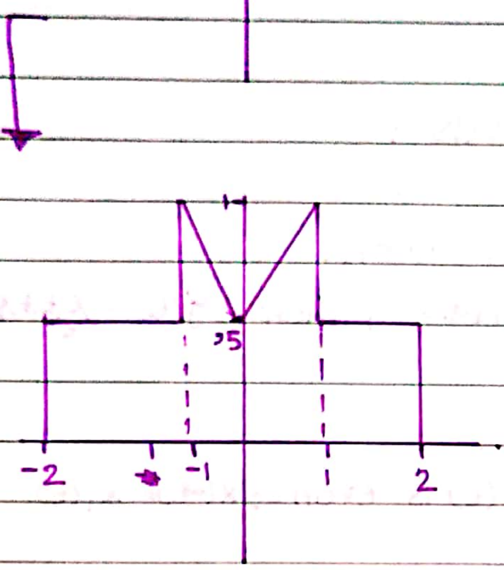
example:- given the signal $x(t)$, shown below, find $x_e(t)$, $x_o(t)$?





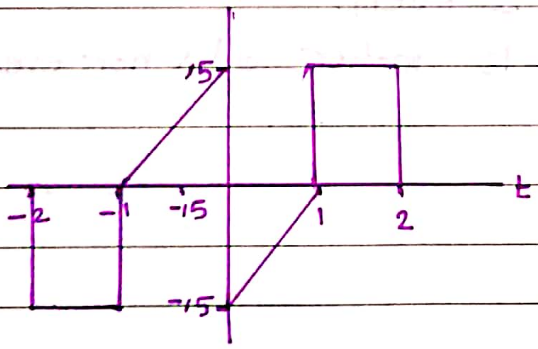
$x_e(t) = \frac{1}{2}[x(t)+x(-t)]$





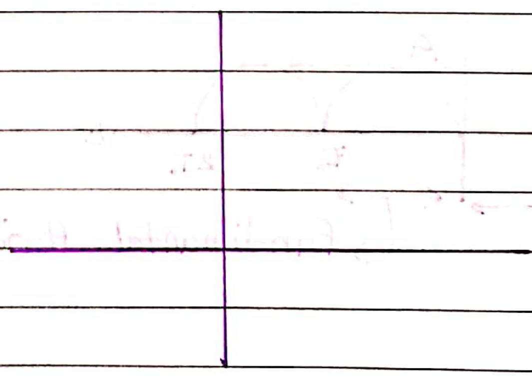
even part of x(t)

$x_o(t) = \frac{1}{2}(x(t) - x(-t))$



$$x(t) = \begin{cases} 0 & , t<-1 \\ 1 & , -1 \le t < 0 \end{cases}$$

$$x(-t) = \begin{cases} 0 & , t<-2 \\ 1 & , -2<t<-1 \\ -t & , -1<t<0 \end{cases}$$



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(2.2) :- Characteristic of Signal.

✓ even and odd signal.

$x_e(t) = \frac{1}{2}[x(t) + x(-t)]$.

$x_o(t) = \frac{1}{2}[x(t) - x(-t)]$.

$x(t) = x_e(t) + x_o(t)$.

* Properties of even and odd signals :-

1- the summation of even signals is even.
(if $x_e(t)$ and $x_e(t)$ are even signals, then $y(t) = x_e(t) + x_e(t)$ is even signals).

$y(t) = y(-t)$ only if $y(t)$ is even, $x(t)$ is even $\rightarrow x(-t) = x_1(t)$

$y(-t) = \underbrace{x_1(-t)}_{} + x_2(-t)$

$x_1(t) + x_2(t)$.

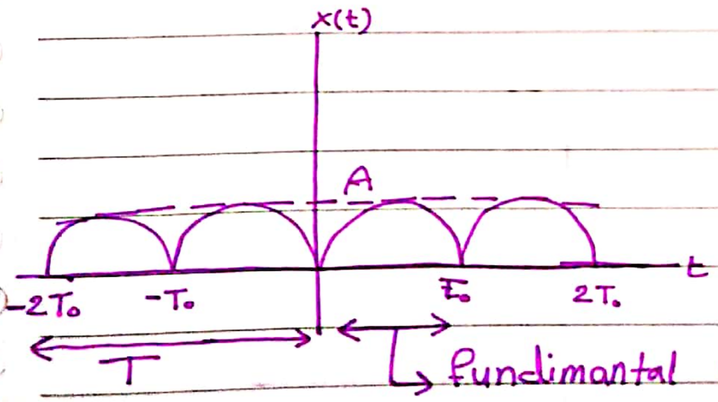
2- the summation of odd signals is odd.

3- the product of even signals is even.

4- the product of odd signals is even.

5- the ~~product~~ summation of even and odd is niether odd nor even.

(2.2) II Periodic signals.

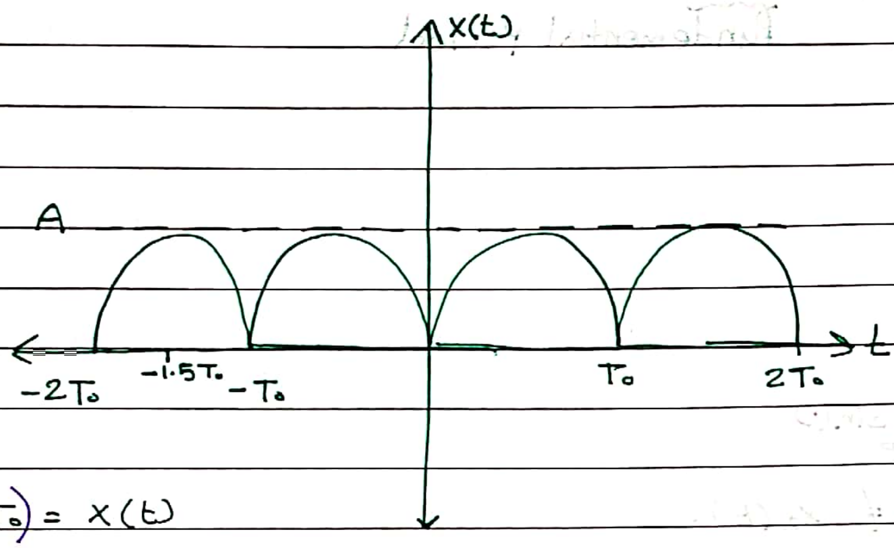


↳ Fundimantal Period a continuous signal.

$x(t)$ is Periodic if :- $x(t) = x(t + nT_o)$

n :- integer

T_o :- fundemantal period.



$x(t + nT_o) = x(t)$

fundemental period :-

the menimum time interval, that the signal replace it self.

* fundemental frequancy :- the number of cycles per time second.

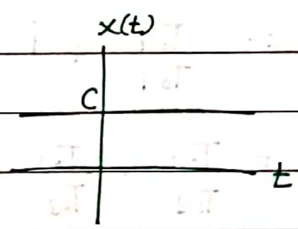
$f_o = \frac{1}{T_o}$ Hertz

* angular frequancy $(\omega_o) = 2\pi f_o = \frac{2\pi}{T_o}$

* أمثلة periodic signals :- * $\cos(\omega_o t)$

* $\sin(\omega_o t)$

* constant.



* Fundementle Period is undefined.

C : constant.

example:- Cheack the following signals are periodic or not?

$x_1(t) = e^{\sin(t)}$ periodic

$\sin(t) = \sin(t) + \underbrace{n T_0}_{\text{fundemental period}})$

$\omega_0 = 1$

$2\pi F_0 = \omega_0$

$F_0 = \frac{1}{2\pi}$

$T_0 = 2\pi$

$x_2(t) = t\, e^{\sin(t)}$

$x_2(t + n T_0) \neq x_2(t)$.

* summation of periodic signals:-
- given three periodic signals, namely $x_1(t)$, $x_2(t)$ and $x_3(t)$, $y(t) = x_1(t) + x_2(t) + x_3(t)$.

1- how ~~way~~ to check whether $y(t)$ is periodic or not.
2- if $y(t)$ is periodic → find the fundemental period.

How to ~~~~ check if periodic or not?
1) the ratio between fundemental period of $x_1(t)$ to the other signals.

ratio :- $\frac{T_{0\,1}}{T_{0\,i}}$

$= \frac{T_{0\,1}}{T_{0\,2}}, \frac{T_{0\,1}}{T_{0\,3}}, \ldots, \frac{T_{0\,1}}{T_{0\,i}}$

if the ratios consists of integers, then $y(t)$ is periodic.

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② How to find the fundemental period of y(t)?

$T_{oy} = K T_{o1}$

K: least common of the denomenators.

$T = \frac{\alpha}{\beta}$ ← numerator, ↳ denomenotor

example:- $\frac{T_{o1}}{T_{o3}} = \frac{16}{4} = 4$.

example:- given $y(t) = x_1(t) + x_2(t) + x_3(t)$, $x_1(t) = \cos(3.5t)$
$x_2(t) = \cos(2t)$
$x_3(t) = \cos(\frac{7}{6}t)$

verfiy wether y(t) is periodic or not, if yes, find the fundemantal periodic.

Solution:-

* $2\pi f_o = \frac{2\pi}{T_{o1}}$

* $\frac{2\pi}{T_{o1}} = 3.5$, $T_{o1} = \frac{2\pi}{3.5} = \frac{4\pi}{7}$

* $\frac{2\pi}{T_{o2}} = 2$, $T_{o2} = \pi$

* $\frac{2\pi}{T_{o3}} = \frac{7}{6}$, $T_{o3} = \frac{12\pi}{7}$

* $\frac{T_{o1}}{T_{o2}} = \frac{4\pi/7}{\pi} = \frac{4}{7}$ (ratio of intger).

* $\frac{T_{o1}}{T_{o3}} = \frac{\frac{4\pi}{7}}{\frac{12\pi}{7}} = \frac{4}{12} = \frac{1}{3}$ (integar).

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$y(t)$ is Periodic signal

$\frac{4}{7} \longleftrightarrow \frac{1}{3}$

K = least common multiple $= 7 * 3 = 21$.

the fundemental Periode of $y(t)$ is $T_{0y} = 21 * \frac{4\pi}{7} = 12\pi$ second.

* For the same signal we add $x_4(t)$.

$y(t) = x_1(t) + x_2(t) + x_3(t) + x_4(t)$.

$x_4(t) = 3 \sin(5\pi t)$.

$5\pi = \frac{2\pi}{T_{04}}$, $T_{0y} = \frac{2}{5}$.

$$\frac{T_{01}}{T_{04}} = \frac{\frac{4\pi}{7}}{\frac{2}{5}} = \frac{10\pi}{7}$$

, then $y(t)$ is non periodic signal , because $x_4(t)$ is not integar.

[2.4] :- singularity function.

def: the functions that are extracted from the impulse function.

① unit step function $u(t)$.

بوديّنا عل → rectangular function , $\text{rect}\left(\frac{t}{T_0}\right)$.

* unit Step function $(u(t))$.

$$u(t) = \begin{cases} 1 & , t > 0 \\ 0 & , t < 0 \end{cases}$$

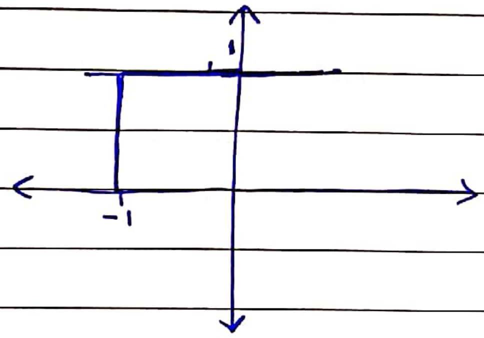
in general :-

$$u(\quad) = \begin{cases} 1 & , (\ ) > 0 \\ 0 & , (\ ) < 0 \end{cases} \qquad \alpha u(\quad) = \begin{cases} \alpha & , (\quad) > 0 \\ 0 & , (\quad) < 0 \\ \text{undefind} & , t = 0 \end{cases}$$

example :- Plot $u(2t+2)$.

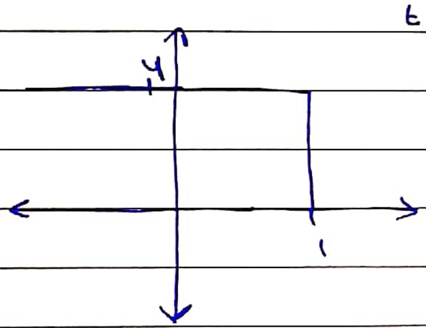
$$u(2t+2) = \begin{cases} 1, & 2t+2>0 \\ 0, & 2t+2<0 \end{cases}$$

$$= \begin{cases} 1, & t>-1 \\ 0, & t<-1 \end{cases}$$



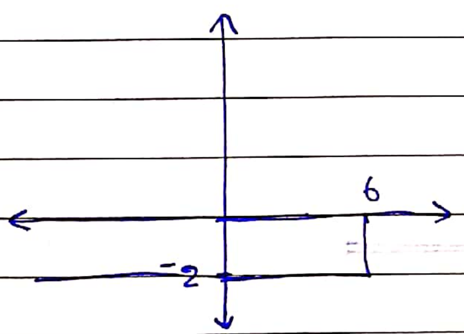
* plot $u(-t+1) \Rightarrow -t+1>0$

$t<1$



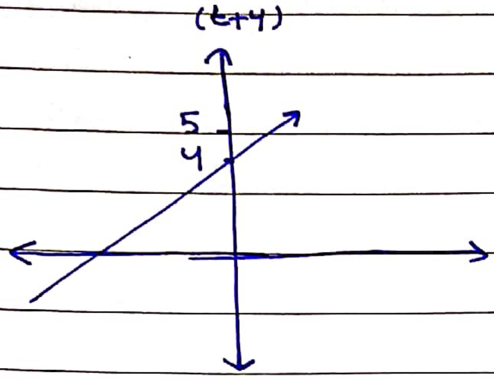
* plot $-2u(\frac{-t}{2}+3) \Rightarrow \frac{-t}{2}+3>0$

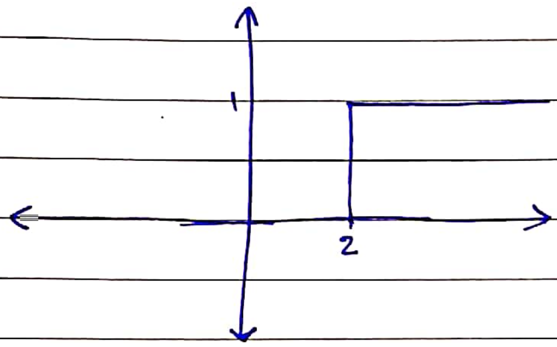
$t<0$

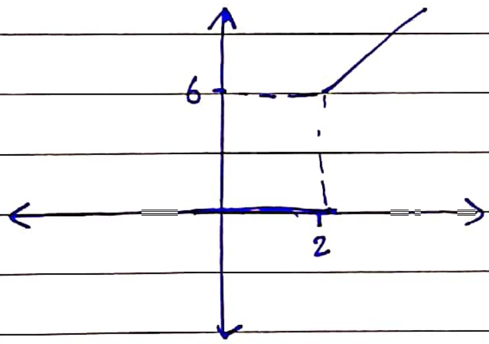


if $y(t) = (t+4)\,u(\frac{t}{2}-1)$, $\frac{t}{2} - 1 > 0$

$t > 2$







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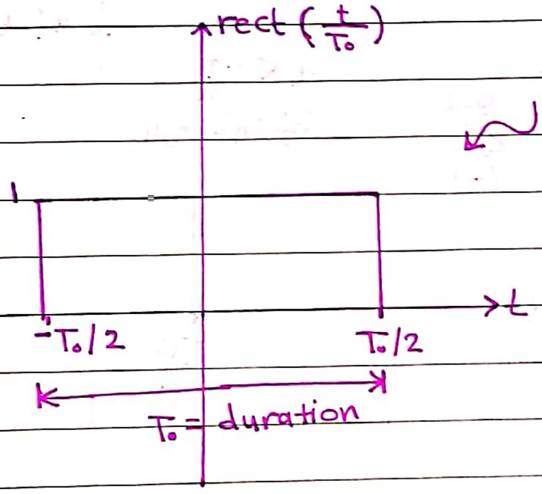
(2.4) :- singulating function :-

1) unit step function $u(t) = \begin{cases} 1 & , t>0 \\ 0 & , t<0 \end{cases}$

rectangular function rect(-1)

2) unit Impulse function .

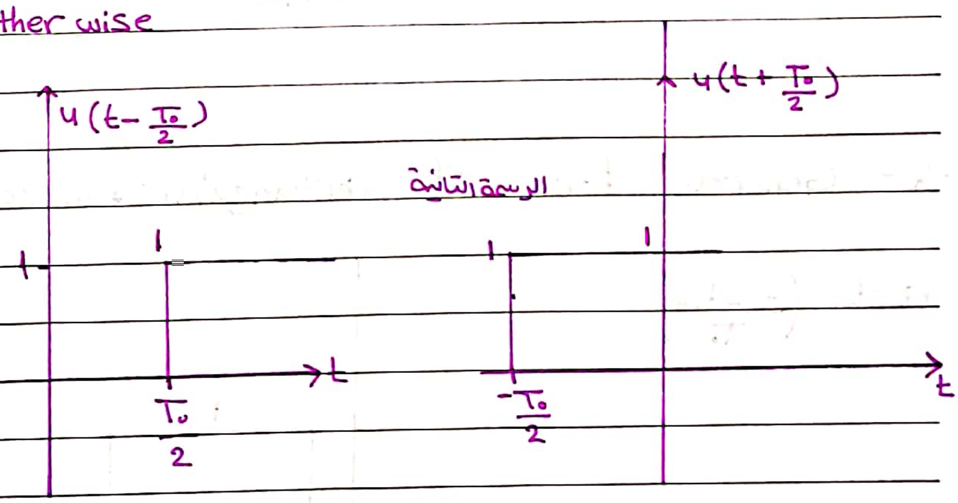
* rectangular function rect ( ) .



$$rect\left(\frac{t}{T_0}\right) = \begin{cases} 1 & , -\frac{T}{2} < t < \frac{T}{2} \\ 0 & , \text{other wise} \end{cases}$$

الرسمة الأولى

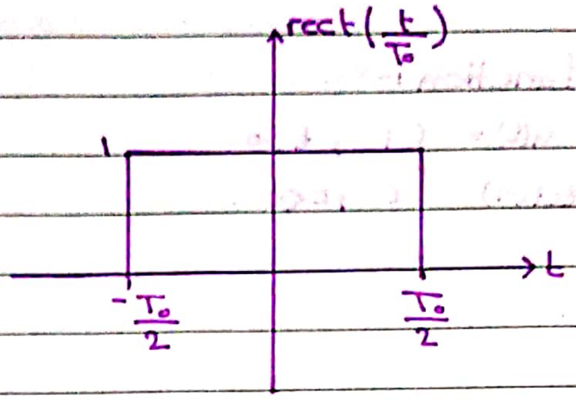
الرسمة التانية



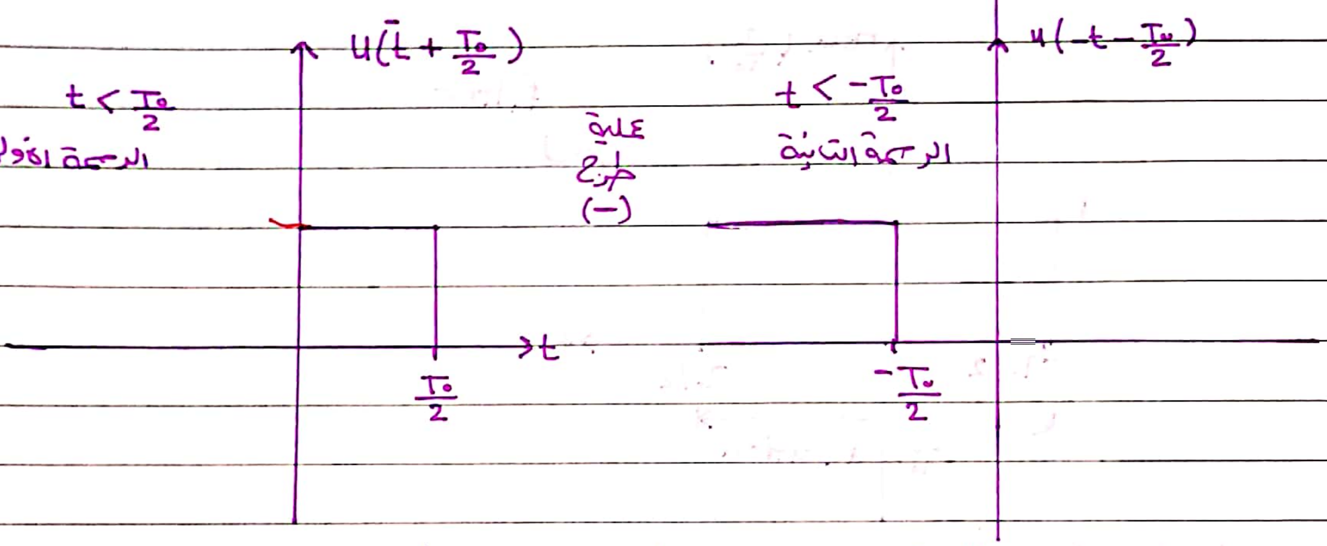
لو طرحت الرسمة الأولى من الرسمة التانية ، ع يطلعلي رسمة rect .

$$u\left(t - \frac{T_0}{2}\right) - u\left(t + \frac{T_0}{2}\right)$$

بتعيد الرسمة ←



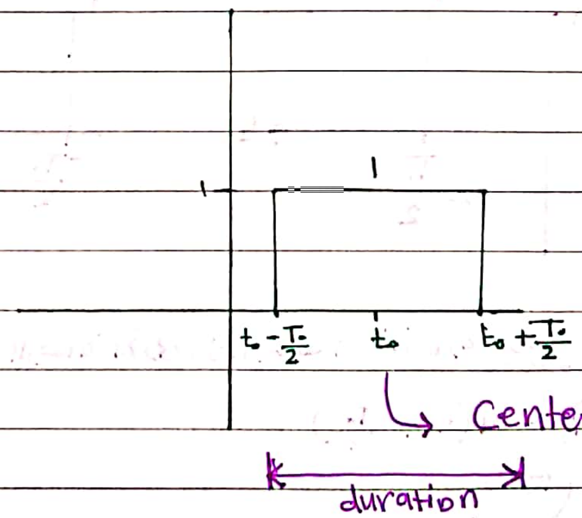
example:-



لو طرحت هادا ال (unit step) من التاني ح يعطيني rect.

* General form of rectangular function.

$rect\left(\frac{t-t_0}{T_0}\right)$



$$\operatorname{rect}\left(\frac{\overset{\pm ve}{a}t+b}{c}\right)$$

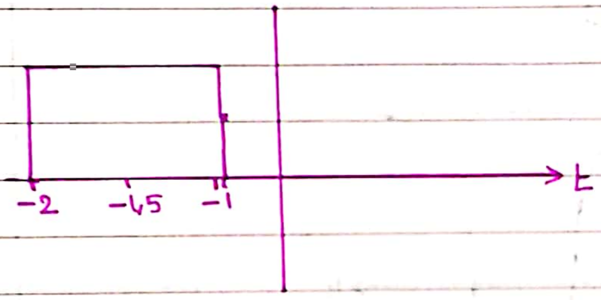
$$= \frac{\frac{1}{a}}{\frac{1}{a}} \rightarrow \operatorname{rect}\left(\frac{t+\frac{b}{a}}{\frac{c}{a}}\right)$$

How to find the center?

$t+\frac{b}{a}=0 \rightarrow t=\frac{-b}{a}$, duration $=\frac{c}{a}$.

* Plot $\operatorname{rect}\left(\frac{2t+3}{2}\right)$?

$\operatorname{rect}\left(\frac{t+\frac{3}{2}}{1}\right) \Rightarrow t+1.5=0 \Rightarrow t=-1.5$ (Center), $\Rightarrow$ duration $=1$.



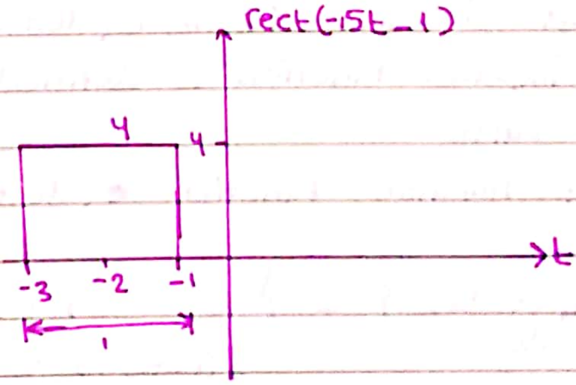
* Plot $4\operatorname{rect}(-.5t-1)$?

نضرب بـ2 ونقسم على 2

$=\operatorname{rect}\left(\frac{-t-2}{2}\right)$

$-t-2=0 \rightarrow t=-2$

duration $=2$



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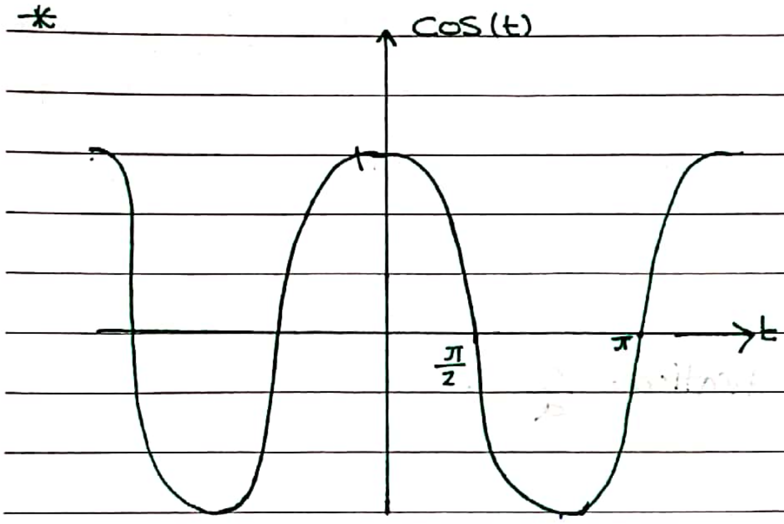
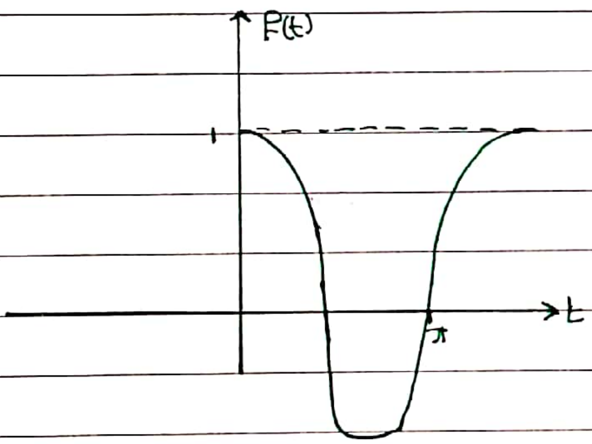
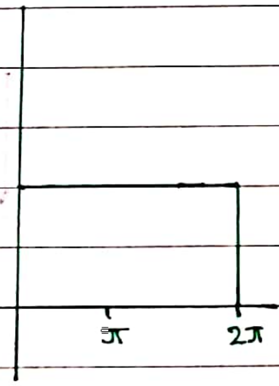


$f(t) = \cos(t)$

$\text{rect}\left(\frac{t-\pi}{2\pi}\right)$, consider $t=\pi$



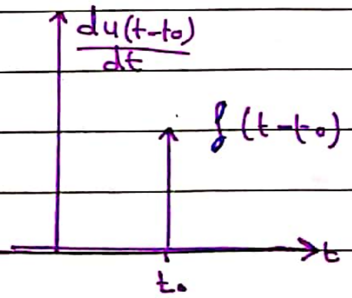


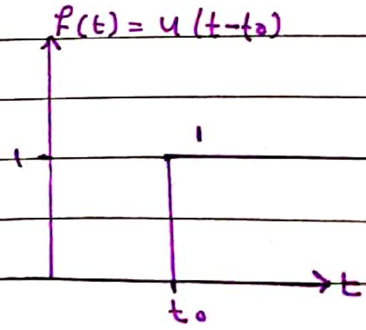
* unit impulse function (delta function).

the impulse function is defined by its properities rather than its value.

$\delta(t)$: Impulse function (delta).

$$d\left\{\frac{u(t-t_0)}{dt}\right\} = \begin{cases} 0 & , t>t_0 \\ \delta(t-t_0) & , t=t_0 \end{cases}$$





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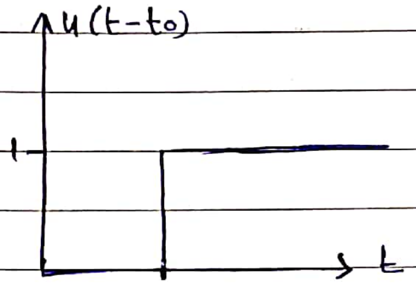
* unit Impulse function :-

Delta function δt

$$\delta(t - t_0) = \begin{cases} \text{undefined} & , t = t_0 \\ 0 & , \text{else where} \end{cases}$$

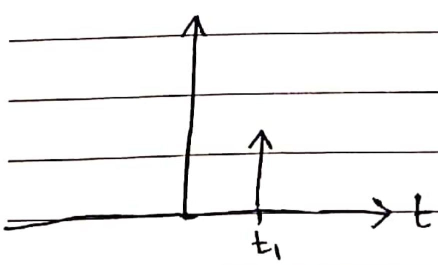


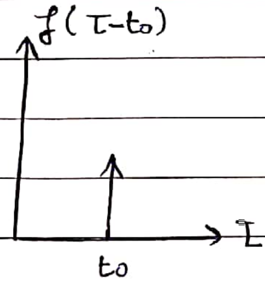
$$\delta(t-t_0) = \frac{du(t-t_0)}{dt}$$

* properties :-

1. $f(t)\ \delta(t-t_0) = f(t_0)\delta(t-t_0)$
2. $\int_{-\infty}^{\infty} \delta(t-t_0)\,dt = 1$
3. $\int_{-\infty}^{\infty} f(t)\,\delta(t-t_0)\,dt = \int f(t_0)\ \delta(t-t_0).dt = f(t_0)\int_{-\infty}^{\infty} \delta(t-t_0).dt$.
4. $u(t-t_0) = \int_{-\infty}^{t} \delta(t-t_0)\,dt$

example :- $t_0 < t_1$, $\int_{-\infty}^{t_0} \delta(t-t_1)\,.dt$,





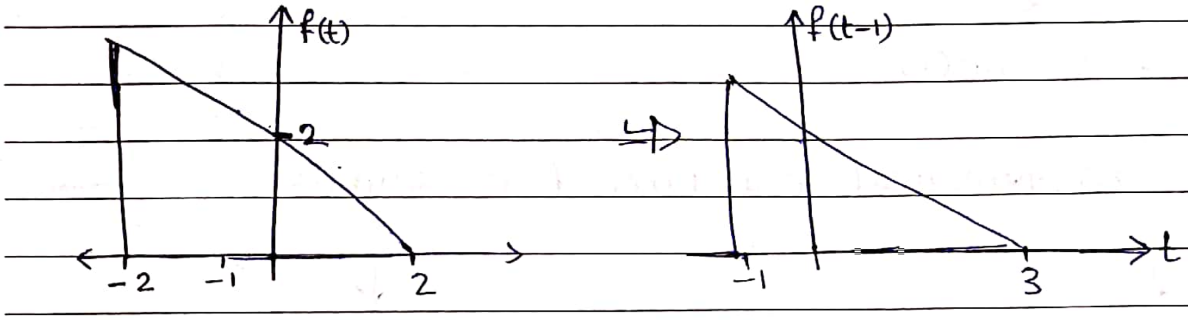
$$u(t-t_0) = \begin{cases} t > t_0 , 1 \\ 0 & , \text{other wise} \end{cases}$$

$\boxed{5}$ $\int_{-\infty}^{\infty} \delta(at+b)\,dt$

$= \frac{1}{|a|} \int_{-\infty}^{\infty} \delta\left(t+\frac{b}{a}\right)\cdot dt = \frac{1}{|a|}$

$\boxed{6}$ $\delta(t) = \delta(-t)$.

example:- given $f(t)$ as shown in figure below, find the following



① $\int_{-\infty}^{\infty} f(t)\cdot\delta(t)\cdot dt$

② $\int_{-\infty}^{\infty} f(t-1)\,\delta(t)\cdot dt$

③ $\int_{-\infty}^{\infty} f(t)\,\delta(t-1)\cdot dt$

④ $\int_{-\infty}^{\infty} f(t-1)\,\delta(t-1)\cdot dt$

⑤ $\int_{-\infty}^{\infty} f(t)\,\delta(4t)\cdot dt$

Solution:-

1) $\int_{-\infty}^{\infty} f(t)\cdot\delta(t)\cdot dt = f(0)\int_{-\infty}^{\infty}\delta(t)\cdot dt = f(0) = 2$

2) $\int_{-\infty}^{\infty} f(t-1)\cdot\delta(t)\cdot dt = f(-1) = 3$

3) $\int f(t)\,\delta(t-1)\cdot dt = f(1) = 1$

4) $\int_{-\infty}^{\infty} f(t-1)\,\delta(t-1)\,dt = f(0) = 2.$

5) $\int_{-\infty}^{\infty} f(t)\,\delta(4t)\,dt = \frac{1}{4}\int_{-\infty}^{\infty} f(t)\,\delta(t)\,dt = \frac{1}{4} f(0) = \frac{2}{4} = .5$

* evaluate $\int_{-\infty}^{\infty} \sin\left[(t-1)\right] \delta(2t-4)\,dt$

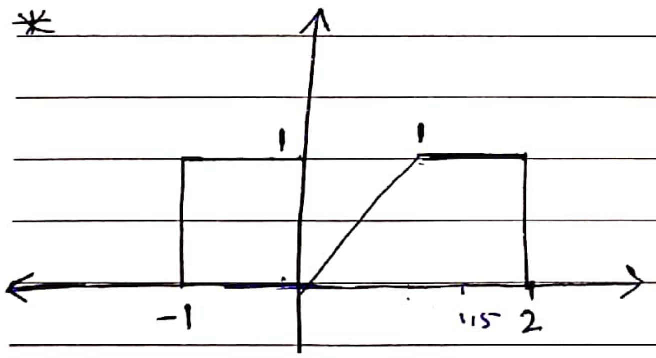
$$= \frac{1}{2}\int_{-\infty}^{\infty} \sin(t-1)\,\delta\left(t-\frac{4}{2}\right)$$

$$= \frac{1}{2}\sin(1).$$

* (2-5) :- mathematical equations for signals

- how to write a mathmatical function of a signal.
- how to plot a given function

*



write a mathematical function of the signal shown in figure?

$$x(t) = \begin{cases} 1 & , -1 < t < 0 \\ t & , 0 < t < 1 \\ 1 & , 1 < t < 2 \end{cases}$$

$$x(t) = 1\,\text{rect}\left(\frac{t+.5}{1}\right) + t\,\text{rect}\left(\frac{t-.5}{1}\right) + 1\,\text{rect}\left(\frac{t-1.5}{1}\right)$$

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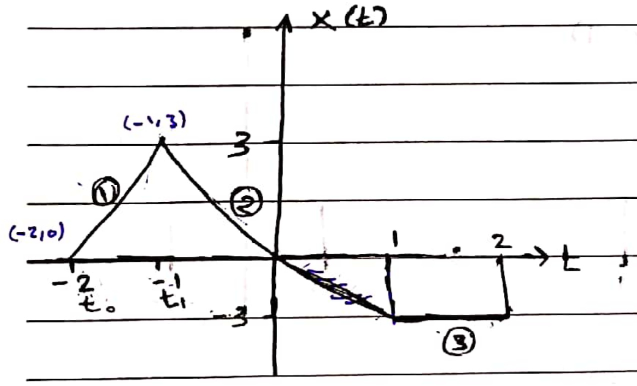
(2.5) : Mathematical equations of signals.

1- how to write mathematical equation of a plotted signals.

2- how to plot a signal.

example :- write the mathematical equation of the signal shown in figure, use unit steps only.



$$x(t) = \begin{cases} \quad , & -2 \le t \le 1 \\ \quad , & -1 \le t \le 1 \\ \quad , & 1 \le t \le 2 \\ 0 \quad , & \text{other wise} \end{cases}$$

* line segment #1 :-

$$\text{slop} = \frac{x(t) - x(t_0)}{t - t_0}$$

$$m = \text{slop} = \frac{3-0}{-1-2} = 3$$

$$\text{slop} = \frac{x(t_1) - x(t_0)}{t_1 - t_0}$$

$$3 = \frac{x(t) - 0}{t - (-2)}$$

$3 * 3 * u(3)$

$9u(9) + -6 * 1 * u(1) + 3 * 1 * u(1)$

$$x(t) = 3t + 6$$

* line segment #2 :-

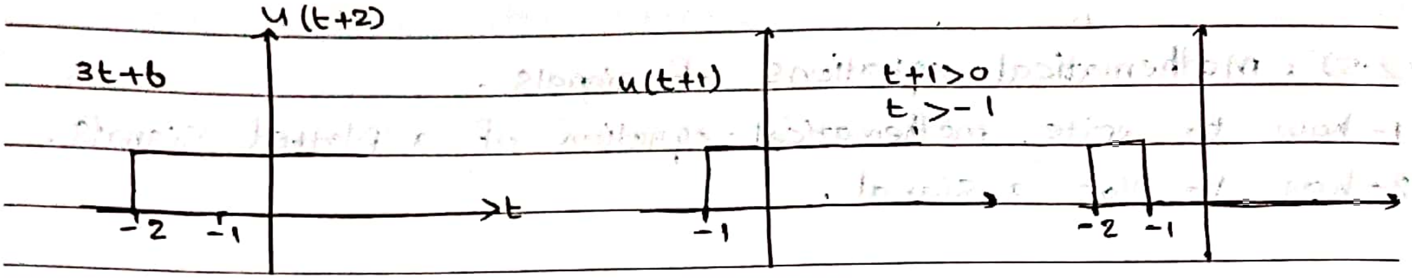
$$\frac{-3-(3)}{1-(-1)} = \frac{-6}{2} = -3$$

$$-3 = \frac{x(t) - 3}{t - (-1)}$$

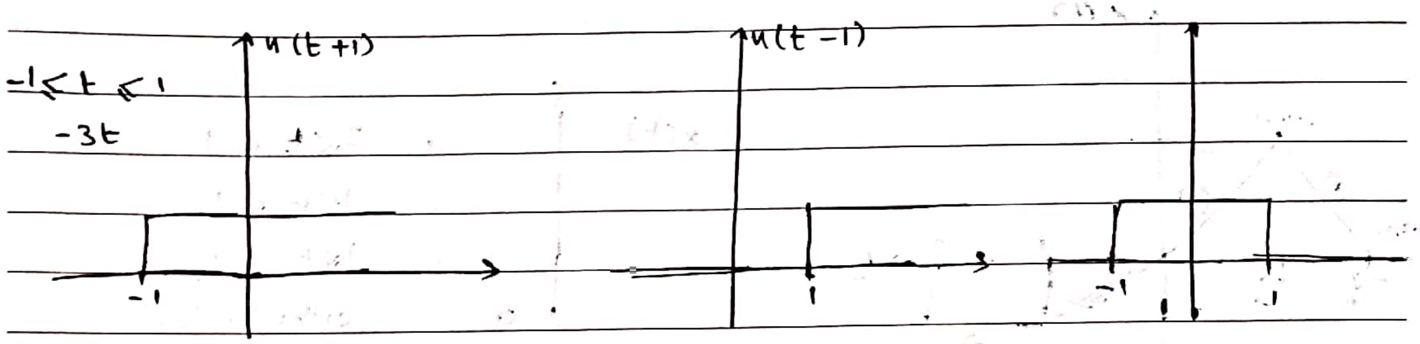
$$-3(t+1) = x(t) - 3$$

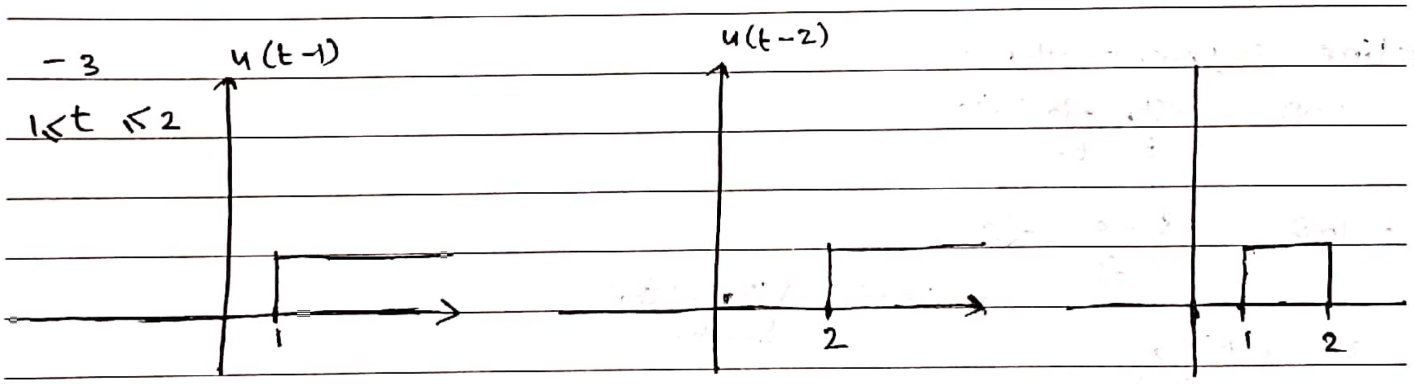
$$x(t) = -3t - 3 + 3$$

$$x(t) = -3t$$



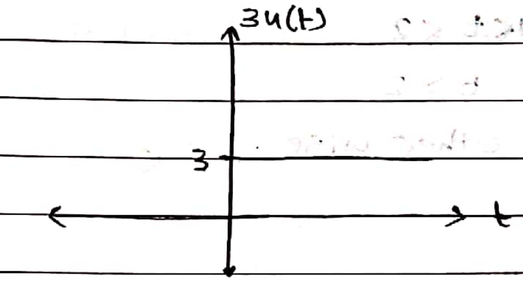
$$x(t) = (3t+6)\left[u(t+2) - u(t+1)\right] - 3t\left[u(t+1) - u(t-1)\right].$$



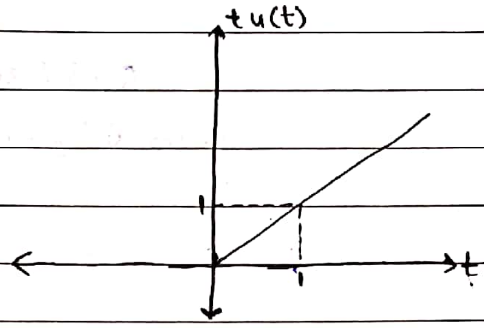


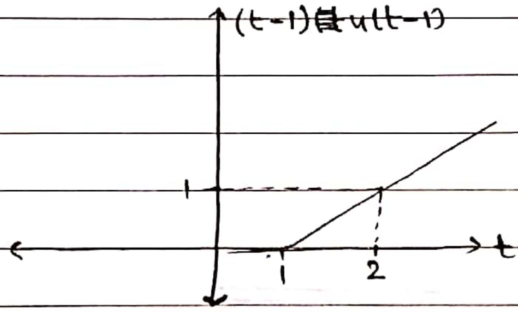
$$x(t) = (3t+6)\left[u(t+2) - u(t+1)\right]$$
$$- 3\left[u(t+1) - u(t-1)\right]$$
$$- 3\left[u(t-1) - u(t-2)\right]$$
$$= (3t+6)\,u(t+2) + (-6-6)\,u(t+1) + 3(3t-3)\,u(t-1) + 3\,u(t-2).$$

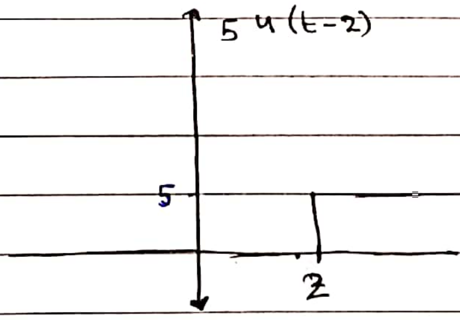
* given a signal $x(t) = 3u(t) + tu(t) - (t-1)u(t-1) - 5u(t-2)$, Plot $x(t)$?



$3+t = 0$

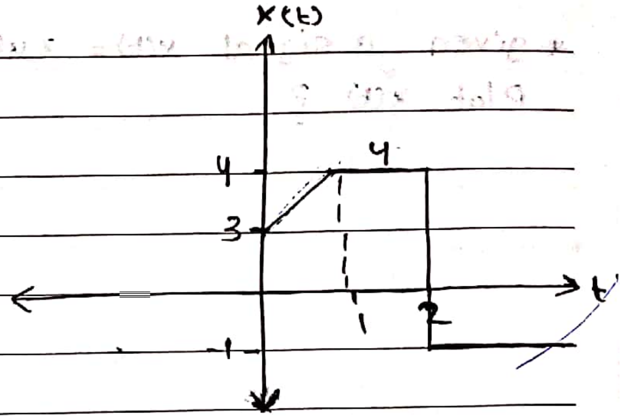






$$x(t) = \begin{cases} 3+t, & 0 \leq t \leq 1 \\ 3+t-(t-1), & 1 \leq t \leq 2 \\ 3+t-(t-1)-5, & t \geq 2 \\ 0, & \text{otherwise} \end{cases}$$

$$x(t) = \begin{cases} 3+t & , 0 \leq t \leq 1 \\ 4 & , 1 \leq t \leq 2 \\ -1 & , t \geq 2 \\ 0 & , \text{other wise} \end{cases}$$



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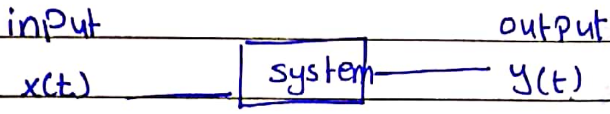
(2.6) Continous Time system.

Properities of continouose time systems:



a system is :- aprocess where a cause - effect relation is exist.

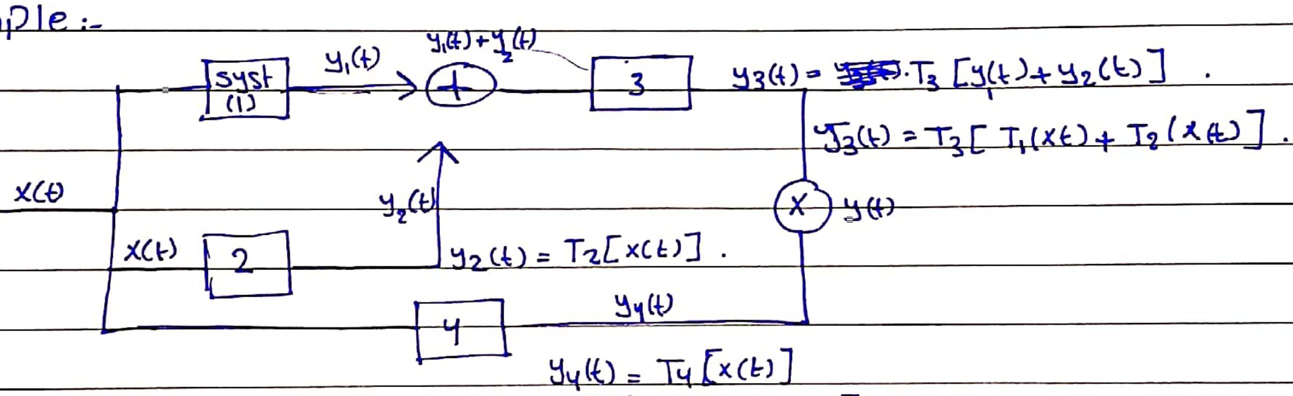
$y(t) = T_1 [x(t)]$.

T ↳ transformation.

* Transformation :- is mathematical modeling.

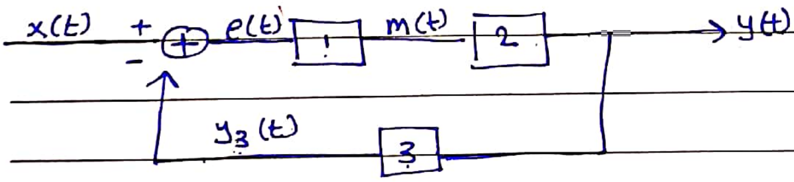
* Interonnection of systems :-

example :-



$y(t) = T_3 [T_1[x(t)] + T_2[x(t)]] [x\ T_4[x(t)]]$.

* feed back system.



$e(t) = x(t) - y_3(t) \longrightarrow x(t) = e(t) + y_3(t)$.

$y_3(t) = T_3 [y(t)]$.

$y(t) = T_2 [m(t)]$.

$m(t) = T_1 [e(t)]$.

$y(t) = T_2 [m(t)] = T_2 [T_1(e(t))]$.

[2.7] :- Properties continous Time systems.

Properities :- 1) systems with memory.

2) invetible systems.

3) causd systems.

4) stable systems.

5) linear systems.

6) time - varying systems.

[I] system with memory :- a system has memory if the out at t_0, $y(x(t))$ depends on the inputs rather than $x(t_0)$.

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(2.7) Properties of continuous-Time systems.

1- memory less.

2- Invertibility :-

a system is invertible if any unique input has only unique output.

$x_1(t) = 1$, $x_2(t) = 2$ → $\boxed{\text{system}}$ → $y_1(t) = 3$, $y_2(t) = 3$

* التكامل هون دايمًا عبارة عن (invertable).

example :- if $y(t) = x^2(t) \Rightarrow$ non-invertible.

if $y(t) = \underbrace{K}_{\text{real number}} x(t) \Rightarrow$ invertible.

* Identity system :- the system in which the output is equal to the input.

3- Causality :- a system is caused if the output at t_0, depends on the values of input $t \leq t_0$.

example :-

- $y(t) = x(t+2)$ $\Rightarrow$ non-causd system.

  $y(t_0) = x(t_0+2)$ $\Big|_{t=t_0}$

- $y(t) = x(t-2) \Rightarrow$ causd

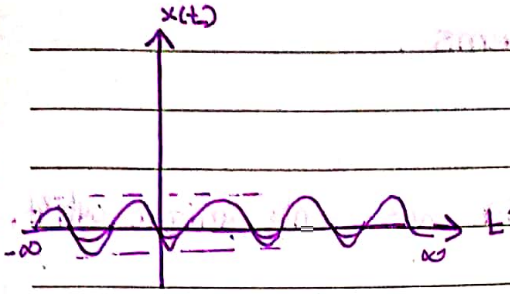
- $y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$.

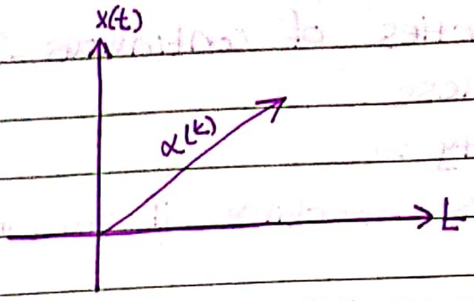
  $y(t_0) = \int_{-\infty}^{t_0} x(\tau) \cdot d\tau$ causd.

4- stability :- bounded-input, bounded-output stability (BIBO).

Bounded-input

$|x(t)| \leq \underbrace{R}_{\text{real number}}$, $|x(t)| < \infty$





Bounded-Input

Not Bounded-Input.

* ملاحظة :- $|x| < \infty$ يعني $\Leftarrow$ $-\infty < x < \infty$

example :-

$y(t) = x^2(t)$, check if this system is (BIBO) steble , for bounded-input?

this system is BIBO steble.

* BIBO steble

$\rightarrow N(t) = t\ i(t)$.

non Bounded-output.

6- time Invariance

$x(t) \rightarrow \boxed{\text{system}} \xrightarrow{y(t)} \boxed{\text{Delay } t_0} \rightarrow y(t-t_0)$

$x(t) \rightarrow \boxed{\text{delay } t_0} \xrightarrow{x(t-t_0)} \boxed{\text{system}} \rightarrow y_d$

$y_d \neq y(t-t_0) \rightarrow$ time-variant system.

$y_d = y(t-t_0)$

$\downarrow$

Time-Invariant system.

* if $y(t) = e^{x(t)}$, Check if time-invarient.

$y(t-t_0) = e^{x(t-t_0)}$

$e^{x(t-t_0)} \rightarrow \boxed{\text{system}} \rightarrow y_d$, $y_d = e^{x(t-t_0)}$

$y(t) = t\ e^{x(t-t_0)}$

$y(t-t_0) = (t-t_0)\ e^{x(t-t_0)}$

$y_d\big|_{x(t-t_0)} = t\ e^{x(t-t_0)}$, Time variant system.

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مكان كل X

$y(x_1+x_2) = (x_1+x_2)$

هل يساوي

$y_1 = x_1$

$y_2 = x_2$

$y_1+y_2 = x_1+x_2$

7- linearty :-

$x_1(t) \to$ system $\to y_1(t)$

$x_2(t) \to$ system $\to y_2(t)$

$\alpha x_1(t) + \beta x_2(t) \to \alpha y_1(t) + \beta y_2(t)$.

Principle of souper position

example :- check the Following system $y(t) = \sin(2t)\, x(t)$.

- memory ?
  memory less.
- causatig ?
  causal
- invertible or not ?
  non- invetible.
- linear or not ?

$$y(t)\Big|_{x(t)=\alpha_1 x_1(t)} = \alpha_1 \sin(2t)\, x_1(t)$$

$a_1 x_1(t) \to$ system $\to a_1 y_1(t)$.

$a_2 x_2(t) \to$ system $\to \alpha_2 y_2(t)$.

$a_1 x_1(t) + a_2 x_2(t) \to$ system $\to$ if $y(t) = \alpha_1 y_1(t) + \alpha_2 y_2(t)$.

$$y_2(t)\Big|_{x=\alpha_2 x_2(t)} = \alpha_2 \sin(2t)\, x_2(t)$$

$$y(t)\Big|_{\alpha_1 x_1(t)+\alpha_2 x_2(t)} = \sin(2t)\left[\alpha_1 x_1(t) + \alpha_2 x_2(t)\right] = \alpha_1 \sin(2t)\, x_1(t) + \alpha_2 \sin(2t)\, x_2(t).$$

* $y(t) = \sin(x(t))$, check ?

Time Inveriant !! $y(t) = \sin(2t)\, x(t)$.

$\to$ syst $\to y(t) = \sin(2t) \to$ delay $\to y_d(t) = \sin(2(t-t_0))\, x(t-t_0)$.

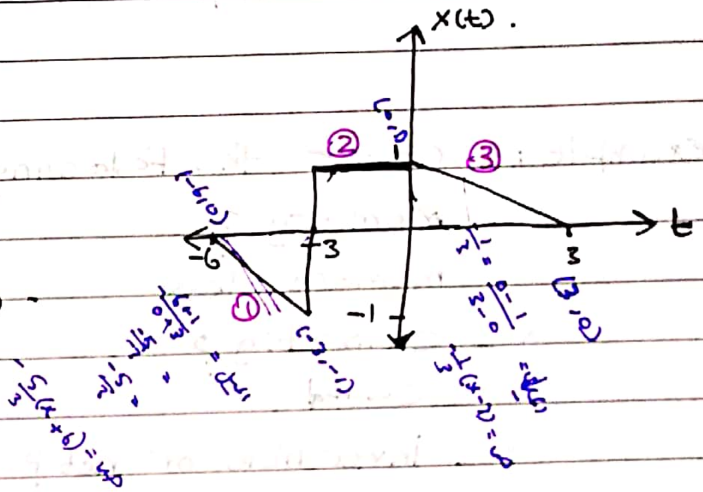
$x(t) \to$ delay $t_0 \to x(t-t_0) \to$ syst $\to y(t) = \sin(2t)\, x(t-t_0)$ $\neq$

the system is time ~~varying~~ variant.

if $y(t) = y_d(t) \to$ time invariant.

* Given the signal shown in Fig.1

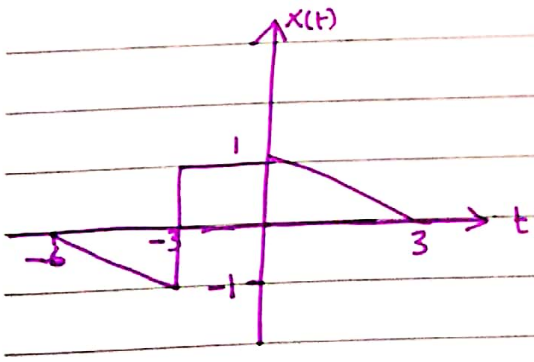
1) write mathematical expression of x(t), use unit step only.
2) plot the even part.
3) plot odd part.
4) plot $y(t) = -2x(-\frac{t}{2}+3) - 1$.
5) evaluate $\int_{-\infty}^{\infty} y(t)\, \delta(t-2) \cdot dt$.
6) plot $y_1(t) = rect\left(\frac{t-2}{3}\right) x(t)$.

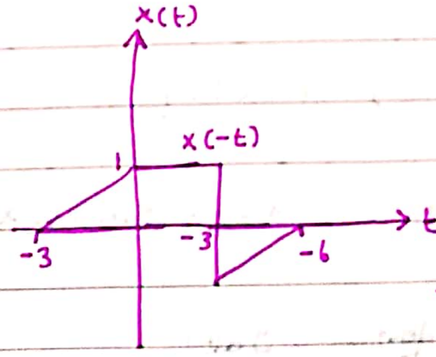


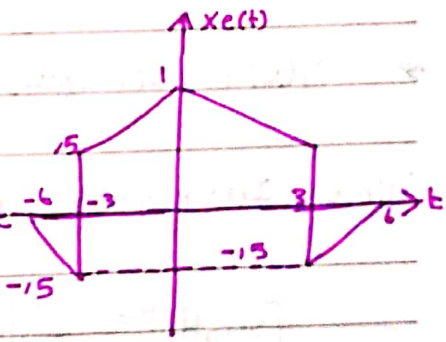
[1] $x(t) = \begin{cases} \frac{-1}{3}(t+6), & -6 < t < -3 \\ 1, & -3 < t < 0 \\ -\frac{1}{3}t + 1, & 0 < t < 3 \\ 0, & \text{other wise} \end{cases}$

$$x(t) = \frac{-1}{3}(t+6)\left[u(t+6) - u(t+3)\right] + \left[u(t+3) - u(t)\right] + \left(1 - \frac{1}{3}t\right)\left[u(t) - u(t-3)\right]$$

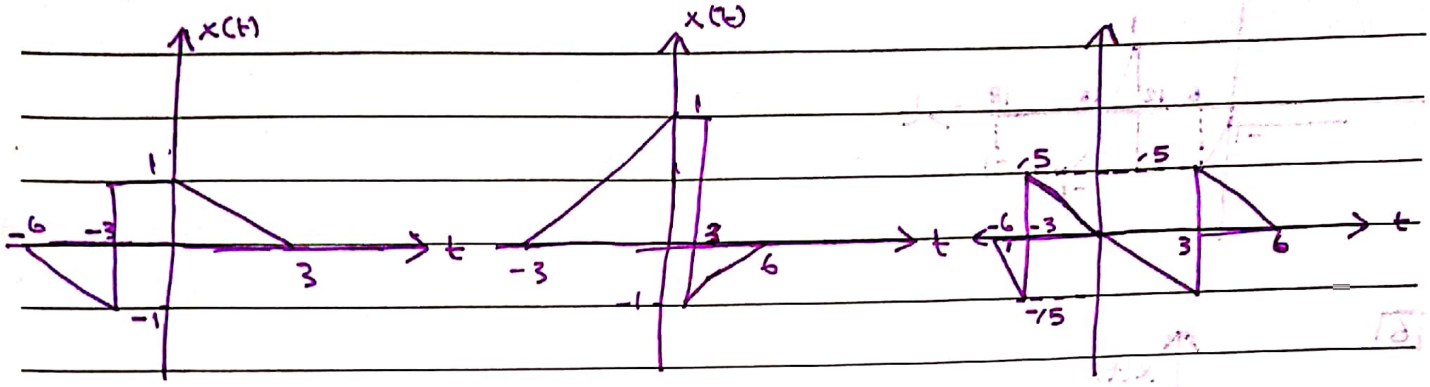
[2] $x_e(t) = \frac{1}{2}\left[x(t) + x(-t)\right]$







3 $x_o(t) = \frac{1}{2}\left[x(t) - x(-t)\right]$.



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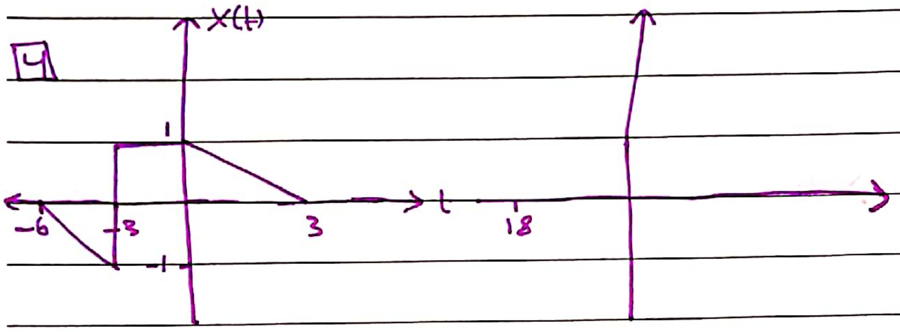
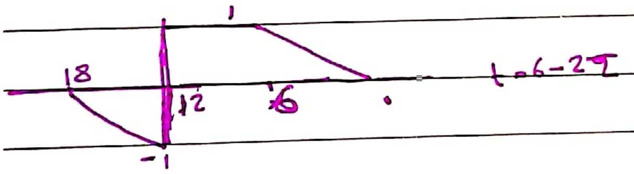


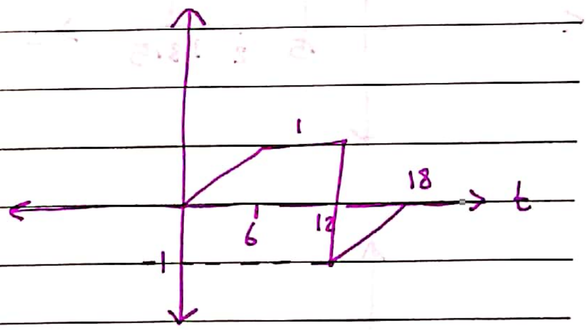
$y(t) = -2\,x\left(\frac{-t}{2}+3\right) - 1$

$y_1(t) = x\left(-\frac{t}{2}+3\right)$

$\tau = \frac{-t}{2}+3$

$\frac{t}{2} = 3-\tau \rightarrow t = 6-2\tau$.





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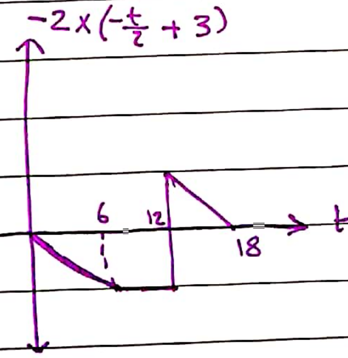
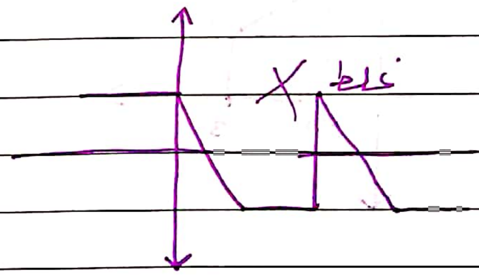
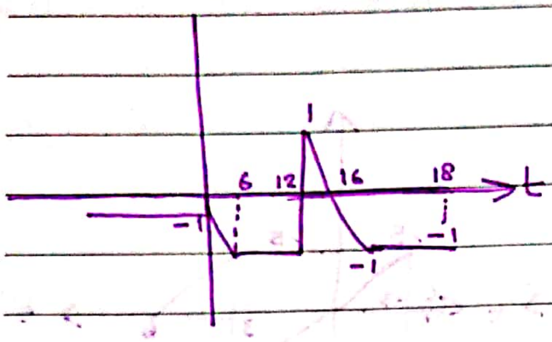


⇒





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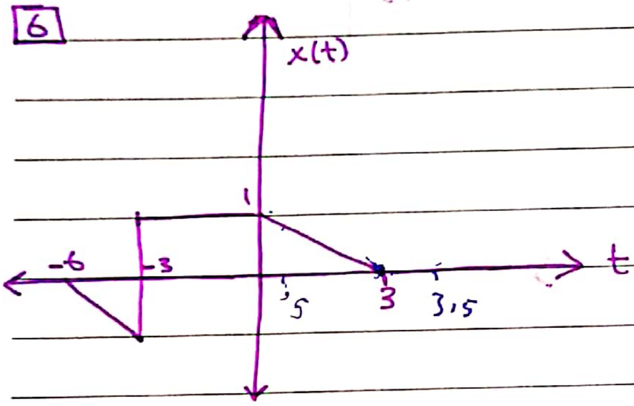
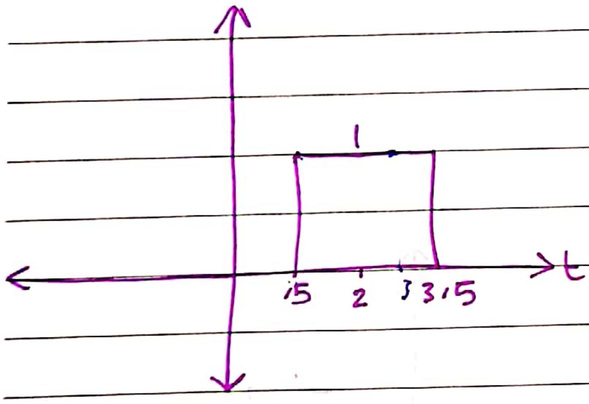
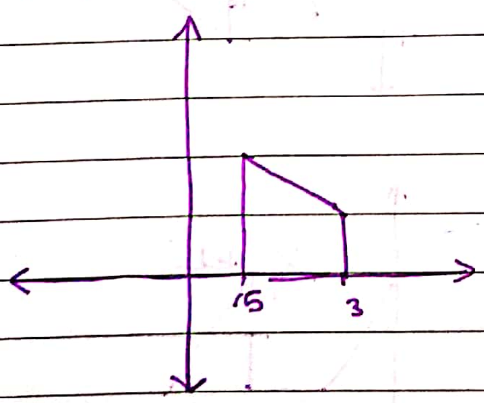






25/2/2020

Chapter (3) :- Continuous linear time-inveriant systems.

$x(t) \rightarrow \boxed{\text{LTI system}} \rightarrow y(t)$

* $y(t) = x(t) * h(t)$.

* $y(t) = \int_{-\infty}^{\infty} x(\tau) * h(t-\tau) * d\tau$

- $h(t)$: impulse response.
- $h(t-\tau)$: convolution.

Time inveriant system.

$x(t) \rightarrow \boxed{\text{system}} \rightarrow y(t)$.

$x(t) \rightarrow y(t)$

① $x(t-t_0) \rightarrow y(t-t_0)$, Time inveriant system.

② $\begin{cases} x_1(t) \rightarrow y_1(t) \\ x_2(t) \rightarrow y_2(t) \end{cases}$

linear $\Rightarrow \alpha_1 x_1(t) + \alpha_2 x_2(t) \rightarrow$ linear $\alpha_1 y_1(t) + \alpha_2 y_2(t)$.

assume that $\delta(t)$ is the input of LTI system.

$x(t) = \delta(t) \rightarrow \boxed{\text{syst..}} \rightarrow h(t)$

$h(t)$:- system impulse response.

$\delta(t-t_0) \rightarrow \boxed{\text{syst..}} \rightarrow h(t-t_0)$.

* any Signal can be written as, summation of simpler function.

$x(t)\,\delta(t-t_0) \rightarrow \boxed{} \rightarrow x(t_0)\,h(t-t_0)$

$x(t)\,\delta(t-t_0)$

$$x(t) = \sum_{k=0}^{\text{constant}} x(t) * \delta(t - kt_0)$$

$$x(t)\,\delta(t - kt_0) = x(kt_0) * \delta(t - kt_0)$$

$$x(t) \longrightarrow \boxed{\text{system}} \longrightarrow y(t)$$

$$\sum x(t)\,\delta(t - kt_0) \qquad \sum_k x(kt)\,h(t - kt_0)$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\cdot d\tau$$

$$= x(t) * h(t)$$ Convolution.

*

$\delta(t) \longrightarrow$ LTI system $\longrightarrow h(t)$: Impulse response

$\delta(t-t_0) \longrightarrow$ LTI system $\longrightarrow h(t-t_0)$; time inv.

$x(t)\,\delta(t-t_0) = x(t_0)\,\delta(t-t_0) \longrightarrow$ LTI system $\longrightarrow x(t_0)\,h(t-t_0)$, linear.

$x(t) \longrightarrow$ LTI system $\longrightarrow \displaystyle\int_{-\infty}^{\infty} x(\tau)\cdot h(t-\tau)\cdot d\tau = \int_{-\infty}^{\infty} h(\tau)\cdot x(t-\tau)\cdot d\tau.$

$\longrightarrow x(t) = \sum_k x(t)\,\delta(t - kt_0)$

$*$ → convolution.

$s = t - \tau$

$\tau = t - s$

29/9/2020

example :- the impulse response of a $\overset{LTI}{\uparrow}$ system is given as $e^{-t}u(t)$

$\boxed{1}$ find $y(t)$ if $x(t) = \delta(t)$.

$\boxed{2}$ find $y(t)$ if $x(t) = \delta(t-1)$.

Solution :-

$$h(t) = e^{-t}u(t)$$

$$\delta(t) \xrightarrow{\text{system}} h(t)$$

if $x(t) = \delta(t)$, then $y(t) = h(t) = e^{-t}.u(t)$.

if $x(t) = \delta(t-1)$, then $y(t) = h(t-1) = e^{-(t-1)}u(t-1)$

$$h(t) = e^{-t}u(t)$$

$$x(t) = \delta(t)$$

$$y(t) = x(t) * h(t)$$

$$= \int_{-\infty}^{\infty} x(\tau) * h(t-\tau) . d\tau .$$

$$= \int_{-\infty}^{\infty} \delta(\tau) \underbrace{e^{-(t-\tau)} u(t-\tau)}_{f(t)} . d\tau .$$

$$= e^{-(t)} . u(t) \; 1$$

$$h(t) = e^{-t}u(t)$$

$$x(t) = \delta(t-1)$$

$$y(t) = x(t) * h(t)$$

$$= \int_{-\infty}^{\infty} x(\tau) * h(t-\tau)$$

$$= \int_{-\infty}^{\infty} \delta(\tau-1) e^{-(t-\tau)} . u(t-\tau) . d\tau .$$

$$= e^{-(t-1)} . u(t-1) \; (1) .$$

$\boxed{3}$ if $x(t) = \sum_{k=0}^{\infty} 0.1\, \delta(t-k)$, find $y(t)$.

$$x(t) = .1\,\delta(t) + .1\,\delta(t-k) + .1\,\delta(t-2k) + \cdots$$

$$y(t) = \sum_{k=0}^{\infty} .1\, h(t-k) .$$

$x_1(t) \rightarrow y_1(t)$

$x_2(t) \rightarrow y_2(t)$

* example 3-2 :- given $y(t) = \int_{-\infty}^{t} x(\tau) . d\tau$, ① find the impulse response.

(graphical convolution)

② find $y(t)$, when $x(t) = t\, u(t)$.

① $x = \delta(t)$ → $\boxed{\text{system}}$ → $y(t) = h(t)$

$$h(t) = \int_{-\infty}^{t} \delta(\tau)\cdot d\tau$$

$$= u(t).$$

② $x(t) = t\,.u(t)$; $h(t) = u(t)$.

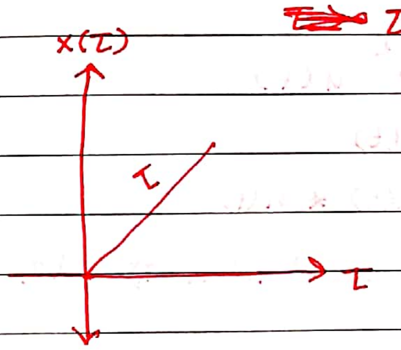
$$y(t) = x(t) * h(t)$$

$$= \int_{-\infty}^{\infty} x(\tau)\cdot h(t-\tau)\cdot d\tau.$$

$$y(t) = \int_{-\infty}^{\infty} \tau u(\tau)\cdot u(t-\tau)\cdot d\tau.$$

$x(\tau) = \tau u(\tau)$

$h(t-\tau) = u(t-\tau)$ $\qquad t-\tau > 0 \Rightarrow \tau < t$



find $y(t)$ at $t=2$

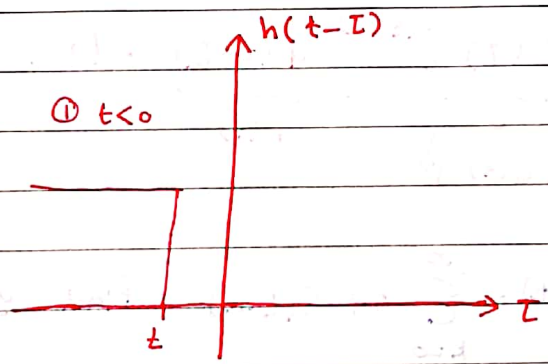
$$y(2) = \int_{-\infty}^{\infty} x(\tau)\, h(2-\tau)$$

① if $t<0$

$$y(t) = \int_{-\infty}^{\infty} \tau u(\tau)\cdot u(t-\tau)\cdot d\tau.$$

$$= 0$$

① $t<0$



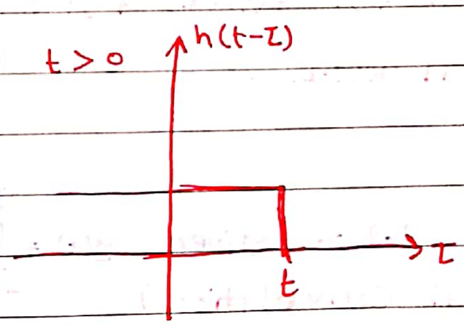
② if $t>0$

$$y(t) = \int_{0}^{t} \tau\cdot d\tau.$$

$$= \left.\frac{\tau^2}{2}\right|_0^t$$

$$= \frac{t^2}{2}.$$

② $t>0$



$$y(t) = \begin{cases} \frac{t^2}{2}, & t>0 \\ 0, & \text{O.W} \end{cases}$$

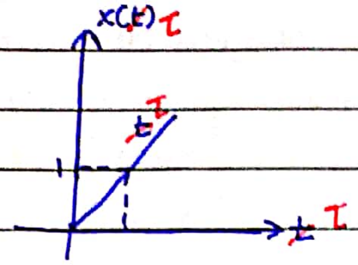
$$y(t) = \frac{t^2}{2}\cdot u(t)$$

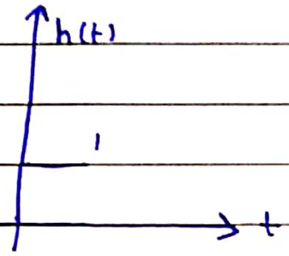
$$y(17) = \frac{(17)^2}{2}$$

$$y(-7) = 0$$

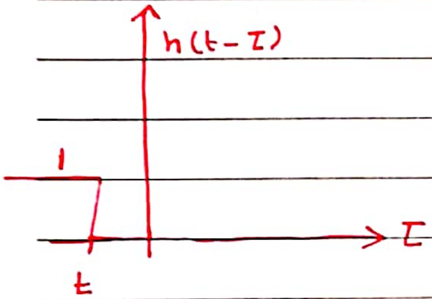
طريقة الحل الثانية

*





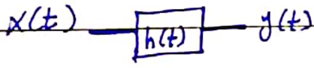
$$y(t) = x(t) * h(t)$$
$$= \int x(\tau) * h(t-\tau) \cdot d\tau$$
$$h(t-\tau) = u(t-\tau)$$
$$t-\tau \geq 0 \rightarrow \tau < t$$

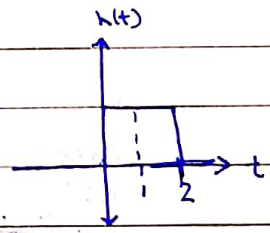


1/3/2020

*example :- given h(t) as shown in figure, find the out if x(t)=

$x(t) = \delta(t+3) + 3e^{-.5t} u(t)$?





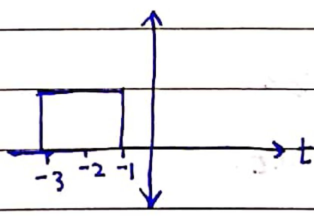
$$y(t) = x(t) * h(t)$$
$$= \int x(\tau) * h(t-\tau) \cdot d\tau .$$

$\delta(t) \rightarrow h(t).$

$h(t) = rect\left(\frac{t-1}{2}\right)$

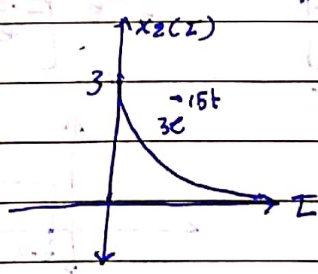
$\delta(t+3) \rightarrow y(t) = h(t+3) = h\left(\frac{t+2}{2}\right).$

$y_1(t) = h(t+3).$



$x_2(t) = 3e^{-.5t}$

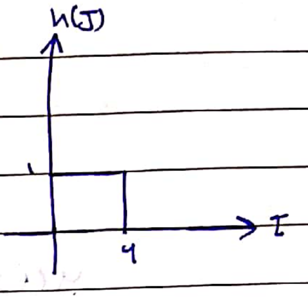
$y_2(t) = \int_{-\infty}^{\infty} x_2(\tau) \cdot h(t-\tau) \cdot d\tau .$

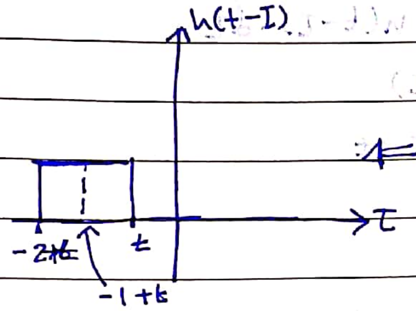


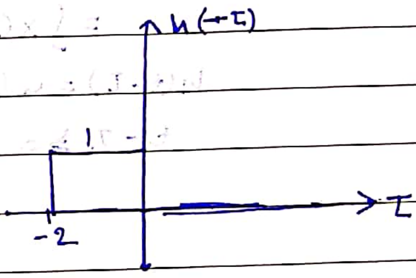
$h(t-\tau) =$

$h(t) = rect\left(\frac{t-1}{2}\right)$

$h(t-\tau) = rect\left(\frac{t-\tau-1}{2}\right)$



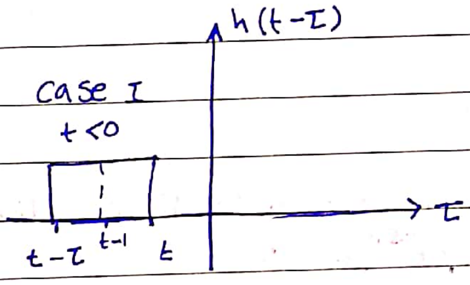




Case(1) :-

$t \leq 0$

$y(t) = 0$



case (2)

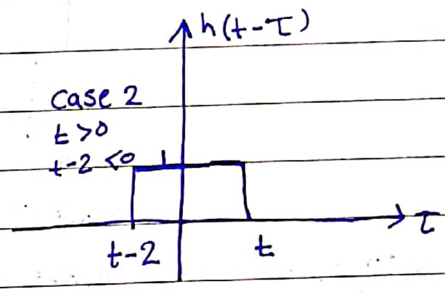
$0 < t \leq 2$

$$y(t) = \int_0^t 3e^{-.5\tau} (1)\, d\tau .$$

$$= -\frac{3}{0.5} e^{-.5\tau}\Big|_0^t$$

$$= -6\left[e^{-.5t} - 1\right]$$

$$= 6\left[1 - e^{-.5t}\right]$$

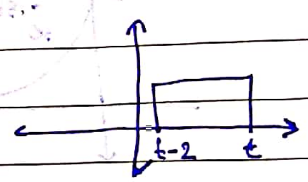


Case (3):-

$t-2 > 0 \; , t \geq 2$

$$y(t) = \int_{t-2}^{t} 3e^{-.5\tau} \cdot d\tau .$$

$$= -6\left[e^{-.5\tau}\Big|_{t-2}^{t}\right] = -6\left[e^{-.5t} - e^{-.5(t-2)}\right] = -6e^{-.5t}\left[1 - e^{1}\right]$$



$h(t+3)$

$y(t) = y_1(t) + y_2(t)$

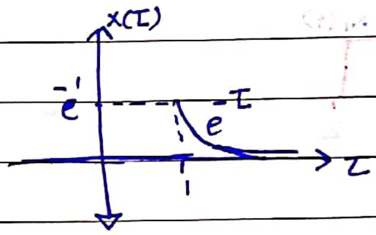
$$y_2(t) = \begin{cases} 0 & , t \leq 0 \\ 6(1-e^{-.5t}) & , 0 \leq t \leq 2 \\ -6e^{-.5t}(1-e) & , t \geq 2 . \end{cases}$$

" انتهى السؤال "

example :- Given $x(t) = e^{-t} u(t-1)$, find $y(t)$?

$h(t) = e^{t} u(-1-t)$

$y(t) = \int_{-\infty}^{\infty} x(\tau) \cdot h(t-\tau) \cdot d\tau$.



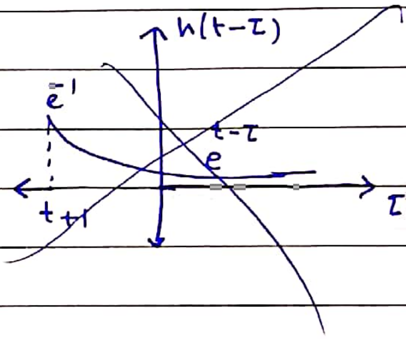
$x(\tau) = e^{-\tau} u(\tau-1)$

$\tau > 1$

$h(t-\tau) = e^{t-\tau} u(-1-(t-\tau))$

$= e^{t-\tau} u(\tau - t - 1)$

$\tau - t - 1 > 0$, $\tau > t+1$



Cases :-

Case (1) :-

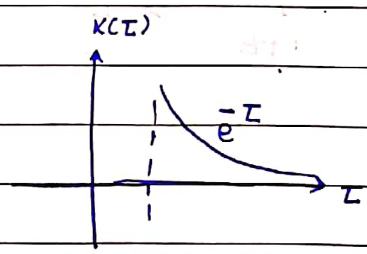
$t+1 < 1$, $t < 0$.

$y(t) = \int_{1}^{\infty} e^{-\tau} \cdot e^{t-\tau} \cdot d\tau$.

$= e^{t} \int_{1}^{\infty} e^{-2\tau} \cdot d\tau = \frac{-e^{t}}{2} e^{-2\tau} \Big|_{1}^{\infty} = \frac{-e^{t}}{2} [0 - e^{-2}] = \frac{e^{t-2}}{2}$.

Case (2) :- $t+1 > 0$, $t > 0$

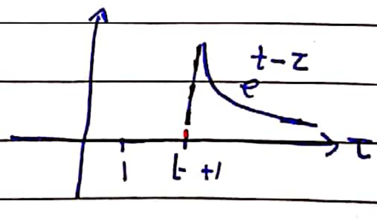
$y(t) = \int_{t+1}^{\infty} e^{-\tau} \cdot e^{t-\tau} \cdot d\tau$.



$= \frac{-e^{t}}{2} e^{-2\tau} \Big|_{t+1}^{\infty}$

$= \frac{-e^{t}}{2} [0 - e^{-2t+-2}]$

$= \frac{1}{2} e^{-t-2}$

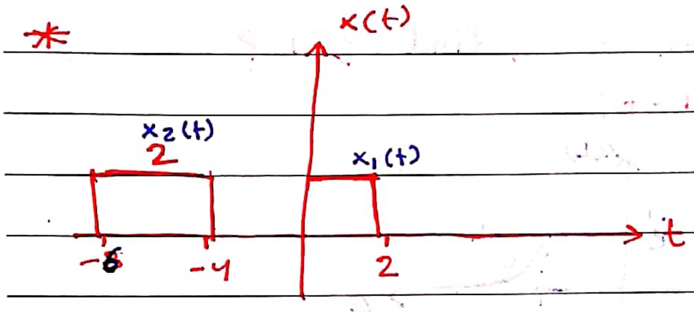


$$y(t) = \begin{cases} \frac{e^{t-\tau}}{2} & , t<0 \\ \frac{e^{-t-2}}{2} & , t>0 \end{cases}$$

$$y(t) = \frac{e^{t-2}}{2} u(-t) + \frac{e^{-t-2}}{2} u(t) .$$

* زنتهی راکل !!

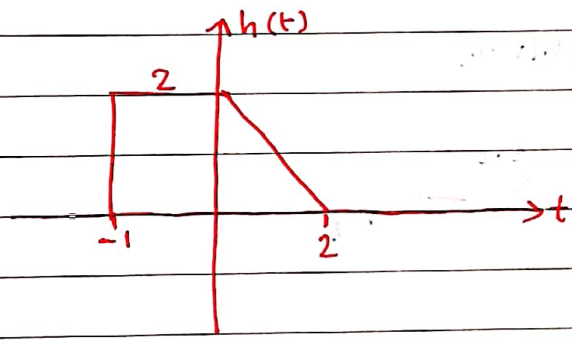
*

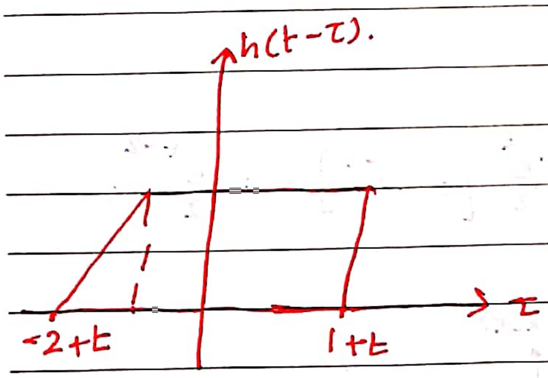


* $y(t) = \int x(\tau) h(t-\tau) d\tau .$

$x_1(t) = \mathrm{rect}\left(\frac{t-1}{2}\right) \rightarrow y_1(t)$

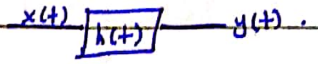
$x_2(t) = 2\ \mathrm{rect}\left(\frac{t+5}{2}\right) \rightarrow \alpha y_1(t+3)$.

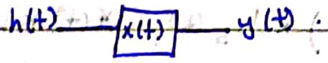




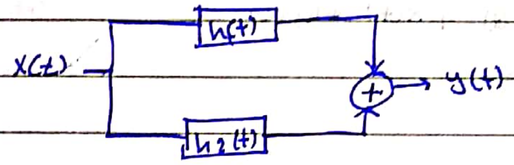
(3.2) Properties of convolution.

1. $x(t) * h(t)$ – equivelent to $h(t) * x(t)$, $y(t) = x(t) * h(t) = h(t) * x(t)$.



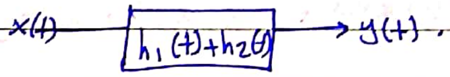


2.



equivelent

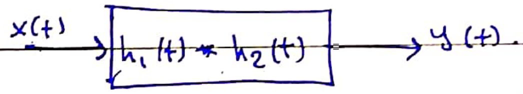
$$y(t) = x(t) * h_1(t) + x(t) * h_2(t).$$
$$= x(t) * [h_1(t) + h_2(t)]$$



3.

equivelent

$$y(t) = (x(t) * h_1(t)) * h_2(t)$$
$$= x(t) * (h_1(t) * h_2(t))$$



3/3/2020

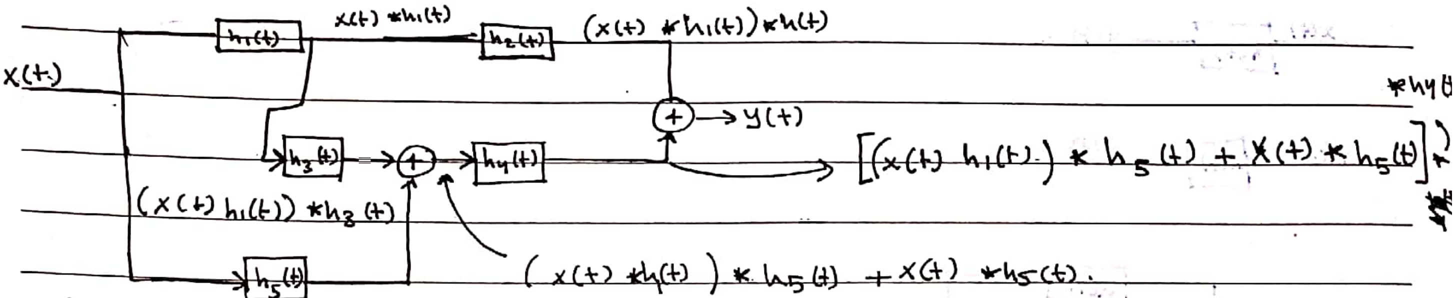
★ $h(t) = \delta(t-4)$

$x(t) = e^{-t} . u(t)$, find $y(t)$?

$y(t) = x(t) * h(t)$.

3/3/2020

Interconnection of sub-systems.



find the equevelent Impulse response.

$x(t) \to \boxed{h_{eq}(t)} \to y(t).$

$$y(t) = (x(t) * h_1(t)) * h_2(t) + \left[(x(t) * h_1(t)) * h_3(t) + x(t) * h_5(t)\right] * h_4(t)$$

$$= x(t) * \left[h_1(t) * h_2(t)\right] + \left[x(t) * (h_1(t) * h_3(t)) + x(t) * h_5(t)\right] * h_4(t).$$

$$\cancel{= x(t) * \left[h_1(t) * h_3(t) + h_5(t)\right] * h_4(t).}$$

$$= x(t) * \underbrace{\left[h_1(t) * h_2(t) + h_1(t) * h_3(t) * h_4(t) + h_5(t) * h_4(t)\right]}_{h_{eq}(t).}$$

تكملة
السؤال

* given $h_1(t) = h_4(t) = u(t).$

$h_2(t) = h_3(t) = 5\,\delta(t).$

$h_5(t) = e^{-2t} \cdot u(t)$

, find $h_{eq}(t)$.

$h_1(t) * h_2(t) = u(t) * 5\,\delta(t)$

$h_1(t) * h_3(t) * h_4(t) = \underline{u(t) * 5\delta(t)} * u(t).$

$5u(t) * u(t).$

(3.3) Properties of Continuous-Time LTI system.

$$y(t) = \int_{-\infty}^{\infty} x(\tau) \cdot h(t-\tau) \cdot d\tau = \int h(\tau) \, x(t-\tau) \cdot d\tau .$$

1. Memory-less.

$y(t_0) \rightarrow x(t_0)$ memory less.

if $h(t) = K\,\delta(t)$, then the system is memory-less, ~~otherwise~~ any other $h(t)$ will make the system with memory.

$$y(t_0) = \int_{-\infty}^{\infty} x(\tau) \, K \, \delta(t_0 - \tau) \, d\tau .$$

$$x(t_0) \int_{-\infty}^{\infty} K(t_0 - \tau) \cdot d\tau .$$

$$y(t_0) = K . x(t_0) . 1$$

2. Invertibily.

$$x(t) \rightarrow \boxed{h(t)} \rightarrow y(t) .$$

$$x(t) \rightarrow \boxed{h(t)} \rightarrow \boxed{h_i(t)} \rightarrow y(t)$$

if I can find $h_i(t)$ (inverse impulse response).

$$x(t) = x(t) * h(t) * h_i(t) .$$

$$= x(t) * \delta(t) .$$

$$\boxed{h(t) * h_i(t) = \delta(t)}$$

3.4 Properties of Continuous LTI systems

Given $x(t) \to \boxed{h(t)} \to y(t)$

$$y = \int x(\tau)\, h(t-\tau)\, d\tau = \int h(\tau)\, x(t-\tau)\, d\tau$$

* Memory-less System

* The system is memoryless if the output at time t_0 requires the input at t_0 only.

$$y(t_0) = \int_{-\infty}^{\infty} x(\tau)\, h(t_0-\tau)\, d\tau$$

if $h(t) = \delta(t)$

$$\to y(t_0) = \int_{-\infty}^{\infty} x(\tau)\, \delta(t_0-\tau)\, d\tau$$

$y(\tau_0) = x(\tau_0)$ $\Rightarrow$ The system with Impulse response $h(t) = k\,\delta(t)$ is memoryless

Invertibility

A system is invertible if each unique ~~output~~ has a unique input

$x(t) \to \boxed{h(t)} \to y(t)$

* The system is invertible if ~~the~~ we can find $h_i(t)$ such that

$y(t) \to \boxed{h_i(t)} \to x(t)$, $h_i(t)$: impulse response of inverse system

$\Updownarrow$ Equivalent

$x(t) \to \boxed{h(t)} \to \boxed{h_i(t)} \to x(t)$ Based on this $h(t) * h_i(t) = \delta(t)$

Causality

The system is causal if the output at time ~~to~~ t_0 depends on only the current and past values of the input.

Note that

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

The system is causal, if the integration can be written as

$$y(t) = \int_{-\infty}^{t} x(\tau)\, h(t-\tau)\, d\tau$$

only past values of $x(\tau)$ are used

Note also

$$y(t) = \int_{-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau$$

if $t-\tau < 0$, then the system is causal

$$y(t) = \int_{-\infty}^{0} x(t-\tau)\, h(\tau)\, d\tau$$

Stability

From Ch2, the system is stable if it is BIBO.

Bounded input $\rightarrow$ $|x(t)| \le M$, M: real constant

Bound Input $|y(t)| \le N$, N: real constant

BIBO stable $\Rightarrow$ $\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$, $h(t)$ is absolutly integrable

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Example 3.8

Determine if $h(t) = e^{-3t} u(t)$ is stable or not.

$$\int_{-\infty}^{\infty} |h(t)|\, dt = \int_{0}^{\infty} e^{-3t}\, dt = -\frac{1}{3} e^{-3t} \Big|_{0}^{\infty}$$
$$= -\frac{1}{3}(0-1) = \frac{1}{3} < \infty$$

the system is BIBO stable

Example 3.9

stability of an integrator

Check the stability of $h(t) = u(t)$

3.5 Differential - Equation models

3.6 Terms in natural response

Example 3.2

if $y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$

1- Find the impulse response of the system

the impulse response $h(t)$ is the output of the system when the input is $\delta(t)$

$$h(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau = u(t)$$

$$\boxed{h(t) = u(t)}$$

2- Find the output if $x(t) = t\, u(t)$

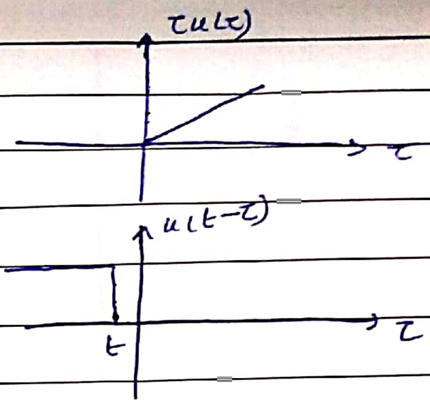
solution $\rightarrow$ $y(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau)\, d\tau$

$u(t-\tau) \Rightarrow t-\tau > 0$
$\tau < t$

$$= \int_{-\infty}^{\infty} \tau u(\tau)\, u(t-\tau)\, d\tau$$

if $t<0 \rightarrow y(t)=0$

if $t>0$.



$$y(t) = \int_0^t \tau \, d\tau$$

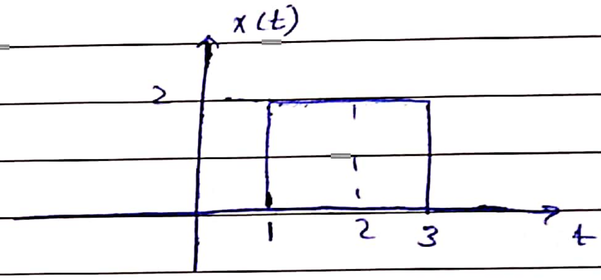
$$= \frac{1}{2} t^2$$

Then $\rightarrow y(t) = \frac{1}{2} t^2 u(t)$.

Problems

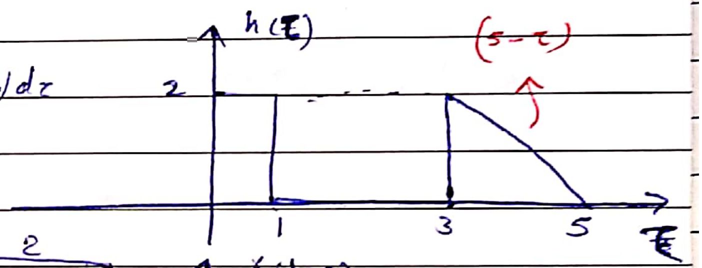
3.4 $\quad x(t) \rightarrow \boxed{h(t)} \rightarrow y(t)$

if



Find and plot $y(t)$

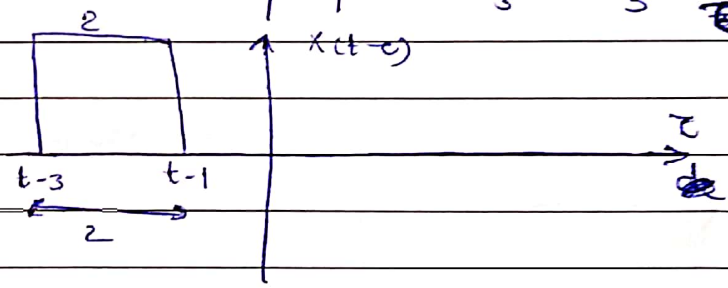
$$y(t) = \int x(\tau) h(t-\tau) d\tau = \int x(t-\tau) h(\tau) d\tau$$



Case #1

$t-1<0 \rightarrow t<1$

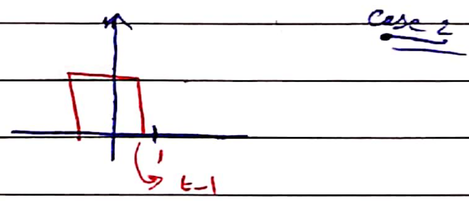
$\boxed{y(t)=0}$



Case # 2.a $\quad t-1>0, \; t<2$

$1 \le t \le 2$

$$y(t) = \int_0^{t-1} (2)(2)\, d\tau$$



$\boxed{y(t) = 4(t-1)}$

Case 3 $\quad t-3 \ge 0, \; t-3 \le 1 \Rightarrow 3 \le t \le 4$

$$y(t) = \int_{t-3}^{1} (2)(2)\, d\tau = 4(1-(t-3))$$

$\boxed{y(t) = 16-4t}$

Case 2.b

$t-1>1, \quad t-3<0$

$2 \le t \le 3$

$$y(t) = \int_1^2 (2)(2)\, d\tau$$

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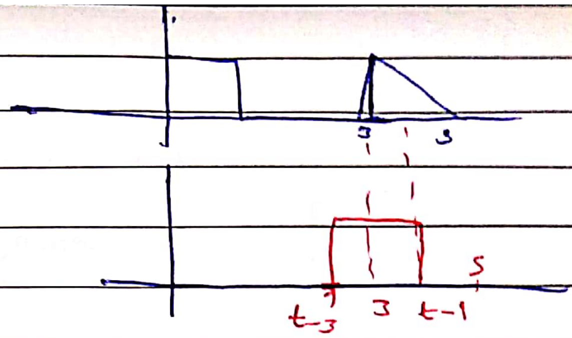
Case #4

$t-1 \geq 3, \quad t-1 \leq 5$

$4 \leq t \leq 6$



$$y(t) = \int_{3}^{t-1} (2)(5-\tau)\, d\tau$$

$$10\tau - \tau^2 \Big|_{3}^{t-1} = \left(10(t-1) - (t-1)^2\right) - (30-9)$$

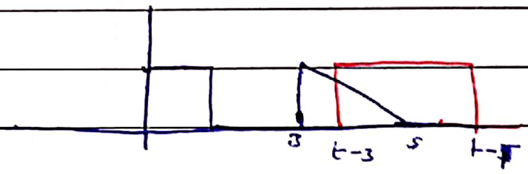
$$= 10t - 10 - t^2 + 2t - 1 - 21$$

$$\boxed{y(t) = -t^2 + 12t - 32}$$

Case 5

$t-3 \geq 3, \quad t-3 \leq 5$

$6 \leq t \leq 8$



$$y(t) = \int_{t-3}^{5} (2)(5-\tau)\, d\tau$$

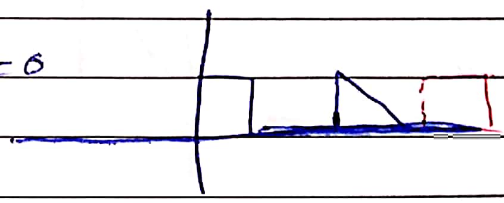
$$= 10\tau - \tau^2 \Big|_{t-3}^{5} = (50-25) - \left[(10t - 30) - t^2 + 6t - 9\right]$$

$$= 25 - \left[16t - 39 - t^2\right]$$

$$y(t) = t^2 - 16t + 64$$

Case 6

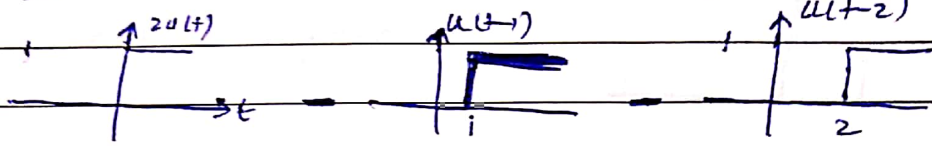
$t-3 > 5 \rightarrow t > 8 \rightarrow y(t) = 0$

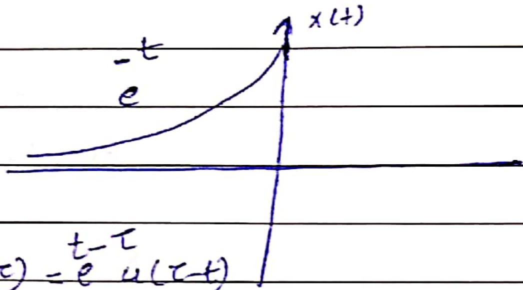


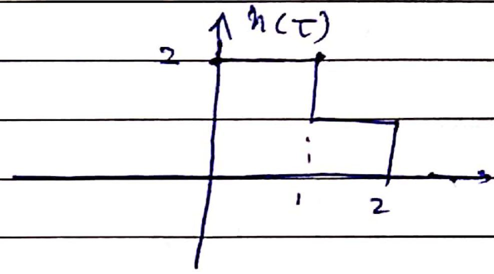
$\boxed{3\text{-}7}$ $x(t) \rightarrow \boxed{h(t)} \rightarrow y(t)$

$x(t) = e^{t} u(-t)$, $h(t) = 2u(t) - u(t-1) - u(t-2)$

$h(t) \Rightarrow$

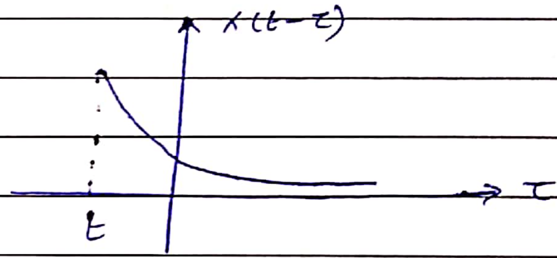






$x(t-\tau) = e^{t-\tau} u(\tau - t)$

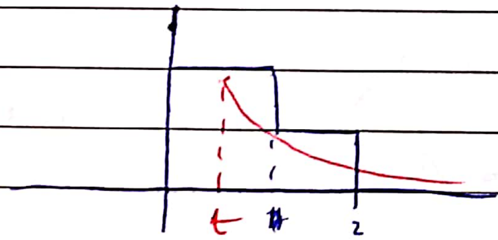
$\hookrightarrow \tau - t > 0 \rightarrow \tau > t$



Case #1

$t < 0$

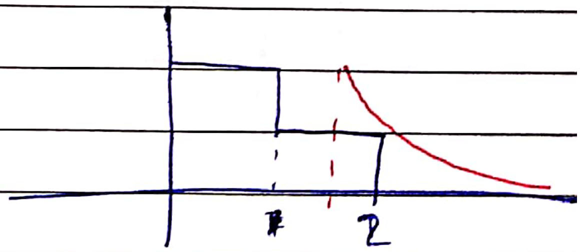
$$y(t) = \int_0^1 e^{t-\tau}(2)\,d\tau + \int_1^2 (1)\,e^{t-\tau}\,d\tau$$



Case #2 $t > 0,\ t < 1$

$0 \leq t \leq 1$

$$y(t) = \int_t^1 e^{t-\tau}(2) + \int_1^2 (1)\,e^{t-\tau}\,d\tau.$$

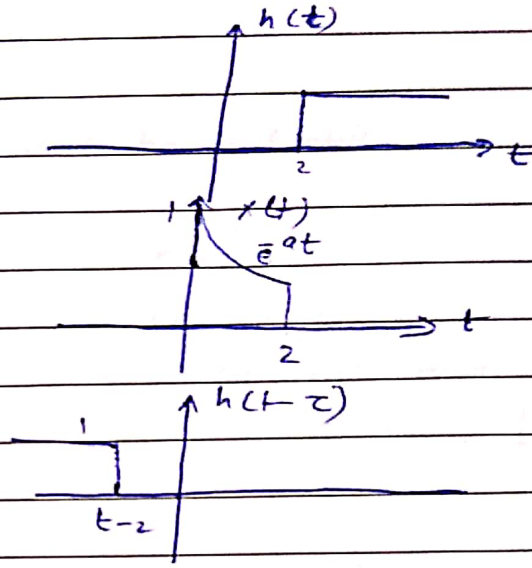


Case #3 $1 \leq t \leq 2$

$$y(t) = \int_t^2 (1)\,e^{t-\tau}\,d\tau$$

Case 4 $t \geq 2 \longrightarrow y(t) = 0$

3-7 (d) $x(t) = e^{-at}[u(t) - u(t-2)]$, $h(t) = u(t-2)$



$h(t-\tau) = u(t-\tau-2)$ $\qquad t-\tau-2>0$

$\qquad\qquad\qquad\qquad\qquad\quad \tau < t-2$

Case #1

$t-2 < 0 \rightarrow t < 2$

$y(t) = 0$

Case # 2 $\qquad t-2>0$, $\quad t-2<2$

$2 \le t < 4$

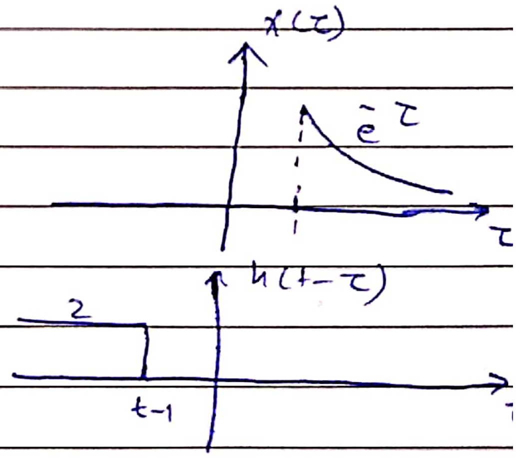
$$y(t) = \int_0^{t-2} e^{-\alpha\tau}(1)\, d\tau$$

Case # 3 $\quad t-2 \ge 2$

$$y(t) = \int_0^{t} (1) e^{-a\tau}\, d\tau$$

3-7 (f) $\quad x(t) = e^{-t}u(t-1) \qquad h(t) = 2u(t-1)$

$h(t-\tau) = 2u(t-\tau-1) = \begin{cases} 2, & \tau \le t-1 \\ 0 & ow \end{cases}$



Case # 1

$t-1 \le 1 \rightarrow t < 2$

$y(t) = 0$

Case # 2 $\qquad t-1 > 1 \Rightarrow t \ge 2$

$$y(t) = \int_1^{t-1} (2)(e^{-a\tau})\, d\tau$$

[2.21] Determine the stability and Causality of

(a) $h(t) = e^{(1-t)} u(1-t)$

stability check

if $\int_{-\infty}^{\infty} h(t)\,dt < \infty$ → then the system is stable

$$\int_{-\infty}^{\infty} e^{1-t} u(1-t)\,dt \xrightarrow{t<1} \int_{-\infty}^{1} e^{1-t}\,dt = -e^{1-t}\Big|_{-\infty}^{1}$$

$$= -(e^{0} - e^{1+\infty}) \to \infty$$

the system $\boxed{\text{is not stable}}$

Causality check

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\,d\tau$$

$$h(t-\tau) = e^{1-(t-\tau)} u(1-(t-\tau)) = e^{\tau+1-t} u(\tau-t+1) = \begin{cases} e^{\tau+1-t}, & \tau > t-1 \\ 0, & \tau < t-1 \end{cases}$$

$$y(t) = \int_{t-1}^{\infty} x(\tau)\, e^{\tau-t+1}\,d\tau$$

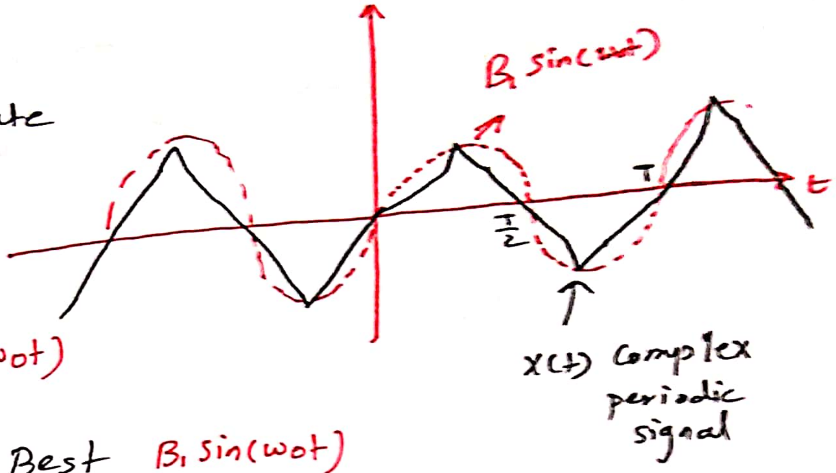
The system is $\boxed{\text{not Causal}}$ because the output requires future value of input

5/3/2020 .

Suggested problems:-

Chapter (2)	Chapter (3)
~~2.1.(a)~~	3.1 (a)
~~2.5~~ .(iv)	3.3
~~2.5~~	3.4
~~2.7 (d)~~	3.6
~~2.16~~	3.7 (e)(f)
~~2.19 (a)~~	3.12
~~2.20 c (iii)~~	3.17
2.26 (a)	3.19
2.27 g	
2.29	

Signals and systems
Lecture #2
19-3-2020



* We aim to approximate the periodic signal $x(t)$ with simpler Sinusoidal Function, namely $B_1 \sin(\omega_0 t)$

* We want to choose Best $B_1 \sin(\omega_0 t)$
* There several techniques to find the Best Sinusoids
* One of the procedure is: To minize the erro function $(e(t))$

$$e(t) = x(t) - B_1 \sin(\omega_0 t)$$

→ when $e(t) = 0 \rightarrow x(t) = B_1 \sin(\omega_0 t)$

* In mathematics: The Function of Error is denoted as $J[e(t)]$

$J[e(t)]$: cost Function of Error.

→ One of well-known Cost functions is minimizing mean-square Error, such that.

$$J[e(t)] = \frac{1}{T_0} \int_0^{T_0} e^2(t)\, dt = \frac{1}{T_0} \int_0^{T_0} [x(t) - B_1 \sin(\omega_0 t)]^2\, dt.$$

→ when minizing $J[e(t)]$ → Then, we evalute B_1 such that $J[e(t)]$ is minimum.

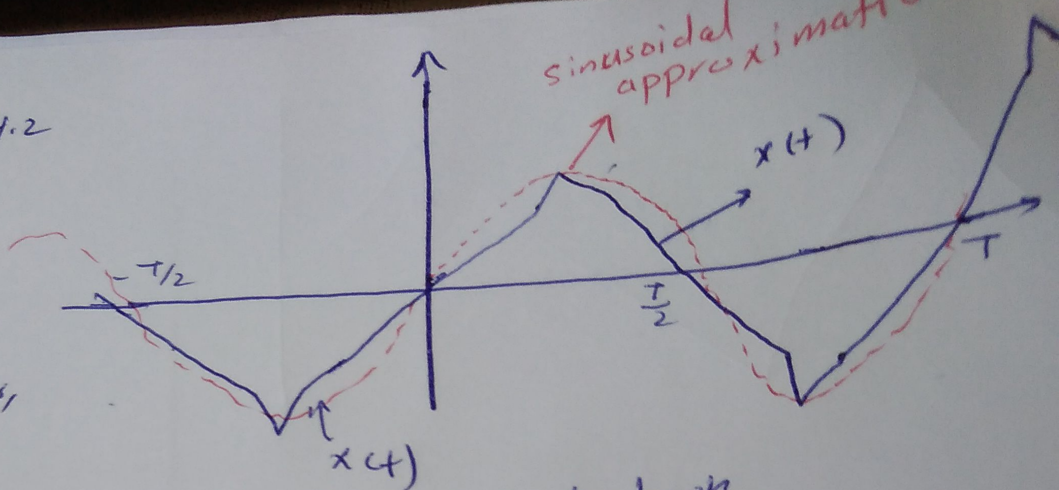
→ $J[e(t)]$ can be minimized through satisificaition of the following

$$\frac{dJ[e(t)]}{dB_1} = 0 = \frac{1}{T_0} \int_0^{T_0} 2\,[x(t) - B_1 \sin(\omega_0 t)](-\sin \omega_0 t)\, d$$

notes about Fig. 4.2

1) $x(t)$ is periodic signal

2) using Fourier series, we can approximate $x(t)$ by sinusoidal function, as shown



4.2 in text book

→ the sinusoidal approximation can be represented as : $B_1 \sin(\omega_0 t)$

→ Next lecture, we will learn how to find best $B_1 \sin(\omega_0 t)$ that approximately represents $x(t)$

End of lecture #1

Thank you

→ In mathematics, there are several techniques that can be used to approximate complex signals with simpler signals. In General, three requirements should be satisfied to consider the approximated signals as good signals to represent the original signals, which are:-

1) The approximated signals should be easier (simpler) than the origrnal signal.
2) The problem must be linear,
3) The contributions of simpler solutions to the total solution must be negligible after considering few terms.

↳ We will explain the above later on.

→ In this chapter, we will consider the "Fourie series"

Fourier series: it is a mathematical procedure that is used to express "Complicated periodic signals" as a sum of "Sinusoidal functions"

In summery

Fourier series: periodic signal ⟶ Summation of Sinusoidal signals

Why Do we use Fourier series?

→ * it is difficult to evalute the output of LTI systems for periodic signals

* So, we write that periodic signals as a sum of sin(–) and Cos(–) functions.

Lecture # 1 15-03-2020

* Signals and system

Ch.4 Fourier Series

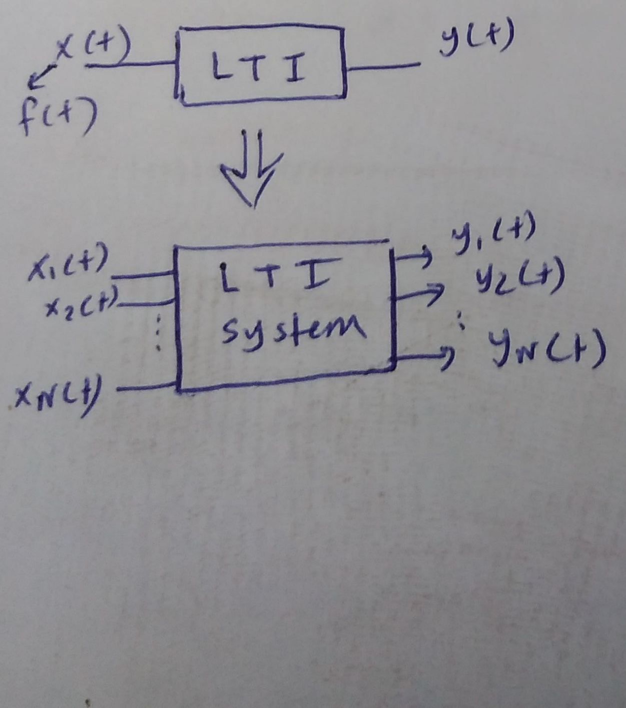
→ Representing a complicated signal as a sum of simpler signals is one of key techniques in engineering.

→ In such techniques, instead of solving Complex signals, dealing with simpler signals is easier than dealing of complicated signals.

→ For example, a function $f(t)$ can be approximated using Taylor's series expansion as a sum of simpler functions, such that

$$\underbrace{f(t)}_{x(t)} = \underbrace{f(0)}_{x_1(t)} + \underbrace{f'(0)t}_{x_2(t)} + \underbrace{f''(0)\frac{t^2}{2!}}_{x_3(t)} + \cdots \quad (1)$$

→ If we cannot evaluate the output of a "LTI" system for the input $f(t)$, we can evaluate the output for each term in (1), and then, the overall output can be determined by summation of each output.



$$\Rightarrow \quad y(t) = y_1(t) + y_2(t) + \cdots + y_N$$

(1) lecture ~~on~~ one

Lecture #2

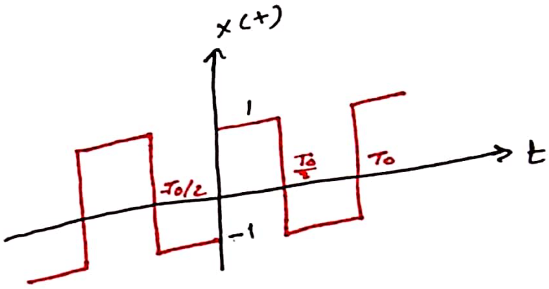
Considering the above

$$B_1 = \frac{2}{T_0}\int_0^{T_0} x(t)\sin(\omega_0 t)\,dt$$

↳ This B_1 minimize the mean square error.

Example (4-1)

* Find the Best approximation, in a mean square error sense, of the square wave given in the figure.



Solution

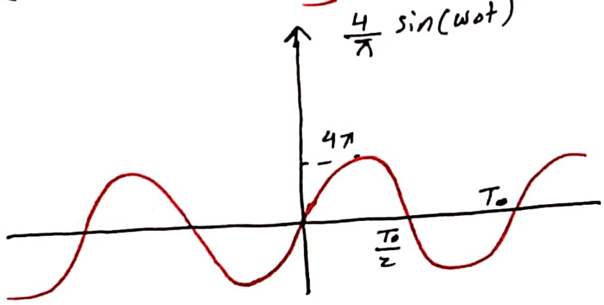
$$B_1 = \frac{2}{T_0}\int_0^{T_0} x(t)\sin(\omega_0 t)\,dt.$$

$$= \frac{2}{T_0}\int_0^{T_0/2}\sin(\omega_0 t)\,dt - \frac{2}{T_0}\int_{\frac{T_0}{2}}^{T_0}\sin(\omega_0 t)\,dt. \qquad \omega_0 = \frac{2\pi}{T_0}.$$

$$= \frac{2}{T_0}\left[\left(-\frac{\cos(\omega_0 t)}{\omega_0}\Big|_0^{T_0/2}\right) - \left(-\frac{\cos(\omega_0 t)}{\omega_0}\Big|_{\frac{T_0}{2}}^{T_0}\right)\right]$$

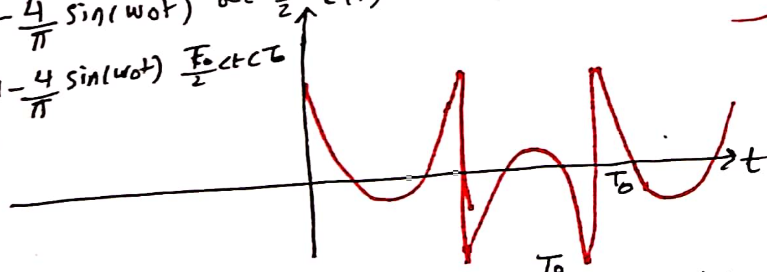
$$= \frac{2}{T_0}\left[-\frac{1}{\omega_0}\left(\cos(\pi) - \cos(0)\right) - \left(-\cos(2\pi) + \cos\pi\right)\right]$$

$$= \frac{4}{\pi}$$



$$\rightarrow e(t) = x(t) - \frac{4}{\pi}\sin(\omega_0 t)$$

$$= \begin{cases} 1 - \frac{4}{\pi}\sin(\omega_0 t) & 0 < t < \frac{T_0}{2} \\ -1 - \frac{4}{\pi}\sin(\omega_0 t) & \frac{T_0}{2} < t < T_0 \end{cases}$$



note that $\frac{1}{T_0}\int_0^{T_0} e(t)\,dt = 0$ (Area)

However, $B_1\sin(\omega_0 t) \neq x(t)$

That why we consider ~~Area~~ mean square error.

another cost function is minimizing the averge value of e(t), such that

$$J[e(t)] = \frac{1}{T_0} \int_0^{T_0} |e(t)| dt$$

→ please not the mean square error is the best among these cost functions.

End of Lecture Two

$\boxed{4-2}$ Fourier series

Given a signal x(t) such that

$$x(t) = 10 + 3\cos(\omega_0 t) + 5\cos(2\omega_0 t + 30^\circ) + 4\sin(3\omega_0 t)$$

→ This will be explained next lecture

In the previous Lecture:-

* Any periodic signal can be represented as a sum of sinusoidal signals
* This sinusoidal representation is Fourier Series (FS)
* The cofficients of sinusoidal signals can be found using average mean square error

4.2 Fourier Series

* Note that $\sin(\omega_0 t)$ and $\cos(\omega_0 t)$ can be written as

$$\cos(\omega_0 t) = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$$

$$\sin(\omega_0 t) = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}$$

~~Exponential~~ Exponential form

* With these expressions, write the following sinusoidal function in exponential form

$$x(t) = 10 + 3\cos(\omega_0 t) + 5\cos(2\omega_0 t + 30) + 4\sin(3\omega_0 t)$$

$$3\cos(\omega_0 t) = \frac{3}{2}\left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right]$$

$$5\cos(2\omega_0 t + 30) = \frac{5}{2}\left[e^{j(2\omega_0 t + 30)} + e^{-j(2\omega_0 t + 30)}\right]$$

$$= 2.5\left[e^{j\frac{\pi}{6}} e^{j2\omega_0 t} + e^{-j\frac{\pi}{6}} e^{-j2\omega_0 t}\right]$$

$$4\sin(3\omega_0 t) = \frac{4}{2j}\left[e^{j3\omega_0 t} - e^{-j3\omega_0 t}\right]$$

$$= 2(-j)\left[e^{j3\omega_0 t} - e^{-j3\omega_0 t}\right]$$

$$= 2e^{-j\frac{\pi}{2}}\left[e^{j\omega_0 3t} - e^{-j3\omega_0 t}\right]$$

$$x(t) = 10 + 1.5e^{j\omega_0 t} + 1.5e^{-j\omega_0 t} + 2.5e^{j\frac{\pi}{6}} e^{j2\omega_0 t} + 2.5e^{j\frac{\pi}{6}} e^{j2\omega_0 t} + 2e^{-j\frac{\pi}{2}} e^{j\omega_0 3t} - 2e^{-j\frac{\pi}{2}} e^{-j3\omega_0 t}$$

General notes

1. $e^{x+y} = e^x e^y$
2. $e^{x-y} = e^x e^{-y}$
3. $\frac{1}{j} = -j$
4. $e^{j\frac{\pi}{2}} = \cos(\frac{\pi}{2}) + j\sin(\frac{\pi}{2}) = j$
5. $e^{-j\frac{\pi}{2}} = \cos(\frac{\pi}{2}) - j\sin(\frac{\pi}{2}) = -j$

$$x(t) = (2e^{j\pi/2})e^{-j3w_0t} + (2.5e^{-j\frac{\pi}{6}})e^{-j2w_0t} + 1.5e^{-jw_0t} + 10 + 1.5e^{jw_0t} + (2.5e^{j\pi/6})e^{j2w_0t} + (2e^{-j\pi/2})e^{j3w_0t}$$

→ This equation can be written in compact form

$$x(t) = C_{-3}e^{-j3w_0t} + C_{-2}e^{-j2w_0t} + C_{-1}e^{-jw_0t} + C_0 + C_1e^{jw_0t} + C_2e^{jw_02t} + C_3e^{j3w_0t}$$

$$= \sum_{k=-3}^{3} C_k e^{jkw_0t}$$

$k=-3 \rightarrow C_k = 2e^{j\frac{\pi}{2}} = 2\angle 90^\circ$

$k=-2 \rightarrow C_{-2} = 2.5e^{-j\frac{\pi}{6}} = 2.5\angle -30^\circ$

$k=-1 \rightarrow C_{-1} = 1.5$

$k=0 \rightarrow C_0 = 10$

$k=3 \rightarrow C_3 = 2e^{-j\frac{\pi}{2}} = 2\angle -90^\circ$

$k=2 \rightarrow C_2 = 2.5e^{j\frac{\pi}{6}} = 2.5\angle 30^\circ$

$k=1 \rightarrow C_1 = 1.5$

Note the following:-

1) $C_k = C_k^*$, $*$: Complex Conjugate

2) Sinusoidal function can be written as

$$\sum_{k=-\infty}^{\infty} C_k e^{jw_0kt}$$

3) Fourier series express any periodic function
↳ as sum of sinusoidal function
↳ any sinusoidal function can be expressed as

$$\sum_{k=-\infty}^{\infty} C_k e^{jkw_0t}$$

↳

$$f(t) \xrightarrow{F.S} \sum_{k=-\infty}^{\infty} C_k e^{jkw_0t}$$

eriodic
unction

This is a complex exponential form of fourier series

Scanned with CamScanner

In general

$f(t)$ "periodic Signal"

Can be written in fourier series in one of these forms

Exponential Form

$$\sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}$$

"Exponential Form"

Combined trigonometric

$$C_0 + \sum_{k=1}^{\infty} 2|C_k| \cos(\omega_0 t + \theta_k)$$

trigonometric

$$A_0 + \sum_{k=1}^{\infty} \left[A_k \cos k\omega_0 t + B_k \sin k\omega_0 t\right]$$

$$2C_k = A_k - jB_k$$

$$C_0 = A_0$$

The big question, How to find Fourier series Cofficient C_k

C_k are found using average mean square error

$$C_k = \frac{1}{T_0} \int_{T_0} x(t)\, e^{-jk\omega_0 t}\, dt$$

When $k = 0$

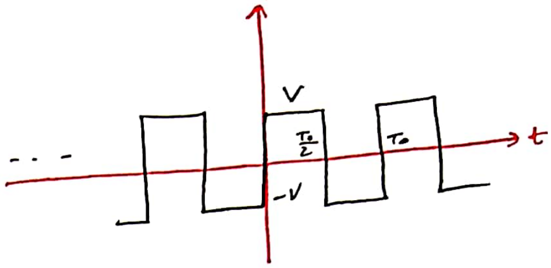
$$C_0 = \frac{1}{T_0} \int_{T_0} x(t)\, dt \Rightarrow \text{average value of } x(t).$$

dc-value.

Examples:-

Calculate Fourier Series Cofficients for the square wave shown in Fig. below

$$x(t) = \begin{cases} V, & 0 \le t \le \frac{T_o}{2} \\ -V, & \frac{T_o}{2} \le t \le T_o \end{cases}$$



$$C_k = \frac{1}{T_o}\int_{T_o} x(t)\, e^{-jk\omega_o t}\, dt \,.$$

$$= \frac{1}{T_o}\int_0^{T_o/2} V e^{-jk\omega_o t}\, dt + \frac{1}{T_o}\int_{\frac{T_o}{2}}^{T_o} (-V)\, e^{-jk\omega_o t}\, dt$$

$$= \frac{V}{T_o}\left(-\frac{1}{jk\omega_o} e^{-jk\omega_o t}\Big|_0^{T_o/2}\right) - \frac{V}{T_o}\left(-\frac{1}{jk\omega_o}\right) e^{-jk\omega_o t}\Big|_{\frac{T_o}{2}}^{T_o}$$

$$= \frac{-V}{jT_o k\omega_o}\left[e^{-jk\omega_o \frac{T_o}{2}} - 1\right] + \frac{V}{jT_o k\omega_o}\left[e^{-jk\omega_o T_o} - e^{-jk\omega_o \frac{T_o}{2}}\right]$$

$$= \frac{-V}{2\pi jk}\left[e^{-jk\pi} - 1\right] + \frac{V}{j2\pi k}\left[e^{-j2\pi k} - e^{-j\pi k}\right]$$

note:- $\omega_o T_o = \frac{2\pi}{T_o} \cdot T_o = 2\pi$

$-\frac{1}{j} = j$

$$C_k = \frac{jV}{2\pi k}\left[e^{-jk\pi} - e^{-j0} - e^{-j2\pi k} + e^{-j\pi k}\right]$$

if k Even, $k=2$ For Example

$$C_k = \frac{jv}{4\pi}\left[1 + 1 - 1 - 1\right] = 0$$

if k odd

$$C_k = -\frac{2jV}{k\pi} = \frac{2V}{k\pi}\angle -90^\circ$$

Then

$$x(t) = \sum_{\substack{k=-\infty \\ k=odd}}^{\infty} \frac{2V}{k\pi}\, e^{-j\frac{\pi}{2}}\, e^{jk\omega_o t}$$

Notes

$e^{-j\pi k} = \cos(\pi k) - j\sin(\pi k)$

if k: Even, $k=2$

$e^{-j2\pi} = \underbrace{\cos(2\pi)}_{1} - j\underbrace{\sin(2\pi)}_{0}$

$= 1$

if $k = odd$, $k=1$

$e^{-j\pi} = \cos(\pi) - j\sin(\pi)$

$= -1$

Euler's Formula and General Expressions

$$\cos(w_0 t) = \frac{e^{jw_0 t} + e^{-jw_0 t}}{2}$$

$$\sin(w_0 t) = \frac{e^{jw_0 t} - e^{-jw_0 t}}{2j}$$

$$e^{jx} = \cos(x) + j\sin(x)$$

$$e^{-jx} = \cos(-x) + j\sin(-x) = \cos(x) - j\sin(x)$$

$$\frac{1}{j} = -j.$$

$$|x| \angle \theta = |x|\cos(\theta) + j|x|\sin(\theta)$$

$$-j = 1 \angle -\pi/2 = \cos(\pi/2) - j\sin(\pi/2) = -j = \frac{1}{j}$$

$$j = 1 \angle \pi/2$$

Any periodic signal $x(t)$ can be written in Fourier Series expansion as

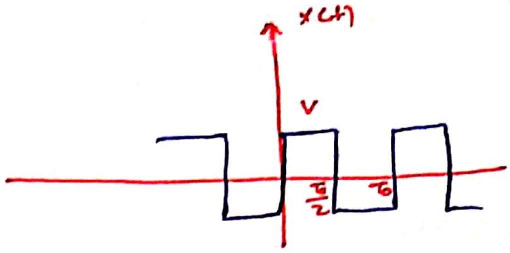
$$x(t) = \sum_{k=\infty}^{\infty} C_k e^{jkw_0 t}$$, This is exponential form

$$C_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-jkw_0 t} dt$$

C_k: fourier series Cofficient

End of Lecture
22/03/2020
Dr. Haitham

→ In the previous lecture, we find Fourier series cofficient for squqre wave shown in fig.



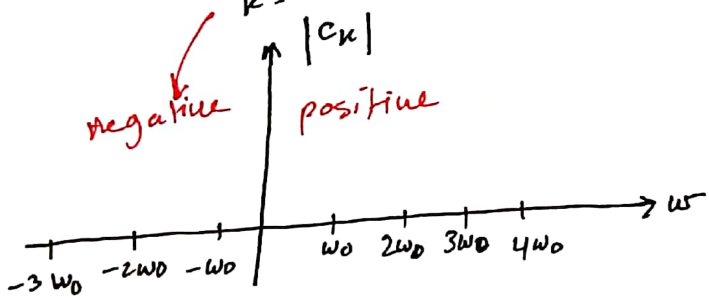
$$C_k = \begin{cases} -\dfrac{2jV}{k\pi} = \dfrac{2V}{k\pi}\angle -90^{\circ}, & k \text{ odd} \\ 0, & k \text{ even} \end{cases}$$

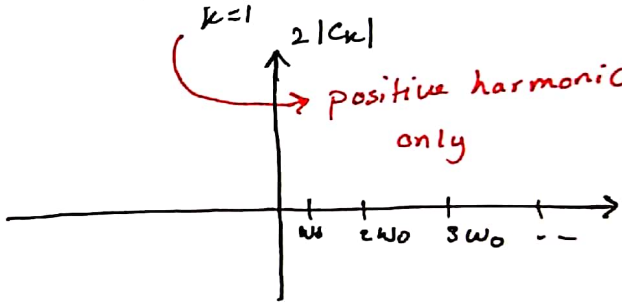
$\boxed{4.3}$ Frequency spectra.

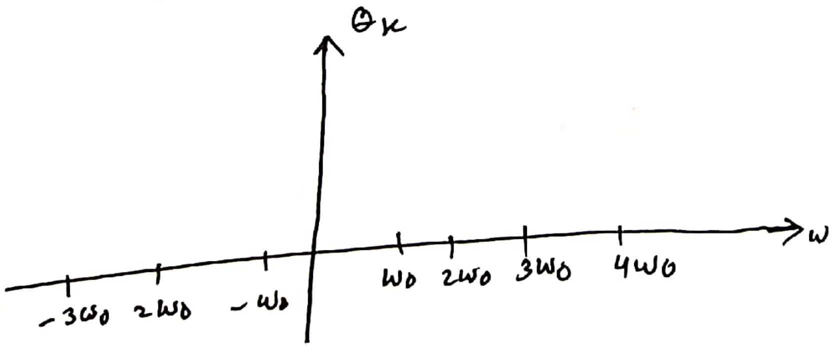
Frequency spectra :- is a graph that shows the amplitudes $(2|C_k|)$ and the phase $(\arg C_k)$ of the harmonic terms (Fourier series cofficients) of a periodic signal.

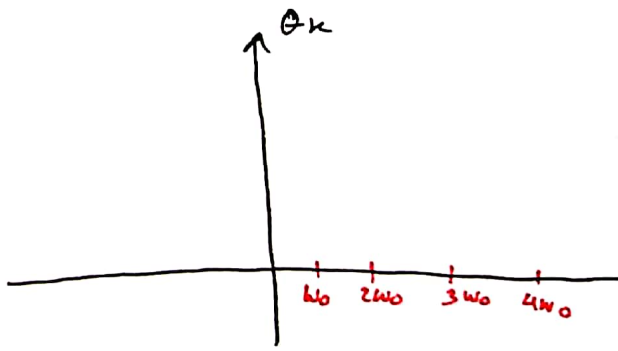
Frequency Spectra

- Exponential $\sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}$
- Combined trigonometric $C_0 + \sum_{k=1}^{\infty} 2|C_k| \cos(k\omega_0 t + \theta_k)$



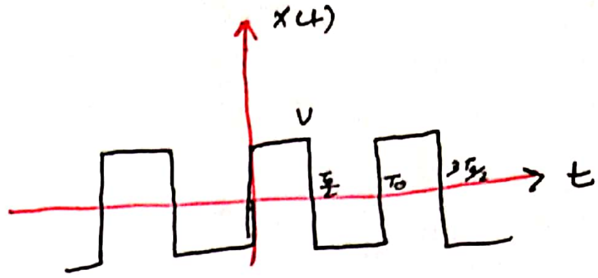






These spectrums are Called line Spectrum

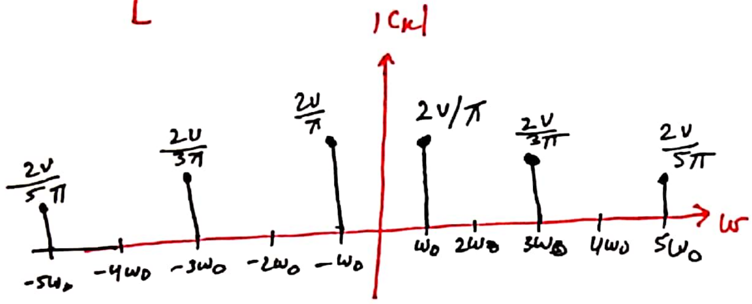
Plot frequency spectrums for the square wave shown

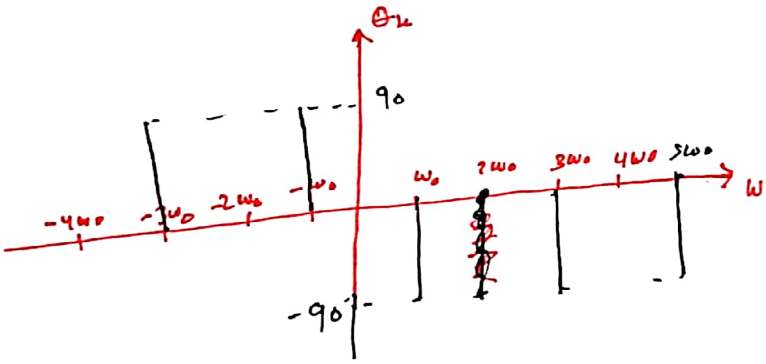


- We found in the previous example the Fourier Series cofficient

$\boxed{\text{In Complex expotential form}}$

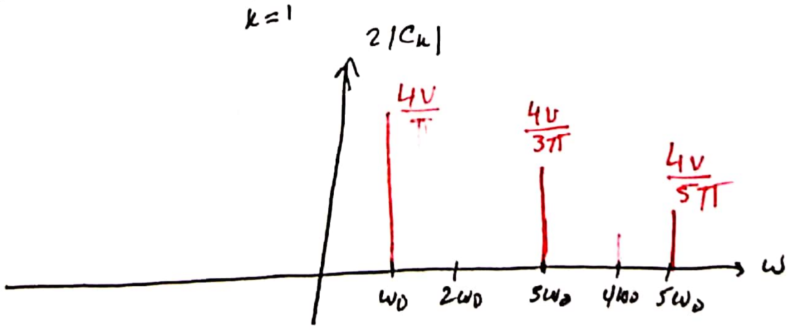
$$C_k = \begin{cases} \frac{-2jV}{k\pi} = \frac{2V}{k\pi} \angle -90^\circ & , k \text{ odd} \\ 0 & , k \text{ even} \end{cases}$$

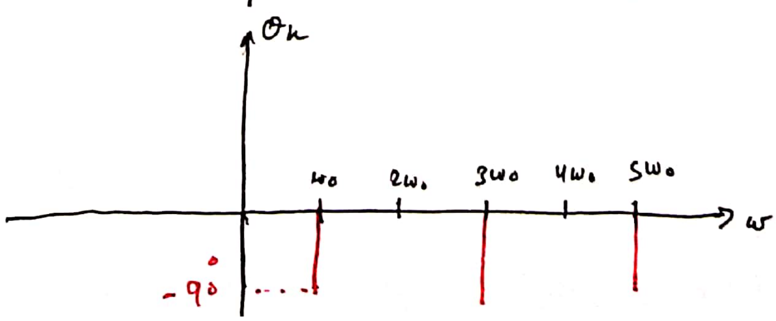




$\boxed{\text{Combined trigonometric}}$

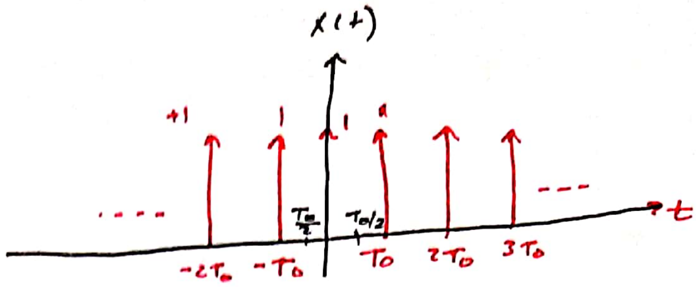
$$x(t) = C_0 + \sum_{k=1}^{\infty} 2|C_k| \cos(k\omega_0 t + \theta_k)$$





Example 4-5 29-3-2020 page(3)

Find the Fourier Series for the impulse train shown in figure. plot frequency spectrum.



Complex exponential

$$x(t) = \sum C_k e^{jk\omega_0 t}.$$

$$C_k = \frac{1}{T_0}\int_0^{T_0} x(t)\, e^{-jk\omega_0 t}\, dt.$$

$$= \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{T_0/2} \delta(t)\, e^{-jk\omega_0 t}\, dt$$

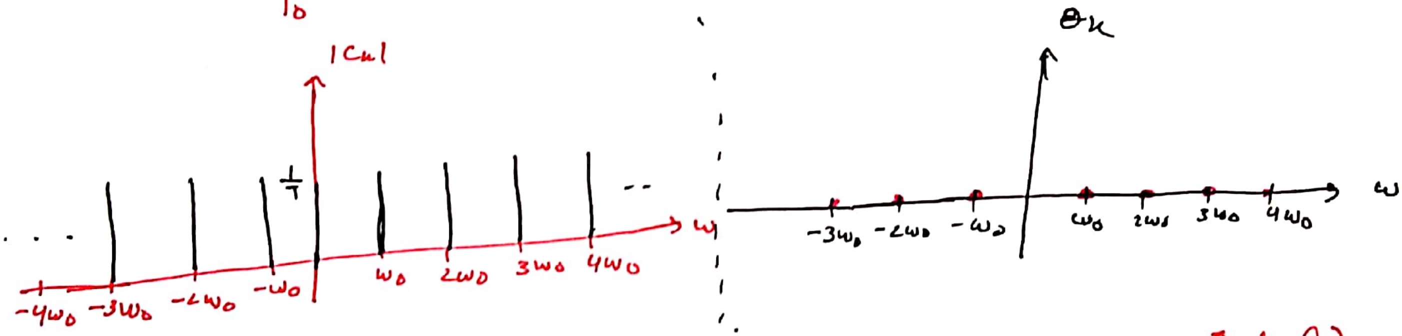
$$= \frac{1}{T_0} e^{-jk\omega(0)} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \delta(t)\, dt \quad (1)$$

$$= \frac{1}{T_0}$$

Notes

$$\int_{-\infty}^{\infty} f(t)\,\delta(t)\,dt = f(0)\int_{-\infty}^{\infty}\delta(t)\,dt = f(0)$$

$$1$$



Frequency Spectrum (Complex exponential)

$$x(t) = \sum_{k=-\infty}^{k} \frac{1}{T_0} e^{jk\omega_0 t}$$

Combined trigonometric

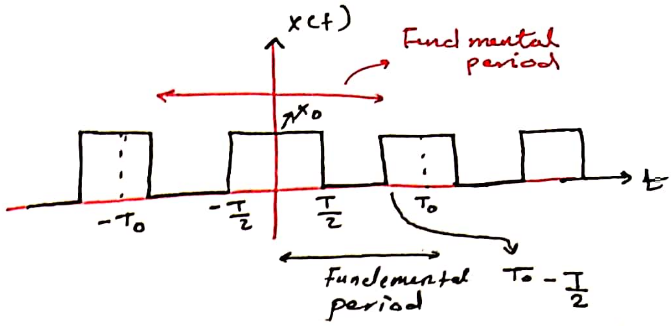
$$x(t) = C_0 + \sum_{k=1}^{\infty} 2|C_k| \cos(k\omega_0 t + \theta_k)$$

$\theta_k = 0$

$$= \frac{1}{T_0} + \sum_{k=1}^{\infty} \frac{2}{T_0}\cos(k\omega_0 t)$$

Example 4-6

* plot frequency spectrum of rectangular pulse train shown in figure



* We have First to Find C_k : Fourier series cofficients.

$$C_k = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T_0}\left[\int_0^{\cancel{T}/2} X_0\, e^{-jk\omega_0 t}\, dt + \underbrace{\cancel{\int_{T/2}^{T_0 - T/2} 0\, e^{-j\omega k t}\, dt}}_{\text{zero}} + \int_{T_0 - T/2}^{T_0} X_0\, e^{-jk\omega_0 t}\, dt\right]$$

$$= \left(\frac{-X_0}{T_0\, jk\omega_0}\right)\left[e^{-jk\omega_0 T/2} - 1\right] + \left(\frac{-X_0}{jk\omega_0 T_0}\right)\left[\overset{1}{\cancel{e^{-jk\omega_0 T_0}}} - e^{-jk\omega_0(T_0 - \frac{T}{2})}\right]$$

$$= \frac{X_0}{jk\omega_0 T_0}\left[1 - e^{-jk\omega_0 \frac{T}{2}}\right] + \frac{X_0}{jk\omega_0 T_0}\left[e^{jk\omega_0 \frac{T}{2}} - 1\right] \qquad \omega_0 T_0 = 2\pi$$

zero

$$= \frac{X_0}{k\omega_0 T_0}\left[\frac{e^{jk\omega_0 \frac{T}{2}} - e^{-jk\omega_0 \frac{T}{2}}}{j}\right]$$

$$= \frac{X_0}{k\omega_0 T_0}\left[2 \sin\left(k\omega_0 \frac{T}{2}\right)\right] * \frac{\frac{T}{2}}{\frac{T}{2}}$$

$$= \frac{2 X_0\, T/2}{T_0}\left[\frac{\sin(k\omega_0 T/2)}{k\omega_0 T/2}\right]$$

$$C_k = \frac{X_0 T}{T_0}\, \mathrm{sinc}\left(k\omega_0 \frac{T}{2}\right)$$

Hence, $x(t)$ can be written as

$$x(t) = \sum_{k=-\infty}^{\infty} \frac{X_0 T}{T_0}\, \mathrm{sinc}\left(k\omega_0 \frac{T}{2}\right) e^{jk\omega_0 t}$$

H.w plot C_k

notes

$$e^{-jk\omega_0 T_0} = e^{-jk2\pi} = \cos(k2\pi) - j\sin(2k\pi) = \boxed{1}$$

$$e^{-jk\omega_0(T_0 - \frac{T}{2})} = e^{-jk\omega_0 T_0}\, e^{jk\omega_0 \frac{T}{2}} = e^{jk\omega_0 \frac{T}{2}}$$

$$\sin(x) = \frac{e^{jx} - e^{-jx}}{2j}$$

$$\mathrm{sinc}(x) = \frac{\sin(x)}{x}$$

4.5 System Analysis

* In this section, we consider LTI systems with periodic inputs

* The system linearity allows use of superposition.
↳ periodic signals can be represented as sum of sinusoidals (complex exponentials)
↳ the system response is a sum of steady-state sinusoidal responses

↳ Due to the fact that sinusoidal and complex exponential function are studied in terms of frequency (frequency spectrum), we consider the variation of the sinusoidal functions with frequency.

* Consider the following LTI system

LTI system

$$x(t) \xrightarrow{h(t)} y(t)$$

periodic signal

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

✓ we define the transfare function $H(s)$

$$H(s) = \int_{-\infty}^{\infty} h(t)\, e^{-st}\, dt$$

- if the ~~output~~ Input of the LTI system with transfare function $H(s)$ is Complex exponential Called $Xe^{s_1 t}$ → Then, the output Can be written as

$$X e^{s_1 t} \longrightarrow X\, H(s_1)\, e^{s_1 t} \quad , \quad X = |X|\, e^{j\phi t}.$$

- If the input is sinusoidal

$$|X| \cos(\omega_1 t + \phi) \longrightarrow |X|\,|H(j\omega_1)| \cos\left[\omega_1 t + \phi + \angle H(j\omega_1)\right]$$

* if $x(t)$ is periodic signal, then the exponential complex form using fourier series can be written as

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{j\omega kt} \qquad C_k : \text{Complex Cofficient.}$$

Therefore

$$\sum_{k=-\infty}^{\infty} C_k e^{j\omega kt} \xrightarrow[\text{system}]{LTI} \sum_{k=-\infty}^{\infty} \underbrace{H(jk\omega_0)}_{\text{Complex}} \underbrace{C_{kx}}_{\text{Complex}} e^{jk\omega_0 t}$$

$$= y_{ss}(t) \leftarrow \text{steady state response.}$$

$$y_{ss}(t) = \sum_{k=-\infty}^{\infty} C_{ky} e^{jk\omega_0 t}, \qquad C_{ky} = H(jk\omega_0) C_{kx}$$

↳ This equation gives Fourier Cofficients of the output signal $y_{ss}(t)$.

$\boxed{\text{Combined trigonometric form}}$

$$x(t) = C_{0x} + \sum_{k=1}^{\infty} 2|C_{kx}| \cos(k\omega_0 t + \theta_{kx}).$$

$$|x| \cos(\omega_1 t + \phi) \xrightarrow[\text{system}]{LTI} |x| \, |H(j\omega_1)| \cos[\omega_1 t + \phi + \angle H(j\omega_1)]$$

Using superposition

$$y_{ss}(t) = C_{0y} + \sum_{k=1}^{\infty} 2|C_{ky}| \cos(k\omega_0 t + \theta_{ky})$$

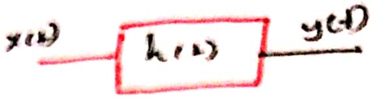
$$C_{ky} = |C_{ky}| \angle \theta_{ky} = H(jk\omega_0) C_{kx}$$

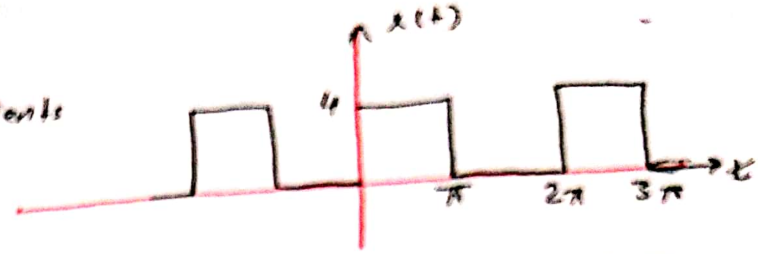
Example 4.7

Given the LTI system shown in figure

if $h(t) = e^{-t} u(t)$

and $x(t)$ is square wave as shown in figure below





Find the fourier series cofficients of the output.

$$H(s) = \int_{-\infty}^{\infty} h(t) e^{-st} dt$$

$$H(s) = \int_{-\infty}^{\infty} e^{-t} u(t) e^{-st} dt = \int_0^{\infty} e^{t(-1-s)} dt = \frac{1}{(-1-s)} e^{-t(1+s)} \Big|_0^{\infty}$$

$$= \frac{-1}{1+s} (0-1) = \boxed{\frac{1}{1+s}}$$ ← ~~impt~~ impulse freq. Response

Now we find the fourier Series cofficients of the input → $x(t)$

- Using table (4.3) and lecture notes

$$x(t) = C_{0x} + \sum_{\substack{n=-\infty \\ k \neq 0}}^{\infty} C_{kx} e^{jk\omega_0 t} = 2 + \sum_{\substack{n=-\infty \\ k: odd}}^{\infty} \frac{4}{\pi k} e^{-j\frac{\pi}{2}} e^{jkt}$$

Exponential form

$$y_o(t) = \sum_{n=-\infty}^{\infty} C_{ky} e^{jk\omega_0 t}$$

$$C_{ky} = H(jk\omega_0) C_{kx}$$

note

$$H(s) = \frac{1}{1+s}$$

$$H(jk\omega_0) = \frac{1}{1+jk\omega_0}$$

$$H(0) = 1$$

$k=0$

$$C_{0y} = H(0) C_{0x} = C_{0x} = 2$$

$k=1$

$$C_{1y} = H(j\omega_0) C_{1x} = \frac{4}{\pi} \angle -90^\circ \left(\frac{1}{1+j\omega} \right)$$

($\frac{1}{1+j\omega_0}$, $\frac{4}{\pi}$)

$$= \frac{4}{\pi} \angle -90^\circ \left(\frac{1-j\omega}{1-j\omega} \right) \left(\frac{1}{1+j\omega} \right) = \frac{4(1-j\omega)}{2}$$

$$= \frac{4}{\pi} \angle -90^\circ \cdot \frac{1}{\sqrt{2}} \angle -45$$

$$= \frac{4}{\sqrt{2}\,\pi} \angle -135^\circ$$

note

$$1 - j\omega = \frac{1}{\sqrt{2}} \angle -45$$

$$= \frac{1}{\sqrt{2}} [\cos(45) - j\sin(45)]$$

$$= 1 - j$$

$$\omega = \frac{2\pi}{T_0}, \quad T_0 = 2$$

$$\omega_0 = 1$$

$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$, LTI system

* The steady-state output signal can be written as

$$y_{ss}(t) = \sum_{k=-\infty}^{\infty} H(jk\omega_0)\, C_{kx}\, e^{jk\omega_0 t}$$ Complex exponential form

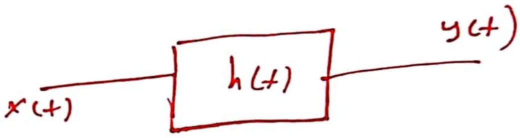
$$y_{ss}(t) = C_{0y} + \sum_{k=1}^{\infty} 2\,|C_{ky}| \cos(k\omega_0 t + \theta_{ky})$$

$$C_{ky} = |C_{ky}| \angle \theta_{ky} = H(jk\omega_0)\, C_{kx}$$

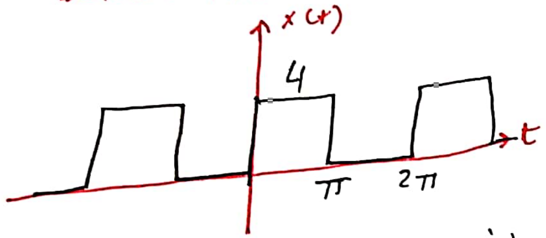
Example 4-7

Suppose that for LTI system of figure below, the impulse response and the transfare function are given by

$$h(t) = e^{-t}u(t) \Leftrightarrow H(s) = \frac{1}{1+s}$$



If $x(t)$ is the square-wave shown below. Find $y_{ss}(t)$



$$y_{ss}(t) = \sum_{k=-\infty}^{\infty} H(jk\omega_0)\, C_{kx}\, e^{jk\omega_0 t}.$$

Therefore, we have to find $H(s)$

$$H(s) = \int_{-\infty}^{\infty} h(t)\, e^{-st}\, dt = \int_{-\infty}^{\infty} e^{-t} u(t)\, e^{-st}\, dt = \int_{0}^{\infty} e^{-t} e^{-st}\, dt$$

$$= \frac{1}{1+s}.$$

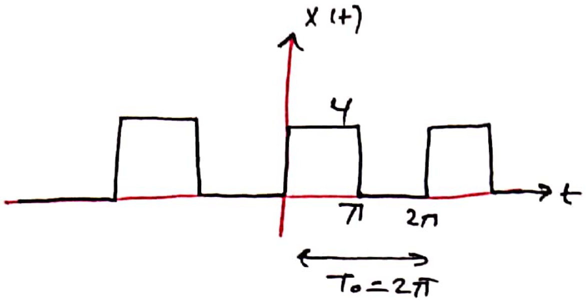
Then, we find fourier series of the input $x(t)$.

$$C_{0x} = 2$$
$$C_{kx} = \frac{4}{\pi k} e^{-j\frac{\pi}{2}} , \quad k: \text{odd}$$

please verify this !!

page(2)

$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}$$



$$C_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-jk\omega_0 t} dt.$$

$$= \frac{1}{2\pi} \int_0^{\pi} 4 e^{-jk\omega_0 t} dt.$$

$$= \frac{4}{2\pi} \left(-\frac{1}{jk\omega_0}\right) e^{-jk\omega_0 t} \Big|_0^{\pi}$$

$\omega_0 = \frac{2\pi}{T_0}$

$\omega_0 = 1$

$$= \frac{-4}{2\pi jk\omega_0} \left[e^{-jk\cancel{\omega_0}\pi} - e^{-j0} \right]$$

2π

$$= -\frac{4}{2\pi jk\omega_0} \left[e^{-jk\pi} - 1 \right]$$

$$\boxed{\begin{array}{l} k: \text{even} \rightarrow e^{-j2\pi} = \cos(2\pi) - j\sin(2\pi) = 1 \\ \text{There fore} \\ C_k = 0 \quad , \quad k \text{ even} \end{array}}$$

if k odd $\rightarrow e^{-j\pi} = \cos(\pi) - j\sin(\pi) = -1$

There fore

$$C_k = \frac{2 \cdot j}{\pi k}(-2) = \frac{-4j}{k\pi} = \frac{4}{k\pi} \angle \frac{-\pi}{2} = \frac{4}{k\pi} e^{-j\pi/2}$$

* Therefore

$$y_{ss}(t) = \sum_{k=-\infty}^{\infty} H(jk\omega_0) C_{kx} e^{jk\omega_0 t} \text{ ., Complex form}$$

$$\boxed{H(jk\omega_0) = \frac{1}{1+jk\omega_0} = \frac{1}{1+jk} \quad , \quad \omega_0 = 1}$$

$$y_{ss}(t) = C_{0y} + \sum_{k=1}^{\infty} 2|C_{ky}| \cos(k\omega_0 t + \theta_{ky}) \text{ , Combined trigonometric}$$

Table 4-6 (Text book)

k	$H(jk\omega_0)$	C_{kx}	C_{ky}	$\lvert C_{kx}\rvert$	$\lvert C_{ky}\rvert$
0	1	2	2	2	2
1	$\frac{1}{\sqrt{2}}\angle -45$	$\frac{4}{\pi}\angle -90^\circ$	$\frac{4}{\pi\sqrt{2}}\angle -135$		
3	$\frac{1}{\sqrt{10}}\angle -71.6$	$\frac{4}{3\pi}\angle -90^\circ$	$\frac{4}{3\pi\sqrt{10}}\angle -161.6$		
5	$\frac{1}{\sqrt{26}}\angle -78.7^\circ$	$\frac{4}{5\pi}\angle -90^\circ$	$\frac{4}{5\pi\sqrt{26}}\angle -168.7^\circ$.		

To explain the ~~are~~ above table, consider $k=3$

$$\boxed{H(j3\omega_0) = \frac{1}{1+j3\omega_0}, \quad \omega_0 = 1 \qquad H(j3) = \frac{1}{1+j3} \Rightarrow \frac{1-j3}{1+9} = \frac{1-j3}{10}}$$

Skip this please !!

In General

$$H(jk\omega_0)\Big|_{\omega_0=1} = \frac{1}{1+jk} = \frac{1}{\sqrt{1+k^2}}\angle \tan^{-1}(-k)$$

let us try it when $k=1$

$$H(j\omega_0)\Big|_{\omega_0=1} = \frac{1}{1+j} = \frac{1}{\sqrt{2}}\angle \tan^{-1}(-1), \quad \tan^{-1}(-1) = -45$$

$$= \frac{1}{\sqrt{2}}\angle -45 = \frac{1}{\sqrt{2}}\left[\cos(45) - j\sin(45)\right]$$

$$\rightarrow \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right]$$

$$= \frac{1}{2}\left[1 - j\right]$$

notes
$\cos(45) = \frac{1}{\sqrt{2}}$
$\sin(45) = \frac{1}{\sqrt{2}}$

note that: $\frac{1}{1+j} = \frac{1}{2}(1-j)$.

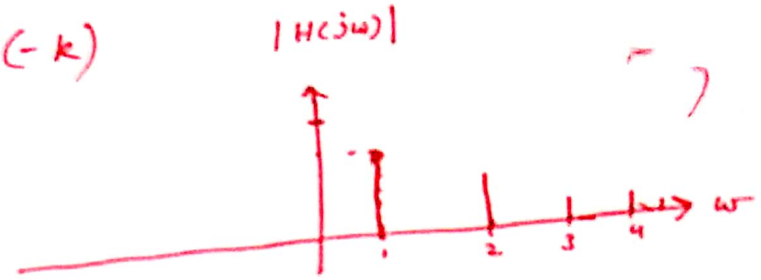
How!!! → Complex Conjugate

$$\frac{1}{1+j}\cdot\frac{(1-j)}{(1-j)} = \frac{1-j}{1+1} = \frac{1}{2}(1-j) \checkmark$$

Now, we plot frequency response for the above example

$$H(jk\omega_0)\Big|_{\omega_0=1} = \frac{1}{\sqrt{1+k^2}} \tan^{-1}(-k)$$

$$= \frac{1}{\sqrt{1+k^2}} \angle \tan^{-1}(-k)$$



4-6 Fourier series transformation

- Given a periodic signal $x(t)$, and its fourier series expanssion is given as

$$x(t) = \sum_{k=-\infty}^{\infty} C_{kx}\, e^{+jk\omega_0 t}$$

- If $y(t) = A\,x(t) + B$, then the Fourier series can be written as

$$y(t) = A\left[C_{0x} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} C_{kx}\, e^{jk\omega_0 t}\right] + B$$

$$= \big(\overbrace{A C_{0x} + B}^{C_{0y}}\big) + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \overbrace{A\, C_{kx}}^{C_{ky}}\, e^{jk\omega_0 t}$$

Example 4.9 Given the Sawtooth signal $x(t)$ as shown in fig.1. Based on it, find Fourier series of signal $y(t)$ shown in fig.2

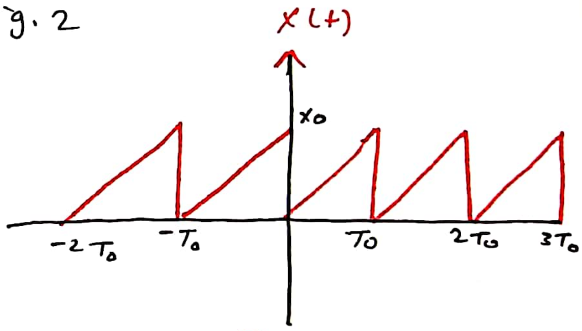


Fig. 1

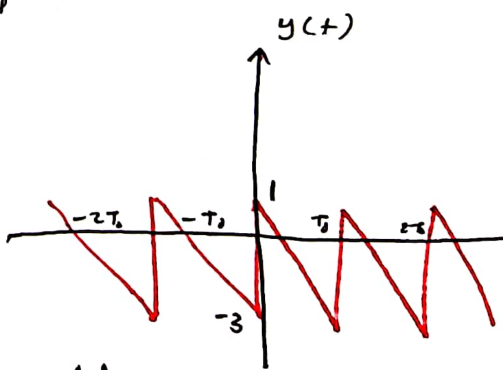
Solution

- From Table (4.3). Fourier Series of $x(t)$ Can be written as

$$x(t) = \frac{x_0}{2} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{x_0}{2\pi k}\, e^{j\frac{\pi}{2}k}\, e^{jk\omega_0 t}$$

- note that $y(t)$ is an amplitude transformed version of $x(t)$.

$y(t) = A\,x(t) + B$, How to Find A & B.



Sorry !!

please see the next page!!

$y(t) = A\,x(t) + B$.

$y(0) = A\,x(0) + B \Rightarrow 1 = A(0) + B \rightarrow B = 1$

$y(T_0) = A\,x(T_0) + B \Rightarrow -3 = A(x_0) + B \rightarrow A = -\frac{4}{x_0}$

Thus,

$$y(t) = -\frac{4}{x_0}\,x(t) + 1$$

There fore,

$$y(t) = [A C_{0x} + B] + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} A\,C_{kx}\,e^{jkw_0t} .$$

$$= \left[\left(\frac{-4}{x_0}\right)\left(\frac{x_0}{2}\right) + 1\right] + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \left(\frac{-4}{x_0}\right)\frac{x_0}{2\pi k}\,e^{j\frac{\pi}{2}}\,e^{jkw_0t}$$

$$= \underbrace{-1}_{C_{0y}} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \underbrace{\frac{2}{\pi k}\,e^{-j\frac{\pi}{2}}}_{C_{ky}}\,e^{jkw_0t}$$

Time Transformation

a general time transformation can be written as

$y(t) = x(at + b)$, we Consider special cases

problem

$\boxed{\text{Case } 1}$ if $a = -1$, $b = 0 \Rightarrow y(t) = x(-t) \rightarrow$ time reversal

~~y(t)~~ Fourier Series of $y(t)$ can be written as

$$y(t) = \sum_{k=-\infty}^{\infty} C_{ky}\,e^{jkw_0t}$$

complex Conjugate

$$C_{ky} = C_{kx}^{*}$$

$\boxed{\text{Case 2}}$ $a=1$, $b=t_0 \rightarrow y(t)=x(t-t_0)$, time shift.

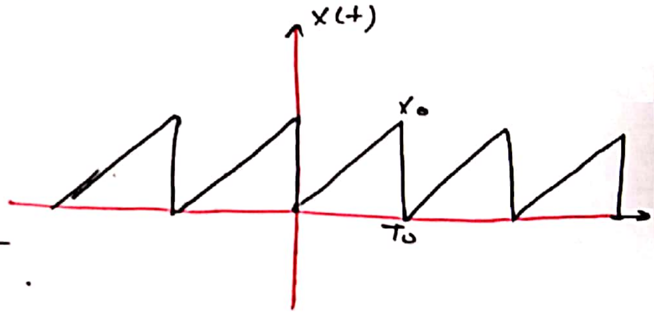
$$y(t)=\sum_{k=-\infty}^{\infty} C_{ky}\, e^{jk\omega_0 t}$$

$$C_{ky} = C_{kx}\, e^{-jk\omega_0 t_0}$$

please see Table 4.8 in the text book

$\boxed{\text{Example 4-10}}$

Consider the Sawtooth signal shown in the previous example



if $y_1(t) = x(-t)$
write Fourier Series of $y_1(t) = x(-t)$

Solution

$$x(t) = \frac{X_0}{2} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{X_0}{2\pi k}\, e^{j\frac{\pi}{2}}\, e^{jk\omega_0 t}.$$

$$y_1(t) = \sum_{k=-\infty}^{\infty} C_{ky}\, e^{jk\omega_0 t}$$

$$C_{ky} = C_{kx}^{*} \Rightarrow C_{0y} = C_{0x} = \frac{X_0}{2}.$$

$$C_{ky} = \left(\frac{X_0}{2\pi k}\, e^{j\frac{\pi}{2}}\right)^{*} = \frac{X_0}{2\pi k}\, e^{-j\frac{\pi}{2}}.$$

* End of chapter 4
* Suggested problems will be announced later.
*

Signals and Systems   Lecture (9)   19-04-2020

Chapter 5

The Fourier Transform

→ The Fourier transform: is a method of representing mathematical models of signals and systems in the frequency domain.

→ The Fourier transform is widely used in the electrical engineering field.

* Given a signal $f(t)$ in time domain

$\mathcal{F}\{f(t)\} = F(\omega)$ is the Fourier transform of $f(t)$ and it is given as

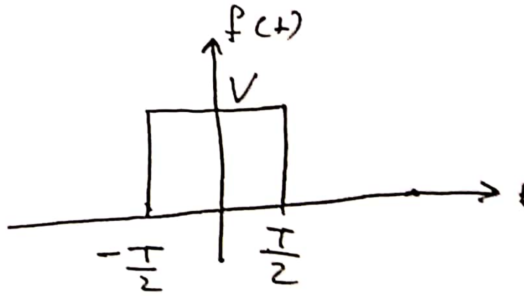
$$\mathcal{F}\{f(t)\} = F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt .$$

* the Inverse fourier transform :-

$$\mathcal{F}^{-1}\{F(\omega)\} = f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega$$

Example

Find the Fourier transform of a single rectangular pulse shown below.



$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt .$$

$$= \int_{-\frac{T}{2}}^{\frac{T}{2}} V\, e^{-j\omega t}\, dt$$

$$= -\frac{V}{j\omega}\left[e^{-j\omega\frac{T}{2}} - e^{+j\omega\frac{T}{2}} \right]$$

$$= \frac{V}{j\omega}\left[e^{j\omega\frac{T}{2}} - e^{-j\omega\frac{T}{2}} \right]$$

$$F(\omega) = \frac{V}{j\omega}\left[e^{j\omega\frac{T}{2}} - e^{-j\omega\frac{T}{2}}\right] = \frac{2V}{\omega}\sin\left(\omega\frac{T}{2}\right)$$

$$= \frac{2V}{W}\sin\left(\frac{TW}{2}\right)\cdot\frac{\frac{T}{2}}{\frac{T}{2}}$$

$$= \frac{2TV}{2}\,\text{Sinc}\left(\frac{TW}{2}\right) = TV\,\text{sinc}\left(\frac{TW}{2}\right)$$

Therefore

$$V\,\text{rect}\left(\frac{t}{T}\right) \xleftrightarrow{\mathcal{F}} TV\,\text{Sinc}\left(\frac{TW}{2}\right)$$

How to plot

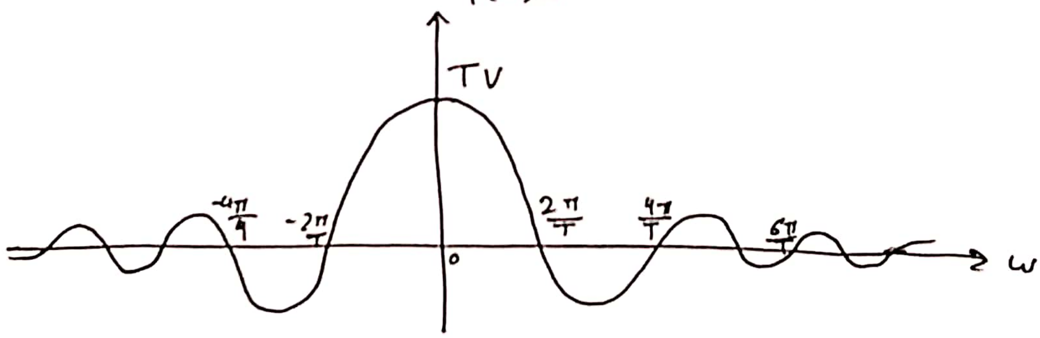
$F(\omega) = TV\,\text{sinc}\left(\frac{TW}{2}\right)$

→ maximum value of $F(\omega)$ is TV

→ $F(\omega)$ has several zeros, i.e., when $\sin\left(\frac{TW}{2}\right) = 0$

$\frac{TW}{2} = n\pi$, n integer.

$F(\omega) = TV\,\text{sinc}\left(\frac{TW}{2}\right)$



$$f(t) \xleftrightarrow{\mathcal{F}} F(\omega)$$

$f(t)$ and $F(\omega)$ is the transform pair

Example.

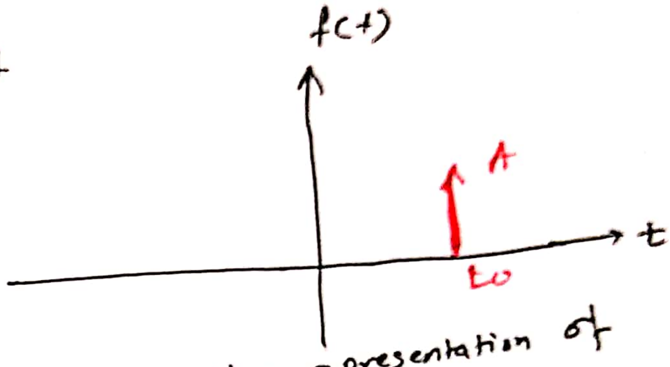
Find Fourier transform of the impulse function defined as,

$$f(t) = A\,\delta(t - t_0)$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} A\,\delta(t - t_0)\, e^{-j\omega t}\, dt$$

$$= A\, e^{-j\omega t_0} \int_{-\infty}^{\infty} \delta(t - t_0)\, dt$$

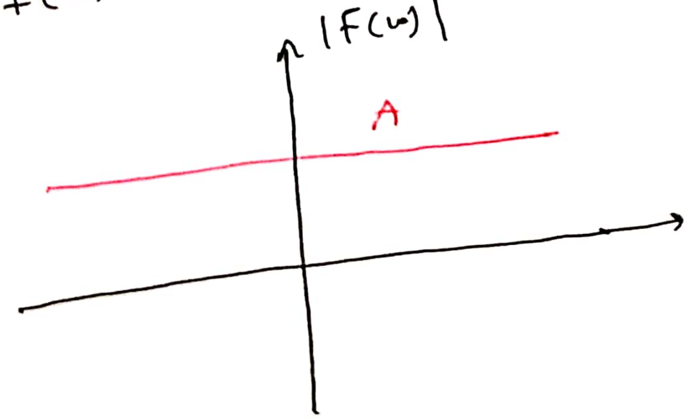
$$= A\, e^{-j\omega t_0}$$



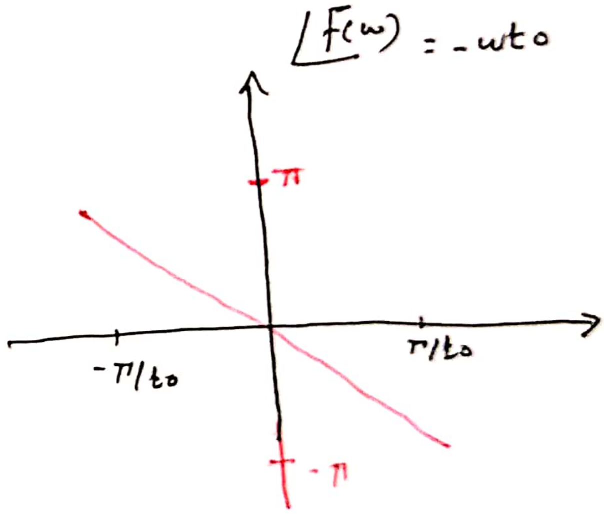
time representation of f(t)

Now, we plot frequency spectrum of $F(\omega)$

$$F(\omega) = A\, e^{-j\omega t_0} = A \angle -\omega t_0$$



Amplitude spectrum



Special Case

$f(t) = \delta(t) \rightarrow F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt$

$$\boxed{F(\omega) = 1}$$

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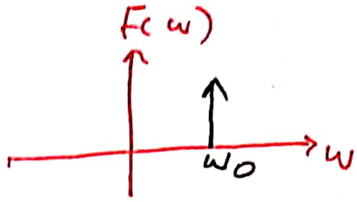
Example

Find the Inverse fourier transform of

$F(w) = \delta(w - w_0)$



Sol –

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(w)\, e^{jwt}\, dw$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(w - w_0)\, e^{jwt}\, dw$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(w - w_0)\, e^{jw_0 t}\, dw$$

$$= \frac{e^{jw_0 t}}{2\pi} \int_{-\infty}^{\infty} \delta(w - w_0)\, dw \quad 1$$

$$= \frac{e^{jw_0 t}}{2\pi} \quad \#$$

$$\delta(w - w_0) \xrightarrow{\mathcal{F}^{-1}} \frac{e^{jw_0 t}}{2\pi}.$$

$$\delta(w) \xrightarrow{\mathcal{F}^{-1}} \frac{1}{2\pi}.$$

$$e^{jw_0 t} \xrightarrow{\mathcal{F}} 2\pi\, \delta(w - w_0)$$

$$1 \xrightarrow{\mathcal{F}} 2\pi\, \delta(w)$$

$\boxed{5.2}$ properties of the fourier transform.

5.2 properties of The Fourier transform

① Linearity

$f_1(t) \xleftrightarrow{\mathcal{F}} F_1(\omega)$ and $f_2(t) \xleftrightarrow{\mathcal{F}} F_2(\omega)$

Then

$(\alpha_1 f_1(t) + \alpha_2 f_2(t)) \xleftrightarrow{\mathcal{F}} \alpha_1 F_1(\omega) + \alpha_2 F_2(\omega)$

where α_1 & α_2 are constants

Example

Find the fourier transform of $f(t) = B\cos(\omega_0 t)$

Solution

$$B\cos(\omega_0 t) = \frac{B}{2}\left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right]$$

$$\frac{B}{2} e^{j\omega_0 t} \xleftrightarrow{\mathcal{F}} \pi B\,\delta(\omega - \omega_0)$$

$$\frac{B}{2} e^{-j\omega_0 t} \xleftrightarrow{\mathcal{F}} \pi B\,\delta(\omega + \omega_0)$$

$$B\cos(\omega_0 t) \xleftrightarrow{\mathcal{F}} \pi B\left[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right]$$

Notes from previous lecture
$e^{j\omega_0 t} \xleftrightarrow{\mathcal{F}} 2\pi\,\delta(\omega - \omega_0)$

② Time Scaling

$$f(t) \xleftrightarrow{\mathcal{F}} F(\omega)$$

$$f(at) \xleftrightarrow{\mathcal{F}} \frac{1}{|a|} F\left(\frac{\omega}{a}\right)$$

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Example 5.3

Find the Fourier Transform of the rectangular waveform

$$g(t) = \text{rect}\left(\frac{2t}{T_1}\right)$$

notes from previous lecture

$$\text{rect}\left(\frac{t}{T}\right) \xleftrightarrow{\mathcal{F}} T\,\text{sinc}\left(\frac{T\omega}{2}\right)$$

* assume $\Rightarrow$ $f(t) = \text{rect}\left(\frac{t}{T_1}\right)$

$\Rightarrow g(t) = f(2t)$

$$\text{rect}\left(\frac{t}{T_1}\right) = f(t) \xleftrightarrow{\mathcal{F}} F(\omega) = T_1\,\text{sinc}\left(\frac{\omega T_1}{2}\right)$$

$$\text{rect}\left(\frac{2t}{T_1}\right) = f(2t) \xleftrightarrow{\mathcal{F}} \frac{1}{2}F\left(\frac{\omega}{2}\right) = \frac{T_1}{2}\,\text{sinc}\left(\frac{\omega T_1}{4}\right)$$

$$\text{rect}\left(\frac{2t}{T_1}\right) \xleftrightarrow{\mathcal{F}} \frac{T_1}{2}\,\text{sinc}\left(\frac{\omega T_1}{4}\right)$$

3 Time shifting

$$f(t - t_0) \xleftrightarrow{\mathcal{F}} F(\omega)\, e^{-j\omega t_0}$$

Example 5.4

Find the Fourier Transform of $f(t) = \delta(t - t_0)$

$$f(t) = \delta(t) \xleftrightarrow{\mathcal{F}} 1$$

$$f(t - t_0) = \delta(t - t_0) \xleftrightarrow{\mathcal{F}} e^{-j\omega t_0}$$

Example 5-5

Find Fourier Transform of $x(t) = 10\cos[200\pi(t - 1.25\times10^{-3})]$

Sol.

$$x(t) = 10\cos\left(200\pi t - \frac{\pi}{4}\right)$$

Assume $\Rightarrow$ $f(t) = 10\cos(200\pi t) \xleftrightarrow{\mathcal{F}} 10\pi\left[\delta(\omega - 200\pi) + \delta(\omega + 200\pi)\right]$

$$x(t) = f\left(t - \frac{\pi}{4}\right) \xleftrightarrow{\mathcal{F}} F(\omega)\, e^{-j\omega\frac{\pi}{4}} = 10\pi e^{-j\omega\frac{\pi}{4}}\left[\delta(\omega - 200\pi) + \delta(\omega + 200\pi)\right]$$

$\Rightarrow \Rightarrow \Rightarrow$

note that

$$F(\omega)\,\delta(\omega-\omega_0) = F(\omega_0)\,\delta(\omega-\omega_0)$$

④ Time Transformation

$$f(at-t_0) \xleftrightarrow{\mathcal{F}} \frac{1}{|a|} F\left(\frac{\omega}{a}\right) e^{-jt_0\left(\frac{\omega}{a}\right)}$$

Example 5-6

Find the Fourier transform of $g(t) = 3\,\text{rect}\left[\frac{(t-4)}{2}\right]$ using properties.

Solution

Assume → $f(t) = 3\,\text{rect}(t) \xleftrightarrow{\mathcal{F}} 3\,\text{Sinc}\left(\frac{\omega}{2}\right)$

$a = 0.5 \quad t_0 = 2$

$$g(t) = f\left(\frac{t}{2}-2\right) \xleftrightarrow{\mathcal{F}} \frac{1}{0.5}(3)\,\text{sinc}\left(\frac{\omega}{2(0.5)}\right) e^{-j(2)\frac{\omega}{0.5}}$$

$$3\,\text{rect}\left[\frac{t-4}{2}\right] \xleftrightarrow{\mathcal{F}} 6\,\text{sinc}(\omega)\, e^{-j4\omega}$$

⑤ Duality

5-2 properties of the fourier transform

(5) Duality

This property states that if

$$f(t) \xleftrightarrow{\mathcal{F}} F(\omega)$$

Then, if we have a function in time domain $F(t)$, such that $F(t) = F(\omega)\big|_{\omega = t}$ then

$$\mathcal{F}[F(t)] = 2\pi f(-\omega), \quad \text{where } f(-\omega) = f(t)\big|_{t=-\omega}$$

Example

Find the fourier transform of $\frac{A\beta}{\pi}\,\text{sinc}(\beta t)$

Solution

note that

$$V\,\text{rect}\left(\frac{t}{T}\right) \xleftrightarrow{\mathcal{F}} TV\,\text{sinc}\left(\frac{T\omega}{2}\right)$$

$$\frac{A}{2\pi}\,\text{rect}\left(\frac{t}{2\beta}\right) \longleftrightarrow \frac{AB}{\pi}\,\text{sinc}(\beta t)$$

$$\mathcal{F}\left[\frac{AB}{\pi}\,\text{sinc}(Bt)\right] = \frac{2\pi A}{2\pi}\,\text{rect}\left(\frac{-\omega}{2B}\right)$$

$$= A\,\text{rect}\left(\frac{\omega}{2B}\right)$$

6. Convolution

$f_1(t) \longleftrightarrow F_1(\omega)$

$f_2(t) \longleftrightarrow F_2(\omega)$

$f_1(t) * f_2(t) \longleftrightarrow F_1(\omega) \cdot F_2(\omega)$

For LTI system $\quad x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$

$y(t) = x(t) * h(t)$

$Y(\omega) = X(\omega) \cdot H(\omega).$

7. frequency shifting

$x(t)\, e^{j\omega_0 t} \longleftrightarrow X(\omega - \omega_0)$

Example 5.9

$g_1(t) = 2\cos(200\pi t)$ $\qquad g_2(t) = 5\cos(1000\pi t)$

$g_3(t) = g_1(t)\, g_2(t)$

Find the Fourier transform of $g_3(t)$.

Solution

$g_3(t) = 2\cos(200\pi t)\, 5\cos(1000\pi t)$

$= \frac{10}{2}\cos(200\pi t)\left[e^{j1000\pi t} + e^{-j1000\pi t}\right].$

* note that $\underbrace{5\cos(200\pi t)}_{x(t)}\, e^{j1000\pi t} \xleftrightarrow{\mathcal{F}} X_1(\omega - 1000\pi).$

$5\cos(200\pi t) \xleftrightarrow{\mathcal{F}} 5\pi\left[\delta(\omega - 200\pi) + \delta(\omega + 200\pi)\right] = X_1(\omega)$

$X_1(\omega - 1000\pi) = 5\pi\left[\delta(\omega - 1200\pi) + \delta(\omega - 800\pi)\right]$

* note that $5\cos(200\pi t)\, e^{-j1000\pi t} \longleftrightarrow X_1(\omega + 1000\pi)$

$X_1(\omega + 1000\pi) = 5\pi\left[\delta(\omega + 800\pi) + \delta(\omega + 1200\pi)\right]$

$G_3(\omega) = 5\pi\left[\delta(\omega - 1200\pi) + \delta(\omega - 800\pi) + \delta(\omega + 800\pi) + \delta(\omega + 1200\pi)\right]$

8. Time Differentiation

if $f(t) \xleftrightarrow{\mathcal{F}} F(\omega)$

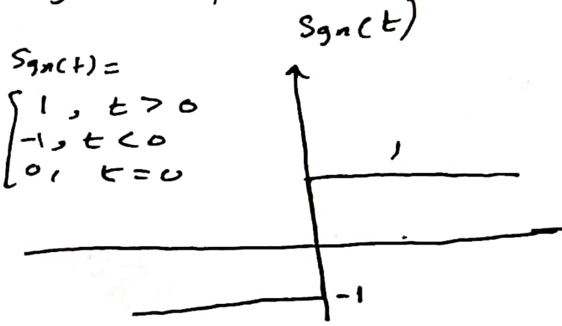
$$\frac{d[f(t)]}{dt} \xleftrightarrow{\mathcal{F}} j\omega F(\omega)$$

$$\frac{d^n[f(t)]}{dt^n} \xleftrightarrow{\mathcal{F}} (j\omega)^n F(\omega)$$

Example 5.10

Find the Fourier transform of the signum function shown in figure

$$\mathrm{sgn}(t) = \begin{cases} 1, & t>0 \\ -1, & t<0 \\ 0, & t=0 \end{cases}$$



Solution

plz note that

$$\frac{d[\mathrm{sgn}(t)]}{dt} = 2\delta(t)$$

$$\frac{d[\mathrm{sgn}(t)]}{dt} \xleftrightarrow{\mathcal{F}} 2$$

$$2 = j\omega F(\omega) \rightarrow F(\omega) = \frac{2}{j\omega}.$$

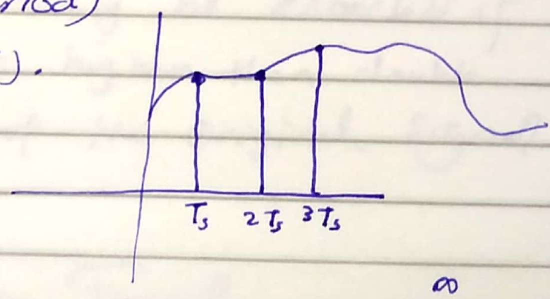
5.4 Sampling Continous_time signals.

- Sampling :- generating an ordered number sequence by taking values of $f(t)$ at spacified instants of time.

- sampling generates a number sequence ~~f(t)~~, $f(t_1), f(t_2), \cdots f(t_n)$
  where t_n represents the instants at which the sampling occures.

- Sampling is very important for Analog-to-digital (A/D) Converter

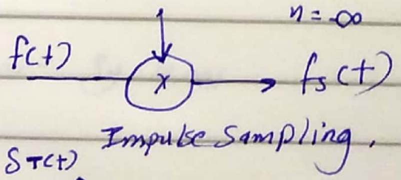
- Continous_time signals are sampled at equal increments of time. Called T_s (sample period)
- The sampled signal is called $f(nT_s)$.
  $n \rightarrow$ integer.



Types of Sampling $\begin{cases} \rightarrow \text{Ideal} \\ \rightarrow \text{practical} \end{cases}$

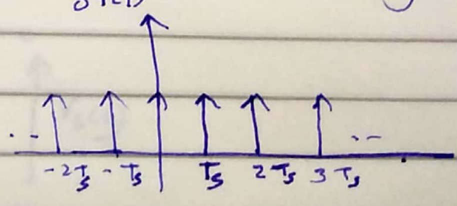
- Ideal Sampling (Impulse Sampling)

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$



$$f_s(t) = f(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s).$$

$$f_s(t) = \sum_{n=-\infty}^{\infty} f(nT_s)\, \delta(t - nT_s)$$

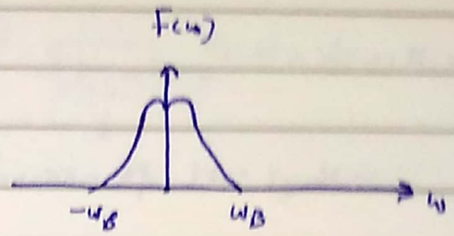


* We study the sampled signal in frequency domain.

$$F_s(\omega) = \frac{1}{2\pi} F(\omega) * \omega_s \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} F(\omega) * \delta(\omega - k\omega_s$$

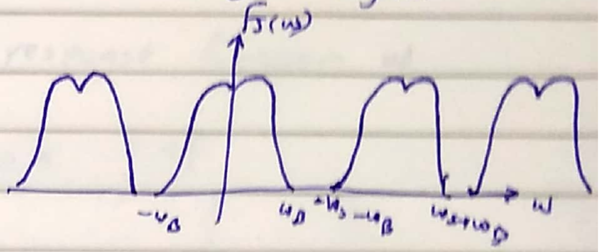
Therefore

$$F_s(\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} F(\omega - k\omega_s)$$



Frequency Spectrum of the original signal.

* We can extract the original signal $F(\omega)$ from the sampled one by multipling $F_s(\omega)$ by a low pass filter (Data reconstruction)



* This can be only achieved when

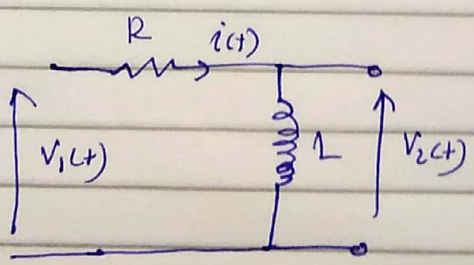
$$\omega_s > 2\omega_B .$$

↳ Nyquist frequency

The original signal can only be extracted if the sampling frequency is higher than double of the frequency of the original signal

5-5 Applications of The fourier Transform

- Frequency Response of Linear Systems

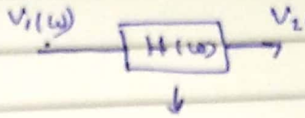


$$V_1(t) = R\,i(t) + L\frac{di(t)}{dt} \quad , \quad V_2(t) = L\frac{di}{dt} .$$

$$V_1(\omega) = R\,I(\omega) + jL\omega\, I(\omega) \quad \cdot \quad V_2(\omega) = Lj\omega\, I(\omega)$$

$$I(\omega) = \frac{V_1(\omega)}{(R + jL\omega)}$$

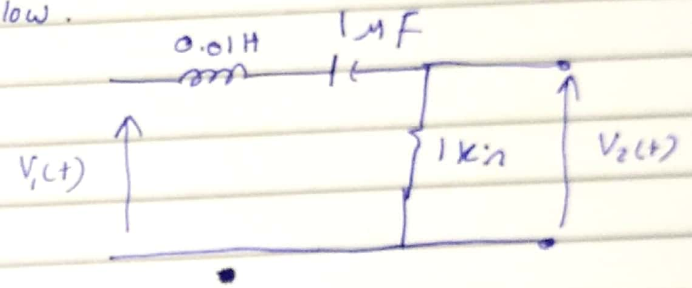
$$V_2(\omega) = \frac{j\omega L}{R + j\omega L} V_1(\omega).$$



$H(\omega)$: frequency response ~~of~~ function of the system.
transfer function.

$$H(\omega) = \frac{V_2(\omega)}{V_1(\omega)}.$$

$\boxed{Hw}$ Find the Frequency ~~Res~~ response function of the ckt shown below.



$$H(\omega) = \frac{V_2(\omega)}{V_1(\omega)},$$

$+ V(\omega) \ldots = \ldots j\omega L\, I(\omega) \ldots$

5.6 Energy and power Density Spectra.

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5.6 Energy and power Density spectral

Energy Density spectrum

a signal $f(t)$ is an energy signal if

$$E = \int_{-\infty}^{\infty} |f(t)|^2 dt < \infty$$

Generally, aperiodic signals are Energy signals.

$$E = \int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega.$$

Note that $|F(\omega)|^2$ is real and Even function of frequency

$$\mathcal{E}_f(\omega) = \frac{1}{\pi} |F(\omega)|^2 = \frac{1}{\pi} F(\omega) F(\omega)^*$$

↳ Energy spectral density.

$$E = \int_{0}^{\infty} \mathcal{E}(\omega) d\omega$$

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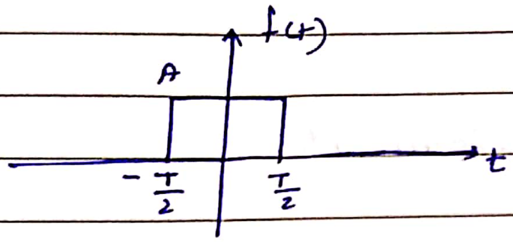
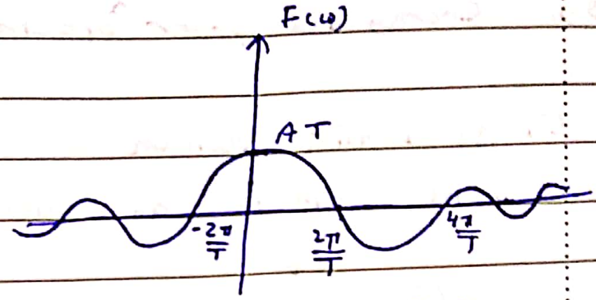
Example 5-19



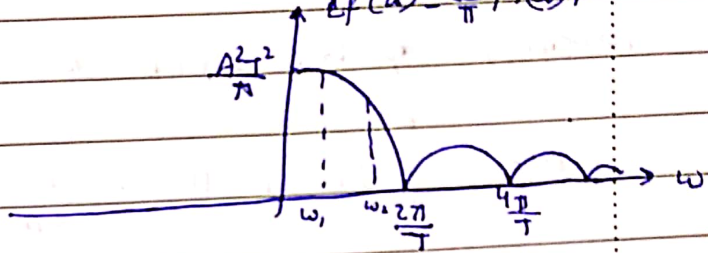


$$F(\omega) = AT\, \text{Sinc}\left(\frac{T\omega}{2}\right)$$

$$E = \int_{-\infty}^{\infty} |f(t)|^2 \, dt$$

$$= \int_{-\frac{T}{2}}^{T/2} A^2 dt = A^2 [T]$$

$$\mathcal{E}_f(\omega) = \frac{1}{\pi} |F(\omega)|^2$$



Energy spectral density

Energy Contained in the band of frequencies between ω_1 and ω_2 is

$$\int_{\omega_1}^{\omega_2} \mathcal{E}(\omega)\, d\omega \quad \text{The area under the Curve}$$

power Density spectrum.

if the signal $f(t)$ satisfies

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{T/2} |f(t)|^2 dt < \infty$$

Then $f(t)$ is power signal and P is the power of the signal.

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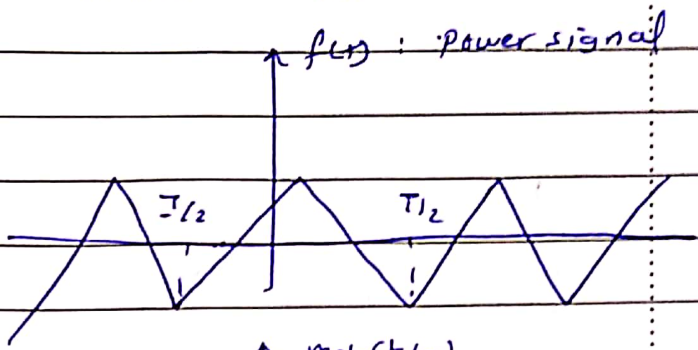
power signals are periodic signals.

power signals have infinite Energy.

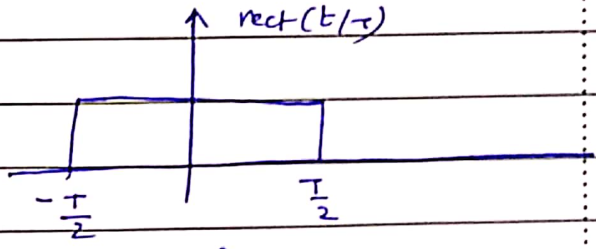
~~power~~

$f_T(t)$:- truncated part of $f(t)$



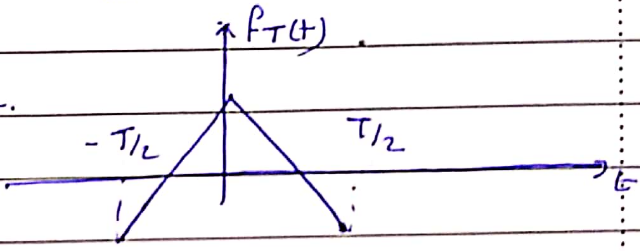
$f_T(t) = f(t)\, \mathrm{rect}(\frac{t}{T})$.

$f_T(t) \xrightarrow{\mathcal{F}} F_T(\omega)$



Note that

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-\infty}^{\infty} |f_T(t)|^2 dt.$$



$S_f(\omega)$: power spectral density of $f(t)$.

$$\cancel{P = \frac{1}{2\pi}\int_{-\infty}} \qquad S_f(\omega) = \lim_{T \to \infty} \frac{1}{T} |F_T(\omega)|^2$$

$$P = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_f(\omega)\, d\omega = \frac{1}{\pi} \int_0^{\infty} S_f(\omega)\, d\omega.$$

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power and Energy Transmission.

The input-output relationship

$$f(t) \rightarrow \boxed{h(t)} \rightarrow g(t)$$

$$G(\omega) = H(\omega)\, F(\omega)$$

$$g(t) = f(t) * h(t)$$

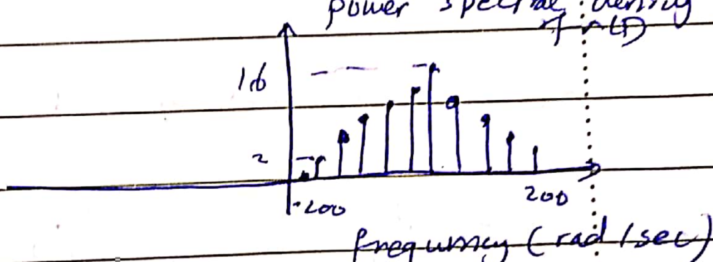
$$\mathcal{E}_g(\omega) = |H(\omega)|^2\, \mathcal{E}_f(\omega)$$ Energy spectral density of $g(t)$

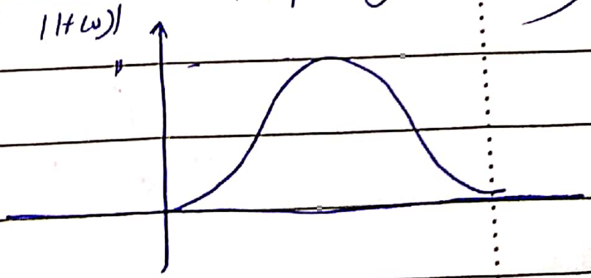
$$P_g(\omega) = |H(\omega)|^2\, P_f(\omega)$$ power spectral density of $g(t)$

Example 5-21

if $x(t)$ has power spectral density shown in figure below. and it is the input to a linear system with frequency response plotted in figure 2 plot the power spectral density of $g(t)$

$$P_y(\omega) = |H(\omega)|^2\, P_x(\omega)$$





Please Solve problem

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CH6 Applications of the Fourier Transform

6.1 Ideal Filters.

$$H(w) = \frac{V_2(w)}{V_1(w)}$$

$H(w)$ transfer function